Acta Mechanica 45, 163—176 (1982)

ACTA MECHANICA

Kinematics of a Multi-Dimensional Shock of Arbitrary Strength in an Ideal Gas

By

P. Prasad, Karlsruhe, Federal Republic of Germany

With 4 Figures

(*Received August 25, 1981*)

Summary

It has been shown that the kinematics of a shock front in an ideal gas with constant specific heat can be completely described by a first order nonlinear partial differential equation (called here — shock manifold equation or SME) which reduces to the characteristic partial differential equation as the shock strength tends to zero. The condition for the existence of a nontrivial solution of the jump relations across the shock turns out to be the Prandtl relation. Continuing the functions representing the state on the either side of the shock to the other side as infinitely differentiable functions and embedding the shock in a one parameter family of surfaces, it has been shown that the Prandtl relation can be treated as a required shock manifold equation for a function Φ, where $\Phi = 0$ is the shock surface. We also show that there are other forms of the SME and prove an important result that they are equivalent. Shock rays are defined to be the characteristic curves of a SME and it has been shown that when the flow on either side of the shock is at rest, the shock rays are orthogonal to the successive positions of the shock surface. Certain results have been derived for a weak shock, in which case the complete history of the curved shock can be determined for a class of problems.

1. Introduction

Kinematics of a wave front when it is not a shock front is clear and well understood, both in the case of hyperbolic and dispersive waves (Hayes, 1970). For a hyperbolic wave, the kinematics of a wave front reduces to the theory of characteristic surfaces of the corresponding hyperbolic system of equations. The rays are related to the bicharacteristic curves. Given the initial position and shape of a wave front, the same can be determined at any time by solving the bicharacteristic (or ray) equations, which are ordinary differential equations. No such theory exists for a shock front in gasdynamics except for a recent work of Maslov (1980). That, there is a need for such a theory is evident from the following consideration taken from Whitham (1956). Consider a shock front which is initially curved and concave to the gas at rest ahead of it (on the right) as shown in the Fig. 1. The normals to the initial surface (or the rays of the linear theory) form an envelop, called a "caustic" at which the ray tube area tends to zero. At such points, the linear theory and also the weakly nonlinear

0001-5970/82/0045/0163/$02.80

theory of Whitham (1956) using linear rays fail. The actual flow pattern at the "nonlinear caustic" is very complex (Sturtevant and Kulkarny, 1976), however, one can hope that in certain situations the rays (now shock rays) deviate so much due to the increase in the intensity of the shock at the centre that the caustic may not be formed as shown in the Fig. 2. But what are these shock rays? In the absence of any mathematical theory, Whitham (1957) defines the "shock-rays" also as curves orthogonal to the successive positions of the shock surface, when the gas ahead of the shock is at rest.

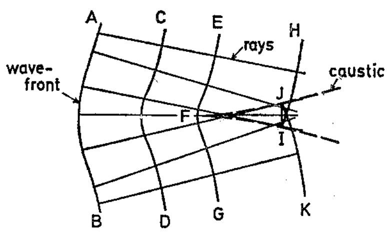

Fig. 1. Linear rays envelop a caustic surface

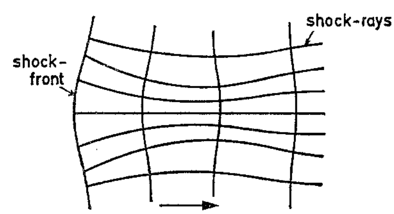

Fig. 2. But actually the rays may deviate due to increase in the intensity of the shock at the centre and the caustic may not be formed

In this paper, we develop a mathematical theory on the kinematics of a shock front in an ideal gas by deriving a shock manifold partial differential equation and give a mathematical definition of shock rays. This has been possible due to an embedding theorem, which is extremely important in the deduction of the characteristic partial differential equation (Courant and Hilbert, 1962, pp. 557—558). Initially the condition that a surface $\chi(x_1, \ldots, x_m, t) = 0$ is a characteristic surface reduces to

$$Q_{\text{ch}}(x_1, \ldots, x_m, t; \chi_{x_1}, \ldots, \chi_{x_m}, \chi_t) = 0 \quad \text{on} \quad \chi(x_1, \ldots, x_m, t) = 0 \tag{1.1}$$

where Q_{ch} is the characteristic determinant of a hyperbolic system. The theorem says that the surface $\chi = 0$ can be embedded in a one parameter family of characteristic surfaces

$$\phi(x_1, \ldots, x_m, t) = c \tag{1.2}$$

such that $\chi = 0$ coincides with a characteristic surface obtained by putting a particular value of c in (1.2) and the function $\phi(x_1, \ldots, x_m, t)$ satisfies the characteristic partial differential equation

$$Q_{\text{ch}}(x_1, \ldots, x_m, t; \phi_{x_1}, \ldots, \phi_{x_m}, \phi_t) = 0. \tag{1.3}$$

There is a little problem in embedding a propagating shock surface in a family of surfaces because the condition for the shock surface (i.e. Eq. (3.18) or (3.19)) contains two sets of functions — one defined only ahead of the shock and other only behind the shock. However, this difficulty is easily removed by continuing these on the other side of the shock as infinitely differentiable functions. In the section 2, we explain the whole procedure of embedding with the help of an example in two independent variables.

Using the theory of generalised functions, Maslov (1980) has developed a successive method of computation of the position of the shock front and also contains some of the ideas of this paper. Maslov treats only an isentropic gas motion without consideration of the energy equation and hence his method of computation, aimed not for an arbitrary shock strength, is correct for weak shocks as long as the entropy changes across the shock can be neglected. But it is also clear that his method is not valid for the caustic problem as the linear wave front has a singularity. Moreover, if one stops at the first approximation in Maslov's method, as generally is the case for most of such problems, one can take into account of the effect of only a linear variation of the quantities in the flow behind the shock. On any account, Maslov's work presents a clear understanding of the problem mathematically and can be regarded as a major break through in an approximate determination, at least in theory, of the shock position. A similar method has been developed by Grinfeld (1978).

The aim of the paper is to give only a simple mathematical theory of the shock-kinematics. We do not attempt here to develop a method of solution, this we do in a subsequent paper (Prasad *et al.*) as mentioned in the last section of this paper.

2. An Example of Shock Embedding

Consider a quasi-linear partial differential equation in the conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{1}{2}\, u^2\right] = 0. \tag{2.1}$$

Across a shock discontinuity of (2.1), the following jump condition must be satisfied

$$[u]\, \Psi_t + \left[\frac{1}{2}\, u^2\right] \Psi_x = 0, \tag{2.2}$$

where the symbol $[f]$ represents the jump in the quantity f across the shock from left to right and $\Psi(x, t) = 0$ represents the curve of discontinuity in (x, t)-plane. If suffixes l and r attach to f represent the values of the quantity f just on the left and just on the right of the discontinuity, then

$$[f] = f_r - f_l. \tag{2.3}$$

A discontinuous solution of the Eq. (2.1) valid for $t > -1$ and containing a shock is

$$u(x, t) = \begin{cases} 0 & \text{for} \quad -\infty < x \leqq 0 \\ \dfrac{x}{1+t} & \text{for} \quad 0 < x \leqq (1+t)^{1/2} \\ 0 & \text{for} \quad (1+t)^{1/2} < x < \infty. \end{cases} \tag{2.4}$$

The equation of the curve of discontinuity can be written in the form

$$\Psi \equiv x - (1+t)^{1/2} = 0. \tag{2.5}$$

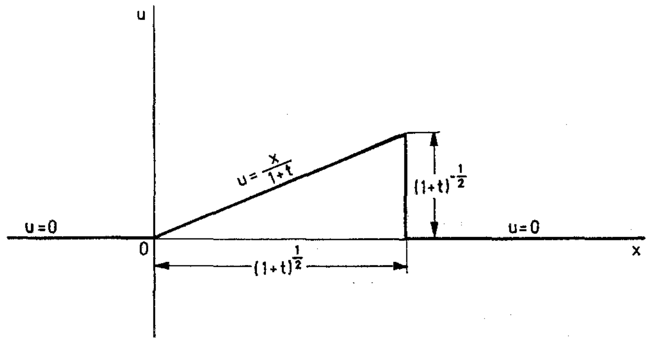

Fig. 3. Graph of the solution (2.4) at a given time t

The solution (2.4) has been graphically shown in the Fig. 3. Along the curve of discontinuity in u, $u_r - u_l \neq 0$ and we can divide (2.3) by $u_r - u_l$ to get

$$\Psi_t + \frac{u_l + u_r}{2} \Psi_x = 0 \quad \text{on} \quad \Psi(x, t) = 0. \tag{2.6}$$

The function u_l appearing in (2.6) is the limiting value of the function $x/(1 + t)$ from left and u_r is that of 0 from right. *We extend the definition of u_r and u_l in such a manner that when embedding is completed, the members of the family of shocks for $x < (1 + t)^{1/2}$ have the state $x/(1 + t)$ on the left and the members of the family of shocks for $x > (1 + t)^{1/2}$ have the state 0 on the right.* Therefore, we extend the definition of these functions in a neighbourhood of the curve (2.5) by (for $t > -1$)

$$u_r = 0, \quad u_l = \frac{x}{1 + t}. \tag{2.7}$$

The functions u_l and u_r defined here satisfy the partial differential Eq. (2.1) as infinitely differentiable functions separately. We note that we have simply extended the solution $u = x/(1 + t)$ on the left of the discontinuity to the right by the same expression in x and t, and similarly the solution $u = 0$ on the right to the left. We now define a function $[u]$ of two independent variables by

$$[u] \equiv u_r - u_l = -\frac{x}{1 + t}. \tag{2.8}$$

With these extended definitions, the Eq. (2.6) i.e.

$$\Psi_t + \frac{x}{2(1 + t)} \Psi_x = 0 \quad \text{on} \quad \Psi(x, t) = 0 \tag{2.9}$$

is still satisfied.

Now we can use the embedding theorem mentioned in the introduction. We find that there exists a function Φ defined by

$$\Phi = \frac{x}{(1 + t)^{1/2}} \tag{2.10}$$

such that Φ satisfies the partial differential equation

$$\Phi_t + \frac{x}{2(1+t)}\,\Phi_x = 0. \tag{2.11}$$

The shock curve $\Psi = 0$ is embedded in a one parameter family of shocks: $\Phi = c$, where c is the parameter. The equations $\Psi = 0$ and $\Phi = 1$ represent the same curve — the curve of discontinuity of our solution (2.4) as shown in the Fig. 4.

We note another important point in this particular example of a *single* conservation law. Since (2.11) is satisfied along each one of the shock curves $\Phi = c$, the jump relation (2.2) is also satisfied by every one of these one parameter shocks i.e.

$$[u]\,\Phi_t + \left[\frac{1}{2}\,u^2\right]\Phi_x = 0 \tag{2.12}$$

identically in x and t provided $[u] = u_r - u_l$ and $\left[\frac{1}{2}\,u^2\right] = \frac{1}{2}\,(u_r^2 - u_l^2)$ are defined with the help of the extended functions u_l and u_r as in (2.7). Since u_l satisfies the partial differential Eq. (2.1) even in its domain of extension, we can solve u_l from $\Phi_t + \frac{1}{2}\,(u_l + u_r)\,\Phi_x = 0$ in terms of u_r, Φ_t and Φ_x and substitute in the Eq. (2.1). We get

$$\begin{aligned}\Phi_x^2\Phi_{tt} - \Phi_x(3\Phi_t + u_r\Phi_x)\,\Phi_{xt} + \Phi_t(2\Phi_t + u_r\Phi_x)\,\Phi_{xx}\\ + \frac{1}{2}\,\Phi_x^2\left\{\Phi_x\,\frac{\partial u_r}{\partial t} + (2\Phi_t - u_r\Phi_x)\,\frac{\partial u_r}{\partial x}\right\} = 0\end{aligned} \tag{2.13}$$

which takes a particularly simple form when $u_r \equiv 0$. Therefore, the set of all functions Φ, giving one parameter family of shocks for which the state on the right is u_r, satisfy a quasilinear second order partial differential Eq. (2.13), which

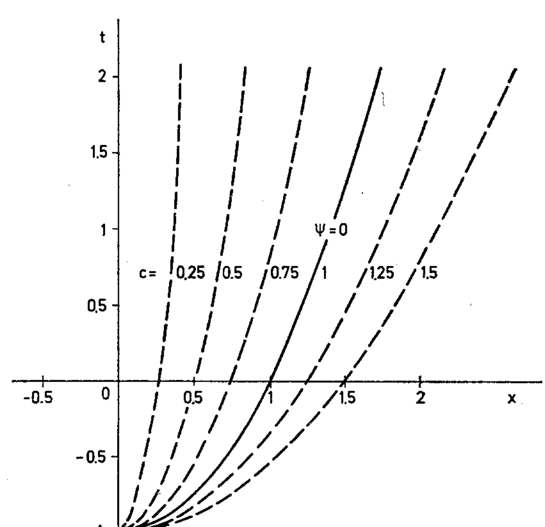

Fig. 4. The shock curve, $\Psi = 0$, of the solution (2.4) is embedded in a family of shock curves $\Phi = c$

is hyperbolic. Given an appropriate initial condition for (2.1) with a shock discontinuity, we can easily write the initial conditions for (2.13) i.e. the values of $\Phi(x, 0)$ and $\Phi_t(x, 0)$. Infact $\Phi(x, 0)$ can be prescribed arbitrarily and then we can set $\Phi_t(x, 0) = -\frac{1}{2}\{u_l(x, 0) + u_r(x, 0)\}\,\Phi_x$. Thus the Eq. (2.13) can be solved in theory, though the solution of this problem is not simpler than the original one. However, what is important here to note is that, unlike the original problem, we need to find only a continuously differentiable (i.e. shock free) solution of (2.13).

Before we pass on to the consideration of gasdynamic shocks we mention that the continuation of the state on the left (or right) of $\Psi = 0$ to the right (or left) of $\Psi = 0$ as infinitely differentiable solution is not unique. Therefore, the embedding described here is also not unique.

3. Derivation of the Shock Manifold Equation for an Ideal Gas

For simplicity we consider the unsteady motion of an ideal gas with constant specific heat in two space dimensions, (x, y)-plane, only. The arguments and the ideas can be easily extended to three dimensional flows. Consider a solution of the equations of motion sufficiently smooth (say for simplicity, infinitely differentiable) in a domain in (x, y, t)-space except for a sufficiently smooth shock surface $\Psi(x, y, t) = 0$ in the flow. The limiting values of the functions representing the state of the flow of the gas and their partial derivatives, as we approach the shock surface from the domains on the two sides of it, exist i.e. all these quantities suffer finite jumps across the surface $\Psi = 0$. Our discussion is based on the existence of such a solution, which we assume to be known. From the conservation form of equations of motion (Zierep, 1976) we can derive the following jump relations across the shock:

$$[\varrho]\,\Psi_t + [\varrho u]\,\Psi_x + [\varrho v]\,\Psi_y = 0 \tag{3.1}$$

$$[\varrho u]\,\Psi_t + [\varrho u^2 + p]\,\Psi_x + [\varrho uv]\,\Psi_y = 0 \tag{3.2}$$

$$[\varrho v]\,\Psi_t + [\varrho uv]\,\Psi_x + [\varrho v^2 + p]\,\Psi_y = 0 \tag{3.3}$$

and

$$\left[\frac{1}{\gamma-1}\,p + \frac{1}{2}\,\varrho q^2\right]\Psi_t + \left[\frac{\gamma}{\gamma-1}\,up + \frac{1}{2}\,\varrho u q^2\right]\Psi_x + \left[\frac{\gamma}{\gamma-1}\,vp + \frac{1}{2}\,\varrho v q^2\right]\Psi_y = 0 \tag{3.4}$$

where u, v are the components of the fluid velocity, ϱ the density, p pressure, γ the ratio of specific heats, $q^2 = u^2 + v^2$ and the symbol [ ] for the jump of a quantity is defined in terms of the quantities on left and right of the shock by (2.3). We can easily show that these are equivalent to the usual form of oblique shock relations normally used in gasdynamics, when we note that the normal and the tangential components of the fluid velocity relative to the shock surface are respectively

$$L = \frac{u\Psi_x + v\Psi_y + \Psi_t}{(\Psi_x{}^2 + \Psi_y{}^2)^{1/2}}, \quad T = \frac{-u\Psi_y + v\Psi_x}{(\Psi_x{}^2 + \Psi_y{}^2)^{1/2}}. \tag{3.5}$$

The oblique shock relations are

$$\varrho_l L_l = \varrho_r L_r \tag{3.6}$$

$$\varrho_l L_l T_l = \varrho_r L_r T_r \tag{3.7}$$

$$\varrho_l L_l^2 + p_l^2 = \varrho_r L_r^2 + p_r \tag{3.8}$$

and

$$\varrho_l L_l \left(\frac{\gamma}{\gamma - 1} \frac{p_l}{\varrho_l} + \frac{1}{2} L_l^2\right) = \varrho_r L_r \left(\frac{\gamma}{\gamma - 1} \frac{p_r}{\varrho_r} + \frac{1}{2} L_r^2\right). \tag{3.9}$$

Using the following formula for the jump of the product of two quantities f and g

$$[fg] = f_r[g] + g_l[f] \tag{3.10}$$

we write the jump relations (3.1)—(3.4) in the form

$$\varrho_r \Psi_x[u] + \varrho_r \Psi_y[v] + K_l[\varrho] = 0 \tag{3.11}$$

$$\varrho_r(u_r \Psi_x + K_l)[u] + \varrho_r u_r \Psi_y[v] + \Psi_x[p] + u_l K_l[\varrho] = 0 \tag{3.12}$$

$$\varrho_r v_r \Psi_x[u] + \varrho_r(v_r \Psi_y + K_l)[v] + \Psi_y[p] + v_l K_l[\varrho] = 0 \tag{3.13}$$

and

$$\begin{aligned} &\left\{\left(\frac{\gamma}{\gamma - 1} p_r + \frac{1}{2} \varrho_r q_r^2\right) \Psi_x + \frac{1}{2} \varrho_r(u_r + u_l) K_l\right\} [u] \\ &+ \left\{\left(\frac{\gamma}{\gamma - 1} p_r + \frac{1}{2} \varrho_r q_r^2\right) \Psi_y + \frac{1}{2} \varrho_r(v_r + v_l) K_l\right\} [v] \\ &+ \left(\frac{1}{\gamma - 1} K_l + u_l \Psi_x + v_l \Psi_y\right) [p] + \frac{1}{2} q_l^2 K_l[\varrho] = 0, \end{aligned} \tag{3.14}$$

where

$$K = u\Psi_x + v\Psi_y + \Psi_t = (\Psi_x^2 + \Psi_y^2)^{1/2} L \tag{3.15}$$

These equations can be treated as four linear homogeneous relations in the quantities $[u]$, $[v]$, $[p]$ and $[\varrho]$. For a discontinuous solution at least one of these four is nonzero. This leads to the condition that the determinant Δ of the matrix of the coefficients must be zero. After some long algebraic operations, we can show that

$$\Delta = -\frac{\gamma + 1}{2(\gamma - 1)} \varrho_r^2 K_l^2 \left\{K_l K_r - \frac{2}{\gamma + 1} a_r^2(\Psi_x^2 + \Psi_y^2) - \frac{\gamma - 1}{\gamma + 1} K_r^2\right\}, \tag{3.16}$$

where a is the local velocity of sound given by $a^2 = \gamma p/\varrho$. Vanishing of the factor $K_l = u_l \Psi_x + v_l \Psi_y + \Psi_t$ corresponds to a contact discontinuity. Hence we can assume that $K_l \neq 0$. For $K_l \neq 0$, from the relations (3.6), (3.7) and (3.9) we deduce

$$\frac{2}{\gamma + 1} a_l^2 + \frac{\gamma - 1}{\gamma + 1} L_l^2 = \frac{2}{\gamma + 1} a_r^2 + \frac{\gamma - 1}{\gamma + 1} L_r^2 = \frac{p_r - p_l}{\varrho_r - \varrho_l} \equiv a_{r,l}^2, \quad \text{say}. \tag{3.17}$$

Therefore, vanishing of Δ on the shock surface implies the following condition

$$(u_l \Psi_x + v_l \Psi_y + \Psi_t)(u_r \Psi_x + v_r \Psi_y + \Psi_t) - a_{r,l}^2(\Psi_x^2 + \Psi_y^2) = 0 \quad \text{on} \quad \Psi(x, y, t) = 0 \tag{3.18}$$

or using (3.6) again

$$(u_r\Psi_x + v_r\Psi_y + \Psi_t)^2 - A_r^2(\Psi_x^2 + \Psi_y^2) = 0 \quad \text{on} \quad \Psi(x, y, t) = 0 \tag{3.19}$$

where

$$A_r^2 = \frac{\varrho_l}{\varrho_r} a_{r,l}^2 = \frac{\varrho_l}{\varrho_r} \frac{p_r - p_l}{\varrho_r - \varrho_l}. \tag{3.20}$$

We note that (3.21) is the well known Prandtl relation for the shocks and the quantity A_r is the normal velocity of a shock front relative to the state on the right.

From (3.6) and (3.7) we get

$$u_l\Psi_x + v_l\Psi_y + \Psi_t = \frac{\varrho_r}{\varrho_l}(u_r\Psi_x + v_r\Psi_y + \Psi_t)$$

and

$$-u_l\Psi_y + v_l\Psi_x = -u_r\Psi_y + v_r\Psi_x.$$

Solving these two for u_l and v_l we get the vector equation

$$(u_l - u_r, v_l - v_r) = \frac{(\Psi_x, \Psi_y)}{(\Psi_x^2 + \Psi_y^2)} K_r \left(\frac{\varrho_r}{\varrho_l} - 1\right). \tag{3.21}$$

From the relations (3.6)—(3.9) with $K_l \neq 0$, we can also derive the well known pressure, density relation across a shock

$$\frac{p_l}{p_r} = \frac{\frac{\gamma+1}{\gamma-1}\frac{\varrho_l}{\varrho_r} - 1}{\frac{\gamma+1}{\gamma-1} - \frac{\varrho_l}{\varrho_r}}. \tag{3.22}$$

As in the example in § 2, we continue the state on the left (right) of the shock into the subdomain on the right (left) as infinitely differentiable solution of the gasdynamic equations as far as possible. However, we shall notice that continuation as *solution* is not necessary for our analysis, they can be continued simply as C^∞ functions. Now we get a three dimensional neighbourhood D of the surface $\Psi(x, y, t) = 0$ where all quantities with suffix l or r and also $a_{r,l}^2$ and A_r are known C^∞ functions of three independent variables x, y and t. Therefore, we can use the embedding theorem to get a one parameter family of surfaces

$$\Phi(x, y, t) = c, \tag{3.23}$$

where c is the parameter. Then the function Φ satisfies the first order partial differential equation

$$Q_{\text{sh}} \equiv (u_l\Phi_x + v_l\Phi_y + \Phi_t)(u_r\Phi_x + v_r\Phi_y + \Phi_t) - a_{r,l}^2(\Phi_x^2 + \Phi_y^2) = 0 \tag{3.24}$$

or

$$\bar{Q}_{\text{sh}} \equiv (u_r\Phi_x + v_r\Phi_y + \Phi_t)^2 - A_r^2(\Phi_x^2 + \Phi_y^2) = 0 \tag{3.25}$$

in the subdomain D. We call a partial differential equation for Φ, such as (3.24) or (3.25), a *shock manifold equation* (SME).

Unlike the case of a single conservation law in the section 2, the condition (3.24) or (3.25) is only a necessary condition for the jump relations (3.1)—(3.4) to be satisfied with the extended functions in D i.e. the conditions (3.1)—(3.4) with Ψ replaced by Φ need not be satisfied when (3.24) or (3.25) are satisfied. Infact, if they were satisfied, the functions would also satisfy (3.22) and we can easily see that this is a too strong condition on arbitrarily extended functions. Therefore, the shock surface $\Psi = 0$ is embedded in a one parameters family or surfaces $\Phi = c$, which in general do not form a family of shock surfaces.

Now that we get two shock manifold equations (3.24) and (3.25) or many others obtained from (3.18) or (3.19) and the relations (3.1)—(3.4), we can ask: "are these partial differential equations equivalent and if so in what sense?". We shall show in the next section that for the given solution of the gasdynamic equations, (3.24) and (3.25) are indeed equivalent for the construction of the shock surface at any time from its position at any other time, say $t = 0$.

Shock manifold equation can also be derived for a general system of hyperbolic equations in the conservation form, provided we assume that the shock strength is small [see Prasad *et al.* (1981)].

4. The Shock Rays

We *define a shock ray* as the projection on (x, y)-plane of a characteristic curve of a shock manifold partial differential equation in (x, y, t) space, starting from a point of a given position of the shock at a given time.

Here we derive the ordinary differential equations of the shock rays only for a shock which is crossed by fluid particles from right to left. For such a shock the shock manifold equation equivalent to (3.25) is

$$\Phi_t + u_r\Phi_x + v_r\Phi_y + A_r(\Phi_x^2 + \Phi_y^2)^{1/2} = 0. \tag{4.1}$$

The characteristic equations of (4.1) or the shock ray equations are

$$\frac{dx}{dt} = u_r + N_1 A_r, \tag{4.2}$$

$$\frac{dy}{dt} = v_r + N_2 A_r, \tag{4.3}$$

$$\frac{d\Phi_x}{dt} = -\left\{\Phi_x \frac{\partial u_r}{\partial x} + \Phi_y \frac{\partial v_r}{\partial x} + \frac{\partial A_r}{\partial x}(\Phi_x^2 + \Phi_y^2)^{1/2}\right\} \tag{4.4}$$

and

$$\frac{\partial\Phi_y}{\partial t} = -\left\{\Phi_x \frac{\partial u_r}{\partial y} + \Phi_y \frac{\partial v_r}{\partial y} + \frac{\partial A_r}{\partial y}(\Phi_x^2 + \Phi_y^2)^{1/2}\right\} \tag{4.5}$$

where N_1 and N_2 are the components of the unit normal to the shock front and satisfy

$$(N_1, N_2) = \left.\frac{(\Phi_x, \Phi_y)}{(\Phi_x^2 + \Phi_y^2)^{1/2}}\right|_{\Psi=0} = \left.\frac{(\Psi_x, \Psi_y)}{(\Psi_x^2 + \Psi_y^2)^{1/2}}\right|_{\Psi=0}. \tag{4.6}$$

The variation of the x component of the unit normal along a shock ray can be derived with the help of (4.4)—(4.6)

$$\frac{dN_1}{dt} = -N_2\left(N_1 \frac{\partial u_r}{\partial \eta} + N_2 \frac{\partial v_r}{\partial \eta} + \frac{\partial A_r}{\partial \eta}\right), \tag{4.7}$$

where

$$\frac{\partial}{\partial \eta} = N_2 \frac{\partial}{\partial x} - N_1 \frac{\partial}{\partial y} \tag{4.8}$$

represents an interior differentiation on $\Psi = 0$ i.e. the rate of change in a direction of a tangent to the shock surface.

Equations (4.2), (4.3), (4.7) (with $N_1^2 + N_2^2 = 1$) are the final form of the shock-ray equations from the SME (4.1). From the theory of first order partial differential equations it follows that a shock manifold in (x, y, t)-space is generated by the shock rays in (x, y, t)-space. Moreover, the appearance of only the interior derivative $\partial/\partial\eta$ in (4.7) shows that the shock rays are determined completely from the distribution of u_r, v_r and A_r on the surface $\Psi(x, y, t) = 0$ alone.

Similarly the shock ray equations from the SME (3.24) are

$$\frac{dx}{dt} = \frac{u_r L_l + u_l L_r - 2N_1 a_{r,l}^2}{L_l + L_r}, \tag{4.9}$$

$$\frac{dy}{dt} = \frac{v_r L_l + v_l L_r - 2N_2 a_{r,l}^2}{L_l + L_r} \tag{4.10}$$

and

$$\begin{aligned} \frac{dN_1}{dt} = -\frac{N_2}{L_l + L_r} \Bigg[&\left\{ N_1 \frac{\partial u_l}{\partial \eta} + N_2 \frac{\partial v_l}{\partial \eta} \right\} L_r \\ &+ \left\{ N_1 \frac{\partial u_r}{\partial \eta} + N_2 \frac{\partial v_r}{\partial \eta} \right\} L_l - \frac{\partial a_{r,l}^2}{\partial \eta} \Bigg] \equiv \frac{G}{L_l + L_r}, \quad \text{say}, \end{aligned} \tag{4.11}$$

where L defined by (3.5) can also be expressed in the form

$$L = N_1 u + N_2 v - S, \tag{4.12}$$

S being the normal velocity of the shock surface.

As in the case of Eqs. (4.2), (4.3) and (4.7) the second set of shock ray Eqs. (4.9) to (4.11) also involve the values of the functions u_r, v_r and $a_{r,l}$ and their interior derivatives only on the shock surface $\Psi(x, y, t) = 0$, where the jump relations (3.1)—(3.4) and hence

$$L_l = \frac{\varrho_r}{\varrho_l} L_r \tag{4.13}$$

and (3.21) or

$$u_l = u_r + L_r N_1 \left(\frac{\varrho_r}{\varrho_l} - 1 \right) \quad \text{and} \quad v_l = v_r + L_r N_2 \left(\frac{\varrho_r}{\varrho_l} - 1 \right) \tag{4.14}$$

are identically satisfied.

Substituting the expressions for K_l, u_l and v_l from (4.13) and (4.14) we get

$$L_l + L_r = \left(1 + \frac{\varrho_r}{\varrho_l} \right) L_r \tag{4.15}$$

$$u_r L_l + u_l L_r - 2N_1 a_{r,l}^2 = \left(1 + \frac{\varrho_r}{\varrho_l} \right) L_r \left(u_r - \frac{\varrho_l}{\varrho_r} a_{r,l}^2 N_1 \frac{1}{L_r} \right) \tag{4.16}$$

$$v_r L_l + v_l L_r - 2N_2 a_{r,l}^2 = \left(1 + \frac{\varrho_r}{\varrho_l} \right) L_r \left(v_r - \frac{\varrho_l}{\varrho_r} a_{r,l}^2 N_2 \frac{1}{L_r} \right) \tag{4.17}$$

and

$$G = -N_2\left(N_1 \frac{\partial u_r}{\partial \eta} + N_2 \frac{\partial v_r}{\partial \eta}\right)\left(1 + \frac{\varrho_r}{\varrho_l}\right) L_r - N_2 L_r^2 \frac{\partial}{\partial \eta}\left(\frac{\varrho_r}{\varrho_l}\right)$$
$$- N_2\left(\frac{\varrho_r}{\varrho_l} - 1\right) L_r \frac{\partial}{\partial \eta}(L_r) + N_2 \frac{\partial}{\partial \eta}(a_{r,l}^2) \tag{4.18}$$

where we have used the result that

$$N_1 \frac{\partial N_1}{\partial \eta} + N_2 \frac{\partial N_2}{\partial \eta} = 0,$$

since $N_1^2 + N_2^2 = 1$ and

$$\frac{\partial}{\partial \eta} L_r = \frac{\partial N_1}{\partial \eta} u + \frac{\partial N_2}{\partial \eta} v + N_1 \frac{\partial u}{\partial \eta} + N_2 \frac{\partial v}{\partial \eta} - \frac{\partial S}{\partial \eta}.$$

Differentiating the relation

$$L_r^2 = \frac{\varrho_l}{\varrho_r} a_{r,l}^2$$

with respect to η and eliminating $L_r \frac{\partial}{\partial \eta} L_r$ from this result and (4.18) we get after some simplification

$$G = -\left(1 + \frac{\varrho_r}{\varrho_l}\right) L_r N_2 \left\{N_1 \frac{\partial u_r}{\partial \eta} + N_2 \frac{\partial v_r}{\partial \eta} - \frac{1}{2L_r} \frac{\partial}{\partial \eta}\left(\frac{\varrho_l}{\varrho_r} a_{r,l}^2\right)\right\}. \tag{4.19}$$

Using the relation $L_r = -A_r$ in (4.16), (4.17) and (4.19) we get the required forms for the expressions on the left hand sides of (4.16) and (4.17), and G. When we substitute these and (4.15) in (4.9)—(4.11) we find that the shock ray equations of the SME (3.25) are exactly the same as the shock ray equations (4.2), (4.3) and (4.7) of the SME (4.1).

Thus we have proved that the shock rays of the two SME (3.24) and (3.25) are the same and either of the two can be used for the construction of the shock surface starting from a given initial shock position.

We have mentioned earlier that the embedding of the shock $\Psi = 0$ in a one parameter family is not unique due to nonuniqueness in continuations of the functions on the two sides. We may then ask "does it mean that the shock rays are also not unique?"

The values of the functions u_l, v_l, p_l and ϱ_l and their partial derivatives are nonunique only in the interior of the subdomain on the right of the original shock. Similarly the values of u_r, v_r, p_r and ϱ_r and their derivatives are uniquely prescribed on the right subdomain including the boundary points up to the shock. Therefore, all the functions, appearing on the right hand sides of the shock ray Eqs. (4.2)—(4.5) are uniquely determined for the original shock $\Psi = 0$ in which we are interested in. The shock rays, for the embedded original shock, are unique and are determined completely from the given solution of the gasdynamic equations.

We derive here an important geometrical property of the shock rays from Eqs. (4.2) and (4.3). When the state ahead of the shock is at rest i.e. when $u_r = 0$, $v_r = 0$ the shock rays form a family of orthogonal curves to the successive positions of the shock surface. From the symmetry of the Eq. (3.24) in suffixes l and r it follows that the same result remains true if the flow behind the shock is at rest. This important relation between the shock rays and the shock surface is in no way evident from the similar property of rays and wave fronts of the linear theory. In his geometrical theory of shock front propagation, Whitham (1957) assumed this property "based on the experience with geometrical optics for linear problems" and defined the shock rays with its help.

Since $\frac{\partial}{\partial \eta}$ represents spatial rate of change of a quantity in a direction along the shock curve, it follows from (4.7) that as the shock propagates the normal to the shock front rotates due to a gradient of the state of the gas and a gradient of the shock strength along the shock curve. If these variations are zero, the normal to the shock has parallel propagation.

5. Kinematics of a Weak Shock and a Method of Solution of Weakly Nonlinear Wave Problems

We define the shock strength δ to be the jump in the pressure

$$\delta = p_l - p_r. \tag{5.1}$$

Then from (3.22) it follows that for small δ, the jump in the density satisfies

$$\varrho_l - \varrho_r = \frac{1}{a_r^2}\,\delta + O(\delta^2). \tag{5.2}$$

The fluid velocity components, from (3.21) become

$$u_l - u_r = -\frac{1}{\varrho_r a_r} N_1 \delta + O(\delta^2), \qquad v_l - v_r = -\frac{1}{\varrho_r a_r} N_2 \delta + O(\delta^2). \tag{5.3}$$

As in the case of ϱ_l, u_l and v_l, we can also approximate A_r up to first order terms in δ and use an approximate form of the shock manifold equation (3.25). However, it is more interesting theoretically to start with the equation (3.24), which for forward facing shock is equivalent to

$$\begin{aligned} &\Phi_t + \frac{1}{2}(u_r + u_l)\,\Phi_x + \frac{1}{2}(v_r + v_l)\,\Phi_y \\ &\quad + a_{r,l}\left[(\Phi_x{}^2 + \Phi_y{}^2) + \frac{1}{4a_{r,l}^2}\{(u_r - u_l)\,\Phi_x + (v_r - v_l)\,\Phi_y\}^2\right]^{1/2} = 0. \end{aligned} \tag{5.4}$$

It is simple to show that for a weak shock, we get

$$a_{r,l} = \frac{1}{2}(a_r + a_l) + O(\delta^2). \tag{5.5}$$

Therefore, for a weak shock if we wish to retain terms only upto the order δ, the approximate shock manifold equation from (5.4) becomes

$$\Phi_t + \frac{1}{2}(u_r + u_l)\,\Phi_x + \frac{1}{2}(v_r + v_l)\,\Phi_y + \frac{1}{2}(a_r + a_l)\,(\Phi_x{}^2 + \Phi_y{}^2)^{1/2} = 0. \tag{5.6}$$

This equation also follows as a particular case of the weak shock manifold equation for an arbitrary hyperbolic system of quasilinear partial differential equations (see Prasad et al. (1982)) and was also obtained by Kluwick (1971).

The Eq. (5.6) is exactly the same as the equation of the forward facing characteristic surface except that the fluid velocity components and the local sound velocity are replaced respectively by the mean of their values ahead of and behind the shock. The equations for the shock rays can be easily written from which we deduce the following result:

"For a weak shock, the shock ray velocity components are equal to the mean bicharacteristic velocity components just ahead and just behind the shock provided we take the wave fronts generating the characteristic surfaces to be instantaneously coincident with the shock surface. Similarly the rate of turning of the shock front is also equal to the mean of the rates of turning of such wave fronts just ahead and just behind the shock."

Solution of an initial or boundary value problem involving a shock wave is still very difficult. The state behind the shock and the motion of the shock influence one another and can not be treated independently. However, in the case of a weak shock propagation there are well known methods of solution by Oswatitsch (1965) and Whitham (1956). The second method has been extensively used in calculating the sonic boom signature but it is valid only when the shock rays do not deviate significantly from the linear rays. Whitham's geometrical shock dynamics (1957) intended to take into account of the deviation of the shock rays from the linear rays, completely decouples the shock motion from the flow behind it. In the introduction we have already remarked on a recent method of Maslov (1980). The kinematics of the shock front developed here can also be used provided the solution behind the shock is known. In a companion paper, we have shown (Prasad et al. (1981)) that in a certain class of weakly nonlinear waves, we can use an earlier method by us (Prasad (1975)) to determine the solution behind the shock. This class consists of those problems in which the waves are produced by the motion of rigid boundaries and the shock wave produced in the flow remains close to the nonlinear wave front initially sent by the boundary. Once the nonlinear solution behind the shock has been determined, it becomes a simple matter to fit in the shock according to the partial differential equation (5.6). Details of this method with application to the flow field produced by sudden introduction of a circular cylinder in otherwise uniform flow of a compressible fluid is available in the companion paper.

6. Remarks

The theory developed here gives a clear picture of the kinematics of a shock front. At present we can not say definitely whether this theory will also be useful in the solution of the problems containing strong shocks. However, there is one example of a transonic flow where the flow behind the moving shock can be approximately predicted due to the freezing property of the sonic flow. Zierep (1968) has calculated the standoff distance of the shock (and its shape) using the freezing property and using the one dimensional theory of shock wave propagation along the axis of symmetric bodies when the flow at infinity is also parallel to the axis. We hope, we can use this theory to find the shape of the shock (see many papers on this topic in Theoretical and Experimental Fluid Mechanics edited by

Müller, Roesner and Schmidt (1979)). We also believe that this theory will make some contribution towards the understanding of the nonlinear caustic problem mentioned in the introduction.

There is one definite possibility of the use of the theory. It can be used to check the accuracy of solutions obtained by other methods such as finite difference schemes. We can use the computed solution behind the shock and verify to what extent the shock shape by the other method and the present theory agree.

Acknowledgement

This work was completed when the author was an Alexander von Humboldt Fellow. The author sincerely thanks Professor J. Zierep for valuable discussion and for providing all facilities.

References

[1] Courant, R., Hilbert, D.: Methods of mathematical physics, Vol. 2. New York: Interscience 1962.
[2] Grinfeld, M. A.: Ray method of calculating the wave front intensity in nonlinear elastic material. PMM J. Appl. Math. & Mech. **42**, 958–977 (1978).
[3] Hayes, W. D.: Kinematic wave theory. Proc. Roy. Soc. (London) **A 320**, 209–226 (1970).
[4] Kluwick, A. von: Zur Ausbreitung schwacher Stöße in dreidimensionalen instationären Strömungen. ZAMM **51**, 225–232 (1971).
[5] Maslov, V. P.: Propagation of shock waves in an isentropic nonviscous gas. J. Soviet Math. **13**, 119–163 (1980).
[6] Müller, U., Roesner, K. G., Schmidt, B. (eds.): Recent developments in theoretical and experimental fluid mechanics. Berlin—Heidelberg—New York: Springer 1979.
[7] Oswatitsch, K.: Propagation problems. In: Contribution to the development of gasdynamic (Schneider, W., Platzer, M., eds.). Braunschweig: Vieweg 1980.
[8] Prasad, P.: Approximation of perturbation equations of a quasilinear hyperbolic system in the neighbourhood of a bicharacteristic. J. Mathematical Anal. and Applic. **50**, 470–482 (1975).
[9] Prasad, P., *et al.*: Kinematics of a weak shock front for a system of conservation laws in multi-dimensions. (To be published.)
[10] Sturtevant, B., Kulkarny, V. A.: The focusing of weak shock waves. J. Fluid Mech. **73**, 651–671 (1976).
[11] Whitham, G. B.: On propagation of weak shock waves. J. Fluid Mech. **1**, 290–318 (1956).
[12] Whitham, G. B.: A new approach to shock dynamics, Part I: Two-dimensional problems. J. Fluid Mech. **2**, 146–171 (1957).
[13] Zierep, J.: Der Kopfwellenabstand bei einem spitzen, schlanken Körper in schallnaher Überschallströmung. Acta Mech. **5**, 204–208 (1968).
[14] Zierep, J.: Theoretical gasdynamics. Berlin—Heidelberg—New York: Springer 1976.

Prof. Dr. P. Prasad
Institut für Strömungslehre und Strömungsmaschinen
Universität Karlsruhe
Kaiserstrasse 12
D - 7500 Karlsruhe 1
Federal Republic of Germany

Permanent address:
Department of Applied Mathematics
Indian Institute of Science
Bangalore - 560012
India