Acta Mechanica 112, 173–186 (1995)

ACTA MECHANICA

Dynamic analysis of a machine model with time-varying mass

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(Received November 3, 1993; revised January 15, 1994)

Summary. In the paper discrete systems with a variable mass and unilateral constraints are considered. The assumed physical and mathematical model enables analysis of large displacements of bodies and also a change of the load mass on the behaviour of the whole structure. Nonlinear equations of motion are solved using numerical integration. The paper contains the testing of equation of motion and computer programmes that generate and solve equations of motion. A special checking function for systems with time varying mass based on energy power balance is introduced. The results of calculations are the proof of correctness of the algorithms that have been used.

1 Introduction

The aim of the work is to present methodology of the solution of dynamical mechanical models that concern a procedure of testing of correctness of a numerical solution. The method proposed has been used in order to solve problems of dynamical systems with time-varying mass. Systems for which the influence of such effects should be considered are machines like cranes or excavators that carry loose loads or fluids (i.e. sand, concrete). The model assumed is analyzed by means of dynamic methods convenient for the nonholonomic systems of variable configuration and mass. To our knowledge multibody programmes, recently quite popular [1], are not applicable to systems with time varying masses.

To derive basic equations, we use Nielsen's equations [2] in the matrix form:

$$\frac{\partial \dot{T}}{\partial \dot{\boldsymbol{q}}} - 2\frac{\partial T}{\partial \boldsymbol{q}} = \boldsymbol{f} - \frac{\partial V}{\partial \boldsymbol{q}}, \tag{1}$$

in which T and V are kinetic and potential energy of the system, $\boldsymbol{f}$ denotes nonconservative applied forces (internal and external), and $\boldsymbol{q}$ is a set of independent generalized coordinates.

Nielsen's equations are a special case of Mangeron-Deleanu equations applied to investigate the motion of a system with nonholonomic nonlinear constraints (due to load motions control). Mangeron-Deleanu equations [3], [4] include a more general class of nonlinear constraints than that which are considered in the methods based on Lagrange's equations.

In order to test results of numerical integration a checking function $C(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t)$ has been introduced – a derivation of which is presented in Appendix A. A constant value of the checking function testifies the correctness of the numerical solution. The derivative of C is defined as

$$\dot{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t) \equiv \dot{E} + \dot{Z}_T + \dot{Z} = 0, \tag{2}$$

where E is the total mechanical energy ($E = T + V$). Z and Z_T are functions of kinetic and potential energies as well as the power of non-conservative applied forces:

$$\dot{Z} = -\dot{\boldsymbol{q}}^T\boldsymbol{f} - \frac{\partial V}{\partial t} - 2\dot{\boldsymbol{q}}^T\frac{\partial T}{\partial \boldsymbol{q}}, \tag{3}$$

$$\dot{Z}_T = -T_0' + T_2' + 2T_3'. \tag{4}$$

T_0', T_2', T_3' denote components of the derivative $\dot{T}$ of the kinetic energy: T_0' is the term independent of the generalized velocities $\dot{\boldsymbol{q}}$, T_2' is dependent on the square of velocities and T_3' is dependent on the generalized velocities in the third power.

2 Derivation of equations of motion

2.1 Kinetic energy and its derivatives

As has been mentioned, to formulate the equations of motion, Nielsen's approach [2] was used. Thus, it is necessary to determine kinetic and potential energy of the systems analyzed. It is assumed that the relative velocity of the mass flow is equal to zero and the reaction force depending on the relative motion of the discharged masses is neglected.

The energy of a particular body can be expressed as

$$T_k = \frac{1}{2}(\boldsymbol{v}_k{}^T\boldsymbol{m}_k\boldsymbol{v}_k + \omega_k{}^T\boldsymbol{j}_k\omega_k) \tag{5}$$

where $\boldsymbol{m}_k, \boldsymbol{j}_k$ are matrices of the mass and moment of inertia, $\boldsymbol{v}_k$ is the mass center velocity, and ω_k the angular velocity of the body k.

The kinetic energy of the system is the sum of the energies of all bodies and – expressed in terms of generalized coordinates – is expressed as follows:

$$T = \frac{1}{2}(m + \boldsymbol{m}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}) \tag{6}$$

in which the scalar function m, the row matrix $\boldsymbol{m}$ and the square matrix $\boldsymbol{M}$ are defined as

$$\begin{aligned} m &= \boldsymbol{v}_t{}^T\boldsymbol{M}_0\boldsymbol{v}_t + \omega_t{}^T\boldsymbol{J}_0\omega_t, \\ \boldsymbol{m} &= 2(\boldsymbol{v}_t{}^T\boldsymbol{M}_1\boldsymbol{U} + \omega_t{}^T\boldsymbol{J}_1\boldsymbol{W}), \\ \boldsymbol{M} &= \boldsymbol{U}^T\boldsymbol{M}_2\boldsymbol{U} + \boldsymbol{W}^T\boldsymbol{J}_2\boldsymbol{W}. \end{aligned} \tag{7}$$

The vectors $\boldsymbol{v}_t$, ω_t and also the matrices $\boldsymbol{U}$, $\boldsymbol{W}$ are defined by Eqs. (A.3), (A.4). Block mass matrices $\boldsymbol{M}_i$ and moment of inertia matrices $\boldsymbol{J}_i$ are symmetric. The terms of $\boldsymbol{M}_i$ and $\boldsymbol{J}_i$ depend on the generalized coordinates $\boldsymbol{q}$ and time t.

Taking into account (A.8) one finds the particular terms of the energy T as: $T_0 = \frac{1}{2}m$, $T_1 = \frac{1}{2}\boldsymbol{m}\dot{\boldsymbol{q}}$, $T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}$.

After differentiation of T with respect to $\boldsymbol{q}$ we obtain

$$\frac{\partial T}{\partial \boldsymbol{q}} = \boldsymbol{L}\dot{\boldsymbol{q}} + \boldsymbol{l}. \tag{8}$$

The matrix $\boldsymbol{L}$ is as follows:

$$\boldsymbol{L} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\boldsymbol{m} + \frac{1}{2}\left(\dot{q}_1\left[\frac{\partial}{\partial q_1}\boldsymbol{M}\right] + \cdots + \dot{q}_n\left[\frac{\partial}{\partial q_n}\boldsymbol{M}\right]\right), \tag{9}$$

where $q_j(j = 1, \ldots, n)$ are the particular generalized coordinates of the model analyzed. The second term in (8) is defined as

$$\boldsymbol{l} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}m. \tag{10}$$

Differentiating the kinetic energy with respect to time, we get

$$\frac{dT}{dt} \equiv \dot{T} = \frac{1}{2}(\dot{m} + \dot{\boldsymbol{m}}\dot{\boldsymbol{q}} + \boldsymbol{m}\ddot{\boldsymbol{q}} + 2\ddot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T\boldsymbol{N}\dot{\boldsymbol{q}}), \tag{11}$$

in which the matrix $\boldsymbol{N}$ denotes the derivative of the mass matrix $\boldsymbol{M}$ with respect to time ($\boldsymbol{N} \equiv \dot{\boldsymbol{M}}$).

Differentiating $\dot{T}$ with respect to $\dot{\boldsymbol{q}}$, we have

$$\frac{\partial \dot{T}}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{f}_0 + \boldsymbol{f}_1 + \boldsymbol{f}_2 + \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{N}\dot{\boldsymbol{q}} + \boldsymbol{P}\dot{\boldsymbol{q}}. \tag{12}$$

The vectors $\boldsymbol{f}_i(i = 0, 1, 2)$ are obtained as follows:

$$\begin{aligned} \boldsymbol{f}_0 &= \operatorname{col}\left[\frac{1}{2}\frac{\partial}{\partial \dot{q}_j}\dot{m}\right], \quad j = 1, \ldots n \\ \boldsymbol{f}_1 &= \operatorname{col}\left[\frac{1}{2}\left(\dot{m}_j + \dot{\boldsymbol{q}}^T\frac{\partial}{\partial \dot{q}_j}\dot{\boldsymbol{m}}^T\right)\right], \quad j = 1, \ldots n \\ \boldsymbol{f}_2 &= \operatorname{col}\left[\frac{1}{2}\ddot{\boldsymbol{q}}^T\frac{\partial}{\partial \dot{q}_j}\dot{\boldsymbol{m}}^T\right], \quad j = 1, \ldots n. \end{aligned} \tag{13}$$

The matrix $\boldsymbol{P}$ is a product of the generalized velocities $\dot{\boldsymbol{q}}$ and the derivative of the matrix $\boldsymbol{N}$ with respect to $\dot{\boldsymbol{q}}$, i.e. $\boldsymbol{P} = \frac{1}{2}\dot{\boldsymbol{q}}^T\left[\frac{\partial}{\partial \dot{\boldsymbol{q}}}\boldsymbol{N}\right]$, and can be expressed as

$$\boldsymbol{P} = \frac{1}{2}\left(\dot{q}_1{}^T\left[\frac{\partial}{\partial \dot{q}_1}\boldsymbol{N}\right] + \dot{q}_2{}^T\left[\frac{\partial}{\partial \dot{q}_2}\boldsymbol{N}\right] + \cdots + \dot{q}_n{}^T\left[\frac{\partial}{\partial \dot{q}_n}\boldsymbol{N}\right]\right). \tag{14}$$

2.2 Potential energy of the system

The total potential energy of the system V is a sum of the gravity potential $V_g(\boldsymbol{q}, t)$ and the energy of elastic deformation of the constraints $V_s(\boldsymbol{q})$,

$$V(\boldsymbol{q}, t) = V_g(\boldsymbol{q}, t) + V_s(\boldsymbol{q}). \tag{15}$$

It has been assumed that the systems analyzed (Figs. 1, 2) have a set of unilateral elastic constraints. The potential energy of the constraints (having bilinear characteristics) can be written in the form

$$V_s(\boldsymbol{q}) = \frac{1}{2}\,\boldsymbol{u}^T\boldsymbol{C}\boldsymbol{J}\boldsymbol{u}, \tag{16}$$

where:
$\boldsymbol{u}$ – the vector of the springs deformation,
$\boldsymbol{J}$ – the matrix indicating the state of constraints,
$\boldsymbol{C}$ – the diagonal springs stiffness matrix.

An expression of the potential gravity energy is:

$$V_g(\boldsymbol{q}, t) = \boldsymbol{G}^T\boldsymbol{r}, \tag{17}$$

in which:

$\boldsymbol{G}$ – the vector containing the gravitational forces,
$\boldsymbol{r}$ – the vector determining positions of mass centers of particular bodies.

The potential forces can be expressed as follows:

$$-\frac{\partial V}{\partial \boldsymbol{q}} = -\frac{\partial}{\partial \boldsymbol{q}}\left(\boldsymbol{G}^T\boldsymbol{r} + \frac{1}{2}\,\boldsymbol{u}^T\boldsymbol{C}\boldsymbol{J}\boldsymbol{u}\right) = -\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{q}}\,\boldsymbol{G} - \frac{1}{2}\left(\frac{\partial \boldsymbol{u}^T}{\partial \boldsymbol{q}}\,\boldsymbol{C}\boldsymbol{J}\boldsymbol{u} + \boldsymbol{u}^T\boldsymbol{C}\boldsymbol{J}\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{q}}\right) = -\boldsymbol{K}(\boldsymbol{q}, t)\,\boldsymbol{q}. \tag{18}$$

The partial derivative of the potential energy that will be used in an expression for $\dot{Z}$ is

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left(\boldsymbol{G}^T\boldsymbol{r} + \frac{1}{2}\,\boldsymbol{u}^T\boldsymbol{C}\boldsymbol{J}\boldsymbol{u}\right) = \frac{\partial \boldsymbol{G}^T}{\partial t}\,\boldsymbol{r} + \boldsymbol{G}^T\,\frac{\partial \boldsymbol{r}}{\partial t}. \tag{19}$$

Putting (19) and (8) into (3) one obtains

$$\dot{Z} = -\dot{\boldsymbol{q}}^T\boldsymbol{f} - \frac{\partial \boldsymbol{G}^T}{\partial t}\,\boldsymbol{r} - \boldsymbol{G}^T\,\frac{\partial \boldsymbol{r}}{\partial t} - 2\dot{\boldsymbol{q}}^T\boldsymbol{L}\dot{\boldsymbol{q}} \equiv h(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{20}$$

3 Equation of model motion and sample problems

Introducing expressions (5)–(18) into (1) for the model analyzed, one obtains equations governing the system. In the symbolic form the equation can be written as follows:

$$\boldsymbol{M}(\boldsymbol{q}, t)\,\ddot{\boldsymbol{q}} + \big(\boldsymbol{P}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - 2\boldsymbol{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\big)\,\dot{\boldsymbol{q}} + \boldsymbol{K}(\boldsymbol{q}, t)\,\boldsymbol{q} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{21}$$

The differential equations of model motions in a general case cannot be solved in a closed form. Matrices in the nonlinear equations (21) are time, displacement and velocity dependent. Practically, the numerical integration is the only possible method to obtain a solution.

The equation used in order to test numerical integration results is as follows:

$$\dot{Z} - h(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0. \tag{22}$$

3.1 Excavator model

The sample planar model of an excavator (Fig. 1) consists of five rigid bodies:

– the body (1) with three degrees of freedom elastically connected to the ground

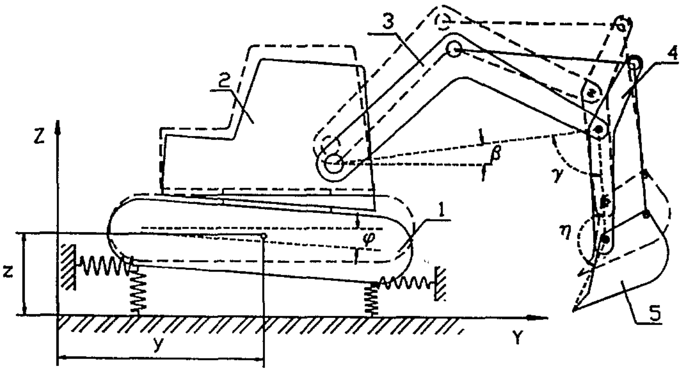

Fig. 1. Excavator model

by means of springs;

- the body (2) mounted on (1);
- the body (3) connected with (2) by means of hinge, the position of which determined the angle β;
- the body (4) connected with (3) by means of hinge – modelling an excavator arm;
- the body (5) which is a model of the excavator scoop and can rotate around the body (4).

The motion of the system (planar model of an excavator) has been described by means of the coordinates:

y, z – displacements of the mass center point of the body (1),
φ – rotation angle of the body (1),
β – angle of jib tilt (3) ($\beta \in \langle \beta_1, \beta_2 \rangle$)[1],
γ – angle of the body deflection (4) with respect to the body (3) ($\gamma \in \langle \gamma_1, \gamma_2 \rangle$),
η – angle of the scoop (5) rotation ($\eta \in \langle \eta_1, \eta_2 \rangle$).

The operation movements can be realized by means of the jib (3), the arm (4), or the scoop (5) movements. It means that the equations of constraints are known and are as follows:

$$\beta - f_1(t) = 0, \quad \gamma - f_2(t) = 0, \quad \eta - f_3(t) = 0. \tag{23}$$

The vector of generalized coordinates of the system is defined by

$$\boldsymbol{q} = [y \ z \mid \varphi]^T = [\boldsymbol{q}_1 \ \boldsymbol{q}_2]^T. \tag{24}$$

3.2 Model of planar crane

The sample planar model of a crane (Fig. 2) consists of four rigid bodies [5]:

- the body (1) with three degrees of freedom, elastically connected to the ground by means of springs;
- the body (2) mounted on (1);
- the body (3) connected with (2) by means of hinge, and a prescribed position angle β;
- the particle (4) with two degrees of freedom, connected to the body (3) with a flexible cord.

[1] Angle limitations result from the length of servo motors and geometrical dimensions of particular elements.

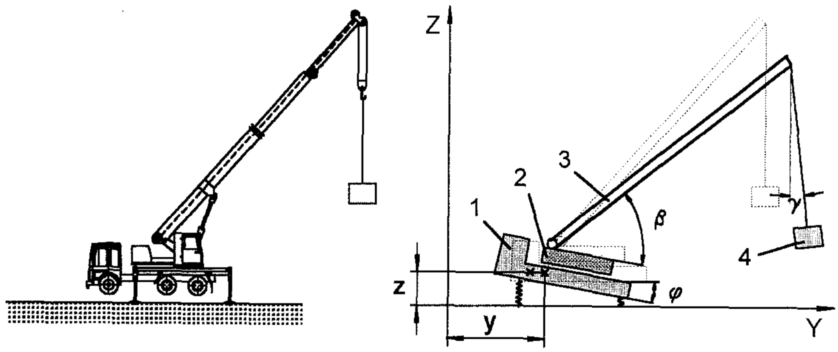

Fig. 2. Planar crane model

The motion of the system has been described by means of coordinates:

y, z – displacements of the mass center point of the body (1),
φ – rotation angle of the body (1),
β – angle of jib tilt (3) ($\beta \in \langle \beta_1, \beta_2 \rangle$),
γ – angle of rotation of the cord relative to the global coordinate system ($\gamma \in \langle \gamma_1, \gamma_2 \rangle$).

The equation of constraints is known and has the form

$$\beta - f_1(t) = 0. \tag{25}$$

The vector of generalized coordinates of the system is defined by

$$\boldsymbol{q} = [y \;\; z \mid \varphi \mid \gamma]^T = [q_1 \;\; q_2 \;\; q_3]^T. \tag{26}$$

4 Numerical simulation of motion

Numerical evaluation of the matrices, forces and the solution of the problem, as well as the graphical presentation of results has been carried out by means of PC-MATLAB system. The programs elaborated allow analyzing the system motions in the case of kinematic excitation. The analysis concerns large displacements of the system, a change in the magnitude of the mass and the moment of inertia of the load as well as in the mass center position.

During numerical simulation it is always advisable to test the correctness of numerical results because:

– the equations may be incorrect,
– the equations may be coded improperly,
– the integration procedure may be flawed.

Numerical integration is carried out in such a way that the equations of motion of the analyzed system are completed with a differential equation (20) defining $\dot{Z}(t)$. Through simultaneous integration of these equations in each time step, the values of the functions $Z(t)$ and $C(t)$ are evaluated. On the basis of time history of $C(t)$ we come to the conclusion about the correctness of the results. The function $C(t)$ should remain constant[2].

Practically, because of finite accuracy of numerical algorithms, various values of $C(t)$ are

[2] In a particular case, in which $E = \text{const} = 0$, the system is conservative and the principle of energy conservation can be used to test the numerical solution.

obtained. We define the correct solution as the solution in which the maximum value of the control function $C(t)$ is several times less than the maximum value of the total mechanical energy E or function Z_T or Z.

Defining the energetic measure of the numerical solution error δ_E as

$$\delta_E = \frac{\sup_t |C|}{\sup_t (\max(|E|, |Z_T|, |Z|))}, \tag{27}$$

integrations can be carried out with given (assumed) accuracy. It means that the time step is reduced as soon as $\delta_E > \varepsilon$, where ε is the assumed accuracy.

4.1 Numerical results for the excavator model

The method presented has been used for the time step optimization in the numerical solution of the motion of the plane excavator model. Numerical simulation has been performed for the kinematic excitation $(f_1(t) = 0.2\,t$ [rad], $f_2(t) = 0, f_3(t) = 0)$.

Figure 3 shows the time history of the functions C, E and Z for two different time step values $h = 0.03$ and $h = 0.005$. The time step $h = 0.005$ is correct (C remains constant, Fig. 3a) but numerical integration with the time step $h = 0.03$ leads to incorrect results (Fig. 3b). It is important to point out that the high frequency oscillations in E are due to the kinematic excitation $f_1(t)$. Assumed constant value of the jib velocity $\dot{\beta}$ and simultaneous vibration of the base means that the energy is transferred in both directions: to and from the system.

A comparison of the results – response curves of y – for the correct ($\delta_E = 0.004$) and incorrect solution ($\delta_E = 0.124$) is shown in Fig. 4. The effect of algorithmic (numerical) dumping has been observed for the second case (Fig. 4b).

The next example shows the response of the excavator model with a time varying mass m_5:

$$m_5 = \begin{cases} 500 & 0 < t < 1 \\ 500 - 100(t-1) & 1 \leqq t \leqq 5, \\ 100 & t > 5 \end{cases}$$

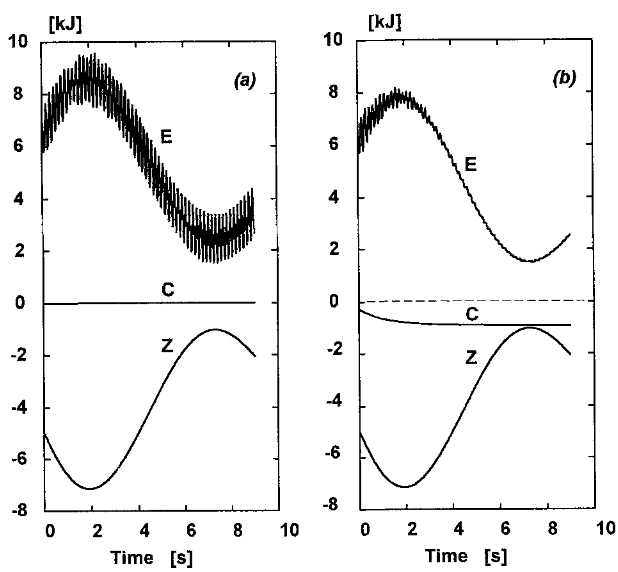

Fig. 3. Functions C, E, Z for two steps of integration h. **a** $h = 0.005$; **b** $h = 0.03$

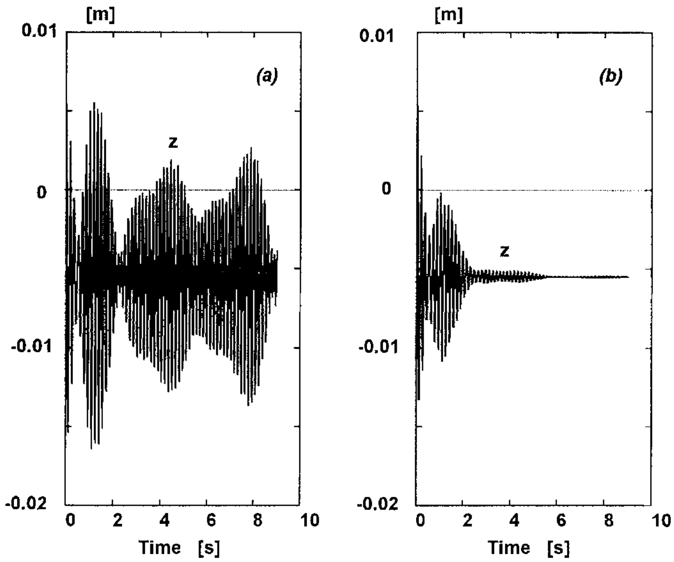

Fig. 4. Displacement $z(t)$ of the excavator model: correct (**a**) and incorrect (**b**) numerical results

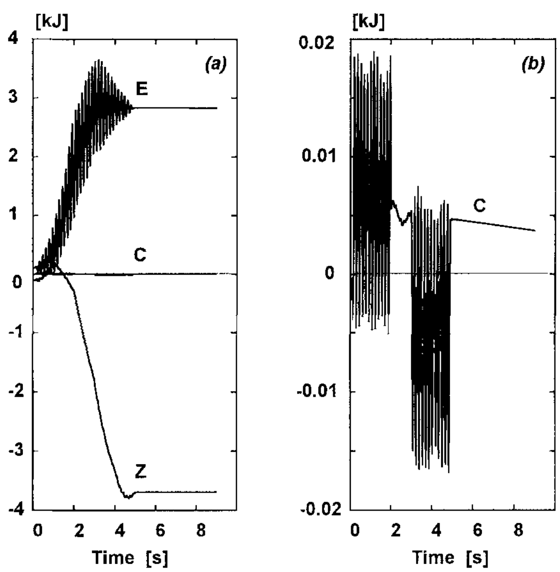

Fig. 5. Time history: **a** energy E, functions Z and C; **b** zoom of function C

to the kinematic excitation determined by velocity $\dot{\beta}$:

$$\dot{f}_1(t) = \dot{\beta} = \begin{cases} \pi/16\,t & t < 2 \\ \pi/8 & 2 \leqq t \leqq 3, \\ \pi/8 - \pi/16(t-3) & t > 5 \end{cases} \quad f_2(t) = 0, \quad f_3(t) = 0.$$

The results obtained — energy E, functions Z and C — are presented in Fig. 5. The plot of the function C shows that the solution error is quite small ($\delta_E = 0.006$). The variations of C in Fig. 5 b (note that Fig. 5 b shows the values of C in a different scale) indicate that disturbances appear for the time steps in which the acceleration $\ddot{\beta}$ or the mass change velocity $\dot{m}_5$ are step-varying functions. Thus, an analysis of variation of the control function C allows to point out the causes of the solution errors.

4.2 Numerical results for the crane model

Figures 6–8 show the results of the crane model movement simulation for the data:

$$e = 14\,[\mathrm{m}], \quad h = 7\,[\mathrm{m}], \quad m_{40} = 3\,500\,[\mathrm{kg}], \quad \gamma_0 = -20^\circ, \quad \beta = 40^\circ,$$

and the linear mass varying function: $m_4 = m_{40} - 350\,t$.

Time history of general coordinates for this system are plotted in Fig. 6. Additionally, in Fig. 7 changes of the rope force (S) are compared with the gravitational component of the rope force ($S_g = m_4 g \cos \gamma$).

Figure 8 shows the reactions in outrigger pads (R_L – left support in Fig. 2; R_R – right support) and the total horizontal outrigger force (Q).

The ground bearing pressures generated by the crane on the left outrigger (R_L) are close to zero; it means the loss of the stability of the crane (it is seen that in small time intervals the support

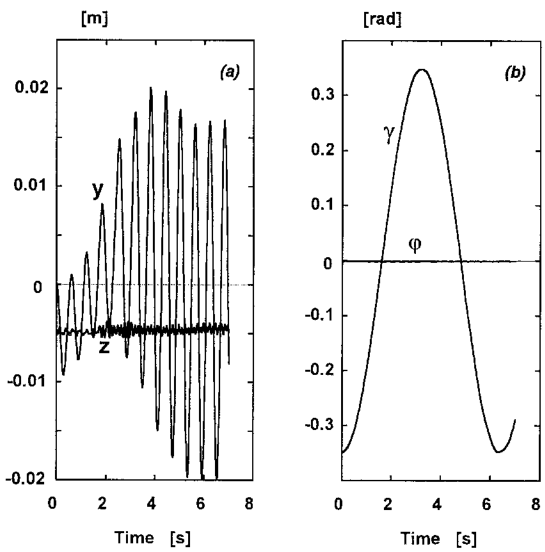

Fig. 6. Response curves; **a** coordinates $z(t)$, $y(t)$, **b** rotations $\varphi(t)$, $\gamma(t)$

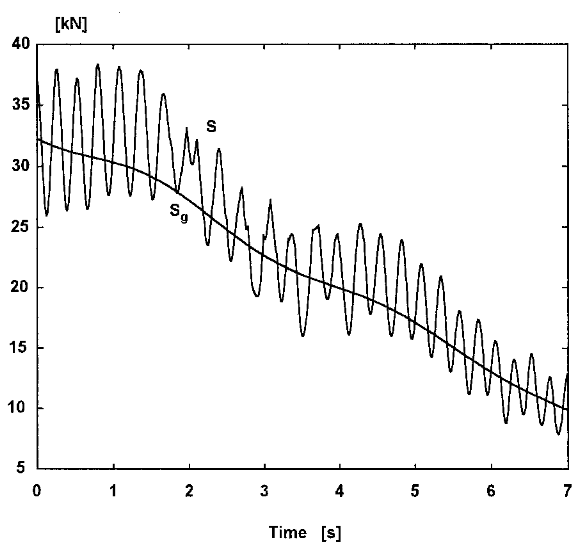

Fig. 7. Rope forces of the crane

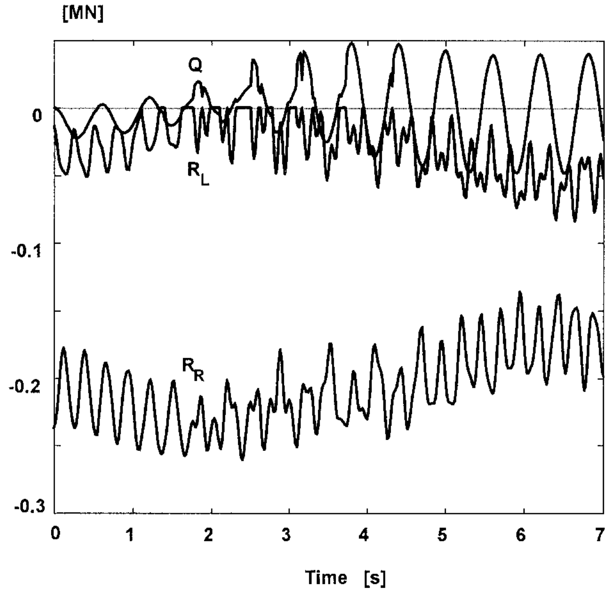

Fig. 8. Outriggers reactions

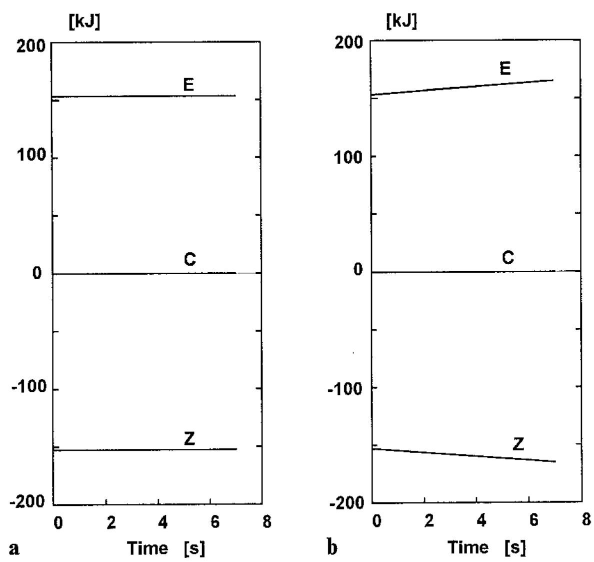

Fig. 9. Results of numerical correctness test. **a** stable motion; **b** unstable motion

looses the contact with the ground.) The correctness of the results is proved, and the control function for this case is shown in Fig. 9a. It should be pointed out that, although the number of active supports changes many times during the analysed time interval, the results are correct (in these regions the integration has been carried out with a varying time step).

The crane movement, as shown in Figs. 6 and 8, is stable for the case of a continuous change of the load mass. The same model of the crane is unstable when the load mass remains constant. Illustrations of such a case are Figs. 10 and 11 where the time histories of the crane coordinates and the trajectories of characteristic points of the crane (base, end of boom, load center of gravity) are plotted. The function C for the unstable tipping motion is shown in Fig. 9b (and suggests correctness of the results).

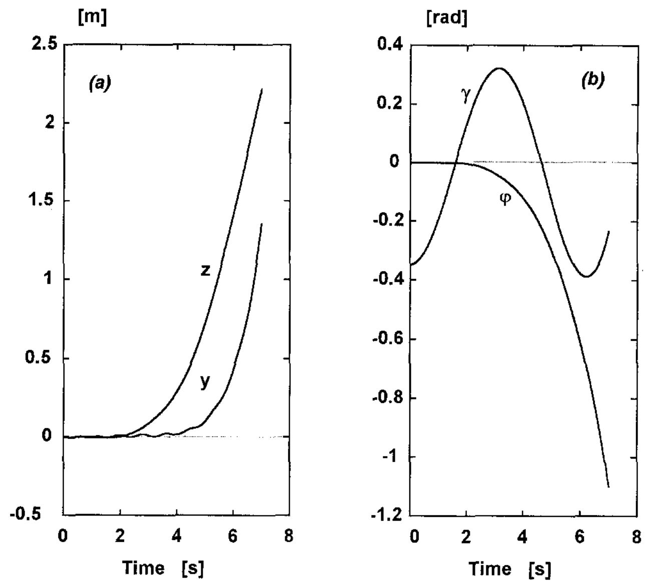

Fig. 10. Time histories of the crane coordinates: **a** displacements y, z; **b** rotations γ, φ

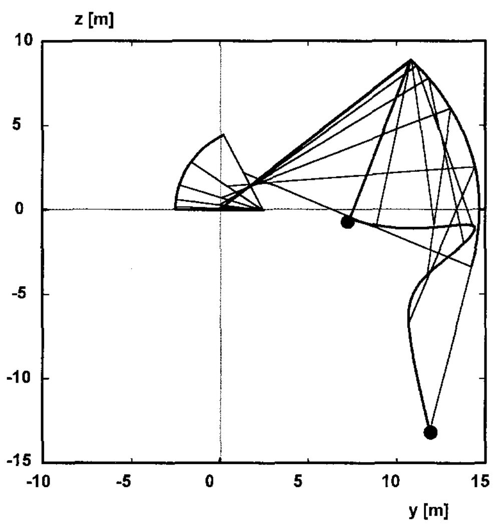

Fig. 11. Trajectories of characteristic points of the crane (base, end of boom, load center of gravity)

5 Concluding remarks

The theoretical considerations presented above — concerning the method of testing numerical solutions of equations of motion for machine models containing bodies with time varying mass — and the computer simulation results lead to the conclusion that the dynamic analysis methodology proposed is very efficient.

The method of testing is a global method — it means that the whole process, from proving the correctness of equations of motion through checking the algorithm and computer code, up to the selection of integration time step, is controlled. Using the control function C in numerical simulation yields a number of practical advantages. At the stage of the program design it allows to verify equations of motion and solving procedures, making it possible to diagnose and identify errors. During computer simulation it permits to choose the integration time step and allows to carry out integration with presumed accuracy. The introduced energetic measure of the errors δ_E is a global measure of calculation accuracy. For a researcher it is a factor of great significance.

Appendix A

Testing of correctness of numerical solutions

To test numerical integration results, an idea presented by Kane and Levinson [6], [7] has been adapted. The authors mentioned have applied Kane's equation to formulate the control function for holonomic and unholonomic systems.

Here the derivation of the control function C for systems carrying time varying mass is presented.

A.1 Assumptions

A holonomic system with time varying mass in formulation proposed by Nielsen is analyzed. The potential energy V is assumed as a time and generalized coordinate dependent function, and the generalized force vector $\boldsymbol{f}$ is a function of time, displacement and velocity, i.e.:

$$V = V(\boldsymbol{q}, t), \tag{A.1}$$

$$\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{A.2}$$

Velocities of points of the system under consideration are linear functions of generalized velocities. Velocity vectors of the system points can be presented as

$$\boldsymbol{v} = \boldsymbol{U}\dot{\boldsymbol{q}} + \boldsymbol{v}_t, \tag{A.3}$$

where

$$\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{q}, t), \qquad \boldsymbol{v}_t = \boldsymbol{v}_t(\boldsymbol{q}, t). \tag{A.4}$$

Angular velocities of the bodies have the form

$$\omega = \boldsymbol{W}\dot{\boldsymbol{q}} + \omega_t, \tag{A.5}$$

where $\boldsymbol{W}$ and ω_t are – like $\boldsymbol{U}$ and $\boldsymbol{v}_t$ – functions independent of the generalized velocities $\dot{\boldsymbol{q}}$.

Global vectors of the velocity of the system point and angular velocities of the rigid bodies can be written in the form:

$$\boldsymbol{v} = [\boldsymbol{v}^{(1)}\boldsymbol{v}^{(2)} \dots \boldsymbol{v}^{(i)}], \tag{A.6}$$

$$\omega = [\omega^{(1)}\omega^{(2)} \dots \omega^{(j)}], \tag{A.7}$$

where i is a number of material points and rigid bodies of the system analyzed, and j is a number of bodies that rotate.

A.2 Energy and power balance

The total energy of the mechanical system E is a sum of potential and kinetic energies. Changes in the total energy can be estimated by investigation of the energy function or its derivative with respect to time $\left(\dot{E} = \frac{d}{dt}(T + V) = \dot{T} + \dot{V}\right)$. Taking into account the above-mentioned assumptions, the kinetic energy $T = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ can be written as the sum of: T_0 – generalized velocity ($\dot{\boldsymbol{q}}$) independent term, T_1 – in first power velocity dependent term and T_2 – square of velocity dependent term, i.e.:

$$T = T_0 + T_1 + T_2. \tag{A.8}$$

The time derivative of the kinetic energy $\dot{T} = T_p(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t)$ of the system analyzed is presented as a four-term sum: T_0', T_1', T_2' – defined in an analogous manner as in the case of the energy T, and T_3' – third power of the generalized velocity dependent term

$$\dot{T} = T_0' + T_1' + T_2' + T_3'. \tag{A.9}$$

It is obvious that

$$\begin{aligned} T_0' &= \frac{\partial T_0}{\partial t} + \ddot{\boldsymbol{q}}^T \frac{\partial T_1}{\partial \dot{\boldsymbol{q}}}, \\ T_1' &= \frac{\partial T_1}{\partial t} + \dot{\boldsymbol{q}}^T \frac{\partial T_0}{\partial \boldsymbol{q}} + \ddot{\boldsymbol{q}}^T \frac{\partial T_2}{\partial \dot{\boldsymbol{q}}}, \\ T_2' &= \frac{\partial T_2}{\partial t} + \dot{\boldsymbol{q}}^T \frac{\partial T_1}{\partial \boldsymbol{q}}, \\ T_3' &= \dot{\boldsymbol{q}}^T \frac{\partial T_2}{\partial \boldsymbol{q}} \end{aligned} \tag{A.10}$$

and hence

$$\dot{T} = \frac{\partial}{\partial t} T + \dot{\boldsymbol{q}}^T \frac{\partial}{\partial \boldsymbol{q}} T + \ddot{\boldsymbol{q}}^T \frac{\partial}{\partial \dot{\boldsymbol{q}}} (T_1 + T_2). \tag{A.11}$$

Premultiplication of Nielsen's equation by $\dot{\boldsymbol{q}}^T$,

$$\dot{\boldsymbol{q}}^T \frac{\partial \dot{T}}{\partial \dot{\boldsymbol{q}}} - 2\dot{\boldsymbol{q}}^T \frac{\partial T}{\partial \boldsymbol{q}} = \dot{\boldsymbol{q}}^T \boldsymbol{f} - \dot{\boldsymbol{q}}^T \frac{\partial V}{\partial \boldsymbol{q}} \tag{A.12}$$

gives a power balance equation of the system during the movements ($\dot{\boldsymbol{q}}^T \boldsymbol{f}$ is the power of all nonpotential forces $\boldsymbol{f}$).

Taking into account Euler's theorem, Eq. (A.12) can be written as follows:

$$T_1' + 2T_2' + 3T_3' - 2\dot{\boldsymbol{q}}^T \frac{\partial T}{\partial \boldsymbol{q}} = \dot{\boldsymbol{q}}^T \boldsymbol{f} - \dot{\boldsymbol{q}}^T \frac{\partial V}{\partial \boldsymbol{q}}. \tag{A.13}$$

Putting in

$$T_1' + 2T_2' + 3T_3' = \dot{T} - T_0' + T_2' + 2T_3' \tag{A.14}$$

and the derivative of the potential energy into (A.12), we obtain

$$\dot{T} + \dot{V} - T_0{}' + T_2{}' + 2T_3{}' - \dot{\boldsymbol{q}}^T \boldsymbol{f} - \frac{\partial V}{\partial t} - 2\dot{\boldsymbol{q}}^T \frac{\partial T}{\partial \boldsymbol{q}} = 0, \tag{A.15}$$

A.3 Control function C(t) for systems with time varying mass

Introducing the following notation:

$$\dot{Z}_T \equiv -T_0{}' + T_2{}' + 2T_3{}', \tag{A.16}$$

$$\dot{Z} \equiv -\dot{\boldsymbol{q}}^T \boldsymbol{f} - \frac{\partial V}{\partial t} - 2\dot{\boldsymbol{q}}^T \frac{\partial T}{\partial \boldsymbol{q}}, \tag{A.17}$$

and

$$\dot{C} \equiv \dot{E} + \dot{Z}_T + \dot{Z} = 0, \tag{A.18}$$

one can state the function C thus defined preserves its value in time.

The function $C(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t)$ is not known in an analytical form. It can be computed in each time step during the numerical integration process, as

$$C \equiv E + Z_T + Z. \tag{A.19}$$

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