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# Computation of angular velocity and acceleration tensors by direct measurements

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**Summary.** The present paper investigates the properties of the angular velocity tensor $\mathbf{\Phi}$ and the angular acceleration tensor $\mathbf{\Psi}$ for rigid body motion. Three vectorial invariants of rigid body kinematics are presented. In case of tensor $\mathbf{\Psi}$ being non-singular, its inverse $\mathbf{\Psi}^{-1}$ is inferred. A novel procedure for automatic computation of tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ based on measured velocity and acceleration data is developed. Depending of the type of available data, three algorithms are suggested. Numerical examples show the application of the method.

## 1 Introduction

The problem of determining the tensors describing the vectorial field of velocities and accelerations is fundamental in rigid body kinematics. For a rigid body $(\mathscr{C})$ in general motion with respect to a reference frame $\{\mathscr{R}\}$, the vectorial field of velocities and that of the accelerations are given by [2], [6], [9]:

$$\vec{v} - \vec{v}_O = \tilde{\boldsymbol{\omega}}(\vec{\varrho} - \vec{\varrho}_O), \tag{1}$$

$$\vec{a} - \vec{a}_O = (\tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}})(\vec{\varrho} - \vec{\varrho}_O), \tag{2}$$

where $\vec{v}_O$ and $\vec{a}_O$ represent the velocity and acceleration of a body-fixed reference point $O \in (\mathscr{C})$ described by its position vector $\vec{\varrho}_O$ in the considered reference frame $\{\mathscr{R}\}$; $\vec{v}$ and $\vec{a}$ stand for the velocity and the acceleration, respectively, of an arbitrary point on the body described by its position vector $\vec{\varrho}$ with respect to $\{\mathscr{R}\}$. Tensors denoted by $\tilde{\boldsymbol{\omega}}$ and $\tilde{\boldsymbol{\varepsilon}}$ represent the skew-symmetric tensors associated with the angular velocity vector $\vec{\omega}$ and angular acceleration vector $\vec{\varepsilon}$, respectively, [2], [3], [10]. The vectors $\vec{\omega}$ and $\vec{\varepsilon}$ (and thus the tensors $\tilde{\boldsymbol{\omega}}$ and $\tilde{\boldsymbol{\varepsilon}}$) do not depend on the choice of $O$. The tensors

$$\mathbf{\Phi} = \tilde{\boldsymbol{\omega}}, \tag{3}$$

$$\mathbf{\Psi} = \tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}}, \tag{4}$$

are called *angular velocity tensor* and *angular acceleration tensor* and completely determine the vectorial field of the velocities and that of the accelerations of the rigid body in motion. Tensor $\mathbf{\Phi}$ is singular, i.e. det $\mathbf{\Phi} = 0$, since it is skew-symmetric. It can be shown that tensor $\mathbf{\Psi}$ is non-singular if and only if vectors $\vec{\omega}$ and $\vec{\varepsilon}$ are non-collinear, e.g. [3]. Classical theoretical mechanics treatises generally avoid the computation of the inverse of tensor $\mathbf{\Psi}$, denoted by $\mathbf{\Psi}^{-1}$. Paper [4] presents a rather intricate calculus for the tensor $\mathbf{\Psi}^{-1}$, and it also contains typing mistakes in the expressions for $\mathbf{\Psi}^{-1}$ and for the second scalar invariant of tensor $\mathbf{\Psi}$. A method based on Cayley-Hamilton's theorem is presented in [7]. An algorithmic procedure for the computation of the screw parameters of rigid body motion is presented in [1].

The present paper depicts a novel procedure for determining the adjugate tensors of those defined by (3) and (4). Next, an expression for tensor $\mathbf{\Psi}^{-1}$ is presented. By means of these tensors, three vectorial invariants for the distribution of the velocities and the accelerations are outlined.

Further on, it is demonstrated that tensors $\mathbf{\Phi}$, $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ can be determined through direct measurements of velocity and acceleration data of certain points of the rigid body in motion. Especially, tensor $\mathbf{\Phi}$ can be determined based on the velocity data of three non-collinear points of the body. Tensor $\mathbf{\Psi}$ (respectively $\mathbf{\Psi}^{-1}$) can be determined: (i) by means of the velocity and acceleration data of three non-collinear points or (ii) by means of the relative accelerations of three non-collinear points with respect to a fourth point non-coplanar with them.

Finally, minimal conditions to be fulfilled by the measured data in order to determine tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are established. It is demonstrated that the tensors can be determined already from the relative velocities and relative accelerations of two points with respect to a third point non-collinear with them.

## 2 Properties of the angular velocity and angular acceleration tensors

According to Cayley-Hamilton's theorem [3], any second order tensor $\mathbf{T}$ satisfies its characteristic equation:

$$\mathbf{T}^3 - T_I\mathbf{T}^2 + T_{II}\mathbf{T} - T_{III}\mathbf{I} = \mathbf{O}\,, \tag{5}$$

where $\mathbf{I}$ denotes the identity tensor and $\mathbf{O}$ stands for the zero tensor. $T_I$, $T_{II}$ and $T_{III}$ represent the scalar invariants [5], [11] of tensor $\mathbf{T}$:

$$T_I = \operatorname{trace}\mathbf{T}\,, \tag{6}$$

$$T_{II} = \frac{1}{2}\left[(\operatorname{trace}\mathbf{T})^2 - \operatorname{trace}\mathbf{T}^2\right], \tag{7}$$

$$T_{III} = \det\mathbf{T} = \frac{1}{3!}\left[(\operatorname{trace}\mathbf{T})^3 - 3(\operatorname{trace}\mathbf{T})\,(\operatorname{trace}\mathbf{T}^2) + 2\operatorname{trace}\mathbf{T}^3\right]. \tag{8}$$

For a given tensor $\mathbf{T}$, its *adjugate tensor* $\mathbf{T}^*$ is uniquely determined by:

$$\mathbf{T}\cdot\mathbf{T}^* = \mathbf{T}^*\cdot\mathbf{T} = (\det\mathbf{T})\,\mathbf{I}\,. \tag{9}$$

From (5), (8) and (9) we get the adjugate tensor $\mathbf{T}^*$ as:

$$\mathbf{T}^* = \mathbf{T}^2 - T_I\mathbf{T} + T_{II}\mathbf{I}\,. \tag{10}$$

Denoting the *dyadic product* of two vectors by $\otimes$ and the skew-symmetric tensor associated with vector $\vec{a}$ by $\tilde{\mathbf{a}}$, the following theorem can be stated:

*Theorem 1*

The adjugate angular velocity tensor and the adjugate angular acceleration tensor are given by:

$$\mathbf{\Phi}^* = \vec{\omega}\otimes\vec{\omega}\,, \tag{11}$$

$$\mathbf{\Psi}^* = (\vec{\omega}\otimes\vec{\omega})^2 - \widetilde{\boldsymbol{\omega}^2\vec{\varepsilon}} + \vec{\varepsilon}\otimes\vec{\varepsilon}\,. \tag{12}$$

*Proof:*

The scalar invariants of tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ may be computed from (6) – (8), e.g. [7], [8]. For the angular velocity tensor (3) we get:

$$\Phi_I = 0\,, \tag{13}$$

$$\Phi_{II} = -\frac{1}{2}\operatorname{trace}\tilde{\boldsymbol{\omega}}^2 = \vec{\omega}^{\,2}\,, \tag{14}$$

$$\Phi_{III} = \det\tilde{\boldsymbol{\omega}} = 0\,. \tag{15}$$

Therefore, its characteristic equation is:

$$\mathbf{\Phi}^3 + \vec{\omega}^{\,2}\mathbf{\Phi} = \mathbf{O} \quad \Longleftrightarrow \quad \tilde{\boldsymbol{\omega}}^3 + \vec{\omega}^{\,2}\tilde{\boldsymbol{\omega}} = \mathbf{O}\,. \tag{16}$$

The first scalar invariant of tensor $\mathbf{\Psi}$ is obtained from (6) and (14) as:

$$\Psi_I = \operatorname{trace}\mathbf{\Psi} = \operatorname{trace}\tilde{\boldsymbol{\omega}}^2 + \operatorname{trace}\tilde{\boldsymbol{\varepsilon}} = -2\vec{\omega}^{\,2}\,. \tag{17}$$

Computing invariant $\Psi_{II}$ with (7) and (17) we get:

$$\Psi_{II} = \frac{1}{2}[(\operatorname{trace}\mathbf{\Psi})^2 - \operatorname{trace}\mathbf{\Psi}^2] = 2\vec{\omega}^{\,4} - \frac{1}{2}\operatorname{trace}\mathbf{\Psi}^2\,. \tag{18}$$

From (4) and (16) we obtain:

$$\mathbf{\Psi}^2 = \tilde{\boldsymbol{\omega}}^4 + \tilde{\boldsymbol{\omega}}^2\tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}}^2 = -\vec{\omega}^2\tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\omega}}^2\tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}}^2\,, \tag{19}$$

hence:

$$\operatorname{trace}\mathbf{\Psi}^2 = -\vec{\omega}^{\,2}\operatorname{trace}\tilde{\boldsymbol{\omega}}^2 + \operatorname{trace}(\tilde{\boldsymbol{\omega}}^2\tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\omega}}^2) + \operatorname{trace}\tilde{\boldsymbol{\varepsilon}}^2\,. \tag{20}$$

Since tensor $\tilde{\boldsymbol{\omega}}^2\tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\omega}}^2$ is skew-symmetric, we get:

$$\operatorname{trace}(\tilde{\boldsymbol{\omega}}^2\tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\omega}}^2) = 0\,. \tag{21}$$

The skew-symmetric tensor $\tilde{\boldsymbol{\varepsilon}}$ satisfies a relation similar to (14), i.e.,

$$\operatorname{trace}\tilde{\boldsymbol{\varepsilon}}^2 = -2\vec{\varepsilon}^{\,2}\,. \tag{22}$$

Substituting (14), (21) and (22) into (20) yields:

$$\operatorname{trace}\mathbf{\Psi}^2 = 2\vec{\omega}^{\,4} - 2\vec{\varepsilon}^{\,2}\,, \tag{23}$$

and with (17) we find:

$$\Psi_{II} = \vec{\omega}^{\,4} + \vec{\varepsilon}^{\,2}\,. \tag{24}$$

Through direct computation, the third scalar invariant of tensor $\mathbf{\Psi}$ is obtained as:

$$\Psi_{III} = \det\mathbf{\Psi} = -(\vec{\omega}\times\vec{\varepsilon})^2\,. \tag{25}$$

From (13) – (15) and (10) we find:

$$\mathbf{\Phi}^* = \mathbf{\Phi}^2 + \vec{\omega}^{\,2}\mathbf{I} = \tilde{\boldsymbol{\omega}}^2 + \vec{\omega}^{\,2}\mathbf{I}\,, \tag{26}$$

or by means of identity (A.5):

$$\mathbf{\Phi}^* = \tilde{\boldsymbol{\omega}}^2 + \vec{\omega}^{\,2}\mathbf{I} = \vec{\omega}\otimes\vec{\omega}\,, \tag{27}$$

proving (11).

From (17), (24), (25) and (10) we obtain:

$$\mathbf{\Psi}^* = \mathbf{\Psi}^2 + 2\vec{\omega}^2\mathbf{\Psi} + (\vec{\omega}^4 + \vec{\varepsilon})\,\mathbf{I} = (\mathbf{\Psi} + \omega^2\mathbf{I})^2 + \vec{\varepsilon}^2\mathbf{I}\,. \tag{28}$$

Taking (4) and (27) into account, we get:

$$\mathbf{\Psi} + \vec{\omega}^2\mathbf{I} = \tilde{\boldsymbol{\omega}}^2 + \vec{\omega}^2\mathbf{I} + \tilde{\boldsymbol{\varepsilon}} = \vec{\omega}\otimes\vec{\omega} + \tilde{\boldsymbol{\varepsilon}}\,, \tag{29}$$

and thus with (A.5):

$$\begin{aligned}\mathbf{\Psi}^* &= (\vec{\omega}\otimes\vec{\omega} + \tilde{\boldsymbol{\varepsilon}})^2 + \vec{\varepsilon}^2\,\mathbf{I} = (\vec{\omega}\otimes\vec{\omega})^2 + \tilde{\boldsymbol{\varepsilon}}\cdot(\vec{\omega}\otimes\vec{\omega}) + (\vec{\omega}\otimes\vec{\omega})\cdot\tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}}^2 + \vec{\varepsilon}^2\,\mathbf{I}\\ &= (\vec{\omega}\otimes\vec{\omega})^2 + \tilde{\boldsymbol{\varepsilon}}\cdot(\vec{\omega}\otimes\vec{\omega}) + (\vec{\omega}\otimes\vec{\omega})\cdot\tilde{\boldsymbol{\varepsilon}} + \vec{\varepsilon}\otimes\vec{\varepsilon}\,.\end{aligned} \tag{30}$$

Making use of identity (A.6) finally proves relation (12). □

## 3 Vectorial invariants in rigid body kinematics

Relations (1) and (2) can be also written as:

$$\vec{v} - \vec{v}_O = \mathbf{\Phi}(\vec{\varrho} - \vec{\varrho}_O)\,, \tag{31}$$

$$\vec{a} - \vec{a}_O = \mathbf{\Psi}(\vec{\varrho} - \vec{\varrho}_O)\,. \tag{32}$$

From (9) we find the identities:

$$\mathbf{\Phi}\cdot\mathbf{\Phi}^* = \mathbf{\Phi}^*\cdot\mathbf{\Phi} = (\det\mathbf{\Phi})\,\mathbf{I} = \mathbf{O}\,, \tag{33}$$

$$\mathbf{\Psi}\cdot\mathbf{\Psi}^* = \mathbf{\Psi}^*\cdot\mathbf{\Psi} = (\det\mathbf{\Psi})\,\mathbf{I}\,. \tag{34}$$

Multiplying Eqs. (31) and (32) with the adjugate tensors from the left yields:

$$\mathbf{\Phi}^*\vec{v} = \mathbf{\Phi}^*\nu_O\,, \tag{35}$$

$$\mathbf{\Psi}^*\vec{a} - (\det\mathbf{\Psi})\,\vec{\varrho} = \mathbf{\Psi}^*\vec{a}_O - (\det\mathbf{\Psi})\,\vec{\varrho}_O\,. \tag{36}$$

Since the reference point $O \in (\mathscr{C})$ has been arbitrarily chosen, these relations prove the existence of the following vectorial invariants for the velocity and acceleration distribution of a rigid body in motion:

$$\vec{I}_1 = \mathbf{\Phi}^*\vec{v}\,, \tag{37}$$

$$\vec{I}_2 = \mathbf{\Psi}^*\vec{a} - (\det\mathbf{\Psi})\,\vec{\varrho}\,. \tag{38}$$

The vectorial characteristics $\vec{I}_1$ and $\vec{I}_2$ have the same value at a given moment of time in every point of the rigid body. Taking *Theorem 1* into account, the invariants can be also written as:

$$\vec{I}_1 = (\vec{\omega}\otimes\vec{\omega})\,\vec{v} = (\vec{\omega}\cdot\vec{v})\,\vec{\omega}\,, \tag{39}$$

$$\begin{aligned}\vec{I}_2 &= [(\vec{\omega}\otimes\vec{\omega})^2 - \widetilde{\boldsymbol{\omega}^2\vec{\varepsilon}} + \vec{\varepsilon}\otimes\vec{\varepsilon}]\,\vec{a} + (\vec{\omega}\times\vec{\varepsilon})^2\,\vec{\varrho}\\ &= \vec{a}\times[\vec{\omega}\times(\vec{\omega}\times\vec{\varepsilon})] + \vec{\omega}^2(\vec{\omega}\cdot\vec{a})\,\vec{\omega} + (\vec{\varepsilon}\cdot\vec{a})\,\vec{\varepsilon} + (\vec{\omega}\times\vec{\varepsilon})^2\,\vec{\varrho}\,,\end{aligned} \tag{40}$$

where indentity $(\vec{\omega}\otimes\vec{\omega})^2 = \vec{\omega}^2(\vec{\omega}\otimes\vec{\omega})$ obtained from (16) and (A.5) has been used. In case of $\vec{\omega}\times\vec{\varepsilon} = \vec{0}$, invariant $\vec{I}_2$ becomes:

$$\vec{I}_2 = \vec{\omega}^2(\vec{\omega}\cdot\vec{a})\,\vec{\omega} + (\vec{\varepsilon}\cdot\vec{a})\,\vec{\varepsilon}\,. \tag{41}$$

To the knowledge of the authors, invariant $\vec{I}_2$ has not been defined in any rigid body kinematics treatise yet.

By using the results of *Theorem 1*, the following theorem can be shown:

*Theorem 2*

If the instantaneous angular velocity vector and instantaneous angular acceleration vector are non-collinear, i.e. $\vec{\omega} \times \vec{\varepsilon} \neq \vec{0}$, then the angular acceleration tensor is non-singular and its inverse can be expressed as:

$$\mathbf{\Psi}^{-1} = \frac{1}{(\vec{\omega} \times \vec{\varepsilon})^2} [\widetilde{\boldsymbol{\omega}^2 \vec{\varepsilon}} - (\vec{\omega} \otimes \vec{\omega})^2 - \vec{\varepsilon} \otimes \vec{\varepsilon}] . \tag{42}$$

*Proof:*

Since $\Psi_{III} = \det \mathbf{\Psi} = -(\vec{\omega} \times \vec{\varepsilon})^2 \neq \vec{0}$, we get from (34):

$$\mathbf{\Psi}^{-1} = \frac{1}{\det \mathbf{\Psi}} \mathbf{\Psi}^* . \tag{43}$$

By means of (12) we find relation (42). □

In case of $\vec{\omega} \times \vec{\varepsilon} \neq \vec{0}$, the explicit form of tensor $\mathbf{\Psi}^{-1}$ allows to uniquely find a point $A \in (\mathscr{C})$ that, at a given moment, has an imposed acceleration $\vec{a}_A$. Denoting by $\vec{\varrho}_A$ the position vector of point $A \in (\mathscr{C})$ with respect to reference frame $\{\mathscr{R}\}$, we get from (32):

$$\vec{\varrho}_A - \vec{\varrho}_O = \mathbf{\Psi}^{-1}(\vec{a}_A - \vec{a}_O) . \tag{44}$$

Taking *Theorem 2* into account, relation (44) becomes:

$$\vec{\varrho}_A = \vec{\varrho}_O + \frac{1}{(\vec{\omega} \times \vec{\varepsilon})^2} \{[\vec{\omega} \times (\vec{\omega} \times \vec{\varepsilon})] \times (\vec{a}_A - \vec{a}_O) - \vec{\omega}^2[\vec{\omega} \cdot (\vec{a}_A - \vec{a}_O)]\,\vec{\omega} - [\vec{\varepsilon} \cdot (\vec{a}_A - \vec{a}_O)]\,\vec{\varepsilon}\} . \tag{45}$$

In particular, from (45) the position vector of the *acceleration pole* $G \in (\mathscr{C})$ characterized by $\vec{a}_G = \vec{0}$, see [2], [6], is obtained as:

$$\vec{\varrho}_G = \vec{\varrho}_O + \frac{1}{(\vec{\omega} \times \vec{\varepsilon})^2} \{\vec{a}_O \times [\vec{\omega} \times (\vec{\omega} \times \vec{\varepsilon})] + \vec{\omega}^2[\vec{\omega} \cdot \vec{a}_O]\,\vec{\omega} + [\vec{\varepsilon} \cdot \vec{a}_O]\,\vec{\varepsilon}\} . \tag{46}$$

From (44) and (31) the velocity $\vec{v}_A$ of point $A \in (\mathscr{C})$ with imposed acceleration $\vec{a}_A$ is found as:

$$\vec{v}_A - \vec{v}_O = \mathbf{\Phi}\mathbf{\Psi}^{-1}(\vec{a}_A - \vec{a}_O) . \tag{47}$$

After elementary computations, taking into account the expressions of tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}^{-1}$, we get:

$$\vec{v}_A = \vec{v}_O + \frac{1}{(\vec{\omega} \times \vec{\varepsilon})^2} \{[\vec{\omega} \cdot (\vec{a}_A - \vec{a}_O)]\,\vec{\omega} \times (\vec{\omega} \times \vec{\varepsilon}) - [\vec{\varepsilon} \cdot (\vec{a}_A - \vec{a}_O)]\,\vec{\omega} \times \vec{\varepsilon}\} . \tag{48}$$

In particular, from (48) the velocity of the acceleration pole is obtained for $\vec{a}_G = \vec{0}$:

$$\vec{v}_G = \vec{v}_O - \frac{1}{(\vec{\omega} \times \vec{\varepsilon})^2} [(\vec{\omega} \cdot \vec{a}_O)\,\vec{\omega} \times (\vec{\omega} \times \vec{\varepsilon}) - (\vec{\varepsilon} \cdot \vec{a}_O)\,\vec{\omega} \times \vec{\varepsilon}] . \tag{49}$$

Relation (48) spotlights another vectorial invariant for the distribution of the velocities and that of the accelerations of a rigid body in motion for $\vec{\omega}\times\vec{\varepsilon}\neq\vec{0}$. Due to the free choice of reference point $O$ we get a constant:

$$\vec{I}_3=\vec{v}-\frac{1}{(\vec{\omega}\times\vec{\varepsilon})^2}[(\vec{\omega}\cdot\vec{a})\,\vec{\omega}\times(\vec{\omega}\times\vec{\varepsilon})-(\vec{\varepsilon}\cdot\vec{a})\,\vec{\omega}\times\vec{\varepsilon}]\,. \tag{50}$$

The tensorial form of invariant $\vec{I}_3$ is:

$$\vec{I}_3=\vec{v}-\frac{1}{(\tilde{\boldsymbol{\omega}}\vec{\varepsilon})^2}[(\tilde{\boldsymbol{\omega}}^2\vec{\varepsilon})\otimes\vec{\omega}-(\tilde{\boldsymbol{\omega}}\vec{\varepsilon})\otimes\vec{\varepsilon}]\,\vec{a}\,. \tag{51}$$

## 4 Computation of angular velocity and acceleration tensors by direct measurements

The tensors $\mathbf{\Phi}$, $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ can be computed from direct measurements of velocities and accelerations of certain points of the rigid body $(\mathscr{C})$. Let $A_k\in(\mathscr{C})$, $k=\overline{1,3}$, be three non-collinear points of the rigid body, non-coplanar to $O\in(\mathscr{C})$. These points are located with respect to the reference frame $\{\mathscr{R}\}$ by their position vectors $\vec{\varrho}_{A_k}$, $k=\overline{1,3}$. Let $\vec{v}_{A_k}$ and $\vec{a}_{A_k}$ be the absolute velocities and accelerations of points $A_k\in(\mathscr{C})$, $k=\overline{1,3}$, respectively (Fig. 1).
In the following, relative quantities will be used:

$$\vec{r}_k=\vec{\varrho}_{A_k}-\vec{\varrho}_O\,, \tag{52}$$

$$\vec{v}_k=\vec{v}_{A_k}-\vec{v}_O\,, \tag{53}$$

$$\vec{a}_k=\vec{a}_{A_k}-\vec{a}_O\,,\qquad k=\overline{1,3}\,. \tag{54}$$

In the hypothesis of non-coplanarity of points $O$ and $A_k\in(\mathscr{C})$, $k=\overline{1,3}$, vectors $\vec{r}_k$ are also non-coplanar. Therefore, the scalar triple product of vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$, denoted by $\langle\vec{r}_1,\vec{r}_2,\vec{r}_3\rangle=\vec{r}_1\cdot(\vec{r}_2\times\vec{r}_3)$, is not zero, and vectors $\vec{r}_k$ make up a basis $\mathscr{B}=\{\vec{r}_1,\vec{r}_2,\vec{r}_3\}$ in the free-vectors set $\mathscr{V}_3$. Let $\mathscr{B}^*=\{\vec{r}^{\,1},\vec{r}^{\,2},\vec{r}^{\,3}\}$ be the reciprocal basis of $\mathscr{B}$. Vectors $\vec{r}^{\,k}$ are given by relation [5] (see Appendix):

$$\vec{r}^{\,k}=\varepsilon^{kpj}\frac{\vec{r}_p\times\vec{r}_j}{2\langle\vec{r}_1,\vec{r}_2,\vec{r}_3\rangle}\,, \tag{55}$$

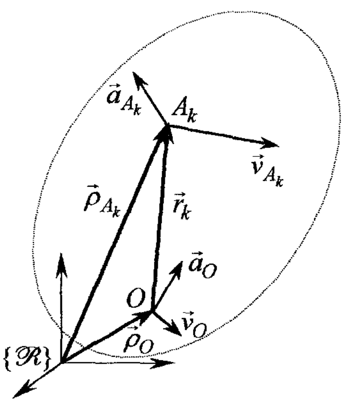


**Fig. 1.** Rigid body with reference point $O$ and observed point $A_k$

where $\varepsilon^{kpj}$ represents Ricci's permutation symbol. Einstein's rule for mute indexes summation has been used. This rule will be used further on for the sake of conciseness. All indexes run from 1 to 3.

Using notations (52)–(54) the following theorem can be shown:

*Theorem 3*

The angular velocity tensor and the angular acceleration tensor are given by:

$$\mathbf{\Phi} = \vec{\nu}_i \otimes \vec{r}^{\,i}\,, \tag{56}$$

$$\mathbf{\Psi} = \vec{a}_i \otimes \vec{r}^{\,i}\,. \tag{57}$$

*Proof:*

From (31) and (32) and notations (52)–(54) we get:

$$\mathbf{\Phi}\vec{r}_i = \vec{\nu}_i\,, \tag{58}$$

$$\mathbf{\Psi}\vec{r}_i = \vec{a}_i\,. \tag{59}$$

Application of (A.12) proves (56) and (57). □

*Remarks*

*(4a)* The measured relative velocities and relative accelerations of the three given non-collinear points of the body $A_k$ with respect to a fourth point $O$ non-coplanar with them should satisfy some necessary and sufficient compatibility conditions. These conditions are derived from the rigidity hypothesis:

$$\vec{r}_i \cdot \vec{r}_j = \text{const}\,. \tag{60}$$

Upon differentiation with respect to time, we derive from (60):

$$\vec{\nu}_i \cdot \vec{r}_j + \vec{r}_i \cdot \vec{\nu}_j = 0\,, \tag{61}$$

$$\vec{a}_i \cdot \vec{r}_j + 2\vec{\nu}_i \cdot \vec{\nu}_j + \vec{r}_i \cdot \vec{a}_j = 0\,. \tag{62}$$

Conditions (61) and (62) should be checked for all direct measurements.

*(4b)* Taking (A.24) into account, the adjugate tensors of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ can be found from (56) and (57):

$$\mathbf{\Phi}^* = \frac{\vec{r}_1 \otimes (\vec{\nu}_2 \times \vec{\nu}_3)}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} + \frac{\vec{r}_2 \otimes (\vec{\nu}_3 \times \vec{\nu}_1)}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} + \frac{\vec{r}_3 \otimes (\vec{\nu}_1 \times \vec{\nu}_2)}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle}\,, \tag{63}$$

$$\mathbf{\Psi}^* = \frac{\vec{r}_1 \otimes (\vec{a}_2 \times \vec{a}_3)}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} + \frac{\vec{r}_2 \otimes (\vec{a}_3 \times \vec{a}_1)}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} + \frac{\vec{r}_3 \otimes (\vec{a}_1 \times \vec{a}_2)}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle}\,. \tag{64}$$

*(4c)* From (56), (57) and (A.18)–(A.20) the scalar invariants of tensors **Φ** and **Ψ** may be obtained, respectively:

$$\Phi_I = \vec{\nu}_i \cdot \vec{r}^{\,i} \,, \tag{65}$$

$$\Phi_{II} = \frac{\langle \vec{r}_1, \vec{\nu}_2, \vec{\nu}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} + \frac{\langle \vec{\nu}_1, \vec{r}_2, \vec{\nu}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} + \frac{\langle \vec{\nu}_1, \vec{\nu}_2, \vec{r}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \,, \tag{66}$$

$$\Phi_{III} = \frac{\langle \vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \,, \tag{67}$$

$$\Psi_I = \vec{a}_i \cdot \vec{r}^{\,i} \,, \tag{68}$$

$$\Psi_{II} = \frac{\langle \vec{r}_1, \vec{a}_2, \vec{a}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} + \frac{\langle \vec{a}_1, \vec{r}_2, \vec{a}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} + \frac{\langle \vec{a}_1, \vec{a}_2, \vec{r}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \,, \tag{69}$$

$$\Psi_{III} = \frac{\langle \vec{a}_1, \vec{a}_2, \vec{a}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \,. \tag{70}$$

By comparing relations (13)–(15) to (65)–(67), and (17), (24), (25) to (68)–(70), the following indentities are obtained:

$$\vec{\nu}_i \cdot \vec{r}^{\,i} = 0 \,, \tag{71}$$

$$\vec{\omega}^{\,2} = \frac{\langle \vec{r}_1, \vec{\nu}_2, \vec{\nu}_3 \rangle + \langle \vec{\nu}_1, \vec{r}_2, \vec{\nu}_3 \rangle + \langle \vec{\nu}_1, \vec{\nu}_2, \vec{r}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \,, \tag{72}$$

$$\langle \vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3 \rangle = 0 \,, \tag{73}$$

$$\vec{\omega}^{\,2} = -\frac{1}{2} \vec{a}_i \cdot \vec{r}^{\,i} \,, \tag{74}$$

$$\vec{\omega}^{\,4} + \vec{\varepsilon}^{\,2} = \frac{\langle \vec{r}_1, \vec{a}_2, \vec{a}_3 \rangle + \langle \vec{a}_1, \vec{r}_2, \vec{a}_3 \rangle + \langle \vec{a}_1, \vec{a}_2, \vec{r}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \,, \tag{75}$$

$$(\vec{\omega} \times \vec{\varepsilon})^2 = -\frac{\langle \vec{a}_1, \vec{a}_2, \vec{a}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \,. \tag{76}$$

These identities do not depend on the choice of the four non-coplanar points $O$ and $A_k \in (\mathscr{C})$.

*(4d)* Tensor **Ψ** is non-singular if and only if $\Psi_{III} \neq 0$. According to (70) we get the condition:

$$\langle \vec{a}_1, \vec{a}_2, \vec{a}_3 \rangle \neq 0 \,. \tag{77}$$

Since $\boldsymbol{\Psi}^{-1} = \frac{1}{\det \boldsymbol{\Psi}} \boldsymbol{\Psi}^*$, we find from (64) and (70) the inverse tensor:

$$\boldsymbol{\Psi}^{-1} = \vec{r}_i \otimes \vec{a}^{\,i} \,, \tag{78}$$

where $\vec{a}^{\,i}$ denote the reciprocal vectors of $\vec{a}_i$ given by:

$$\vec{a}^{\,i} = \varepsilon^{ijk} \frac{\vec{a}_j \times \vec{a}_k}{2\langle \vec{a}_1, \vec{a}_2, \vec{a}_3 \rangle} \,. \tag{79}$$

*(4e)* Relations (71) and (73) represent the necessary compatibility conditions for rigid body motion. Further, from relations (72), (74), (75) and (76) four necessary inequalities for the relative velocities $\vec{\nu}_k$ and relative accelerations $\vec{a}_k$ are obtained:

$$\frac{\langle \vec{r}_1, \vec{\nu}_2, \vec{\nu}_3 \rangle + \langle \vec{\nu}_1, \vec{r}_2, \vec{\nu}_3 \rangle + \langle \vec{\nu}_1, \vec{\nu}_2, \vec{r}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \geq 0\,, \tag{80}$$

$$\vec{a}_i \cdot \vec{r}^{\,i} \leq 0\,, \tag{81}$$

$$\frac{\langle \vec{r}_1, \vec{a}_2, \vec{a}_3 \rangle + \langle \vec{a}_1, \vec{r}_2, \vec{a}_3 \rangle + \langle \vec{a}_1, \vec{a}_2, \vec{r}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \geq 0\,, \tag{82}$$

$$\frac{\langle \vec{a}_1, \vec{a}_2, \vec{a}_3 \rangle}{\langle \vec{r}_1, \vec{r}_2, \vec{r}_3 \rangle} \leq 0\,. \tag{83}$$

These inequalities are valid for any four non-coplanar points of the rigid body.

*(4f)* According to (A.4) the vectorial invariants of tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ given by relations (3) and (4) are:

$$\operatorname{vect} \mathbf{\Phi} = \vec{\omega}\,, \tag{84}$$

$$\operatorname{vect} \mathbf{\Psi} = \vec{\varepsilon}\,. \tag{85}$$

By means of relations (58), (59) and (A.22) these invariants become:

$$\operatorname{vect} \mathbf{\Phi} = \frac{1}{2} \vec{r}^{\,i} \times \vec{\nu}_i\,, \tag{86}$$

$$\operatorname{vect} \mathbf{\Psi} = \frac{1}{2} \vec{r}^{\,i} \times \vec{a}_i\,. \tag{87}$$

By comparison, we directly find the instantaneous angular velocity and the instantaneous angular acceleration as:

$$\vec{\omega} = \frac{1}{2} \vec{r}^{\,i} \times \vec{\nu}_i\,, \tag{88}$$

$$\vec{\varepsilon} = \frac{1}{2} \vec{r}^{\,i} \times \vec{a}_i\,. \tag{89}$$

## 5 Minimal conditions for algebraic computation of tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$

In *Theorem 3*, the angular velocity tensor $\mathbf{\Phi}$ and the angular acceleration tensor $\mathbf{\Psi}$ are computed from relative velocities and relative accelerations of three non-collinear points of the rigid body. As a consequence of *Theorem 3* and relations (1)–(2), the absolute velocities and absolute accelerations of four non-coplanar points uniquely determine the vectorial velocity field (acceleration field). These conditions, however, are not minimal.

### *5.1 Angular velocity tensor*

For the computation of the angular velocity tensor $\mathbf{\Phi}$ it suffices to know only the absolute velocities of three non-collinear points $A_k \in (\mathscr{C})$, $k = \overline{1,3}$, as defined above. Let $O \in (\mathscr{C})$ be now an arbitrary point of the rigid body non-coplanar to points $A_k$. Then relation (1) with notation (52) yields:

$$\vec{v}_{A_k} - \vec{v}_O = \tilde{\boldsymbol{\omega}}\vec{r}_k \,. \tag{90}$$

Computing the scalar product with vector $\vec{r}_k$ results in:

$$\vec{v}_{A_k} \cdot \vec{r}_k = \vec{v}_O \cdot \vec{r}_k \,, \quad \text{no summation.} \tag{91}$$

As described above, vectors $\vec{r}_k$ make up a basis $\mathscr{B} = \{\vec{r}_1, \vec{r}_2, \vec{r}_3\}$ with its reciprocal basis $\mathscr{B}^* = \{\vec{r}^{\,1}, \vec{r}^{\,2}, \vec{r}^{\,3}\}$ defined by relation (55). Vector $\vec{v}_O$ may then be expressed in basis $\mathscr{B}^*$ as:

$$\vec{v}_O = (\vec{v}_O \cdot \vec{r}_k)\vec{r}^{\,k} \,. \tag{92}$$

By means of (91), this becomes

$$\vec{v}_O = (\vec{v}_{A_k} \cdot \vec{r}_k)\vec{r}^{\,k} \,. \tag{93}$$

Relation (93) allows to compute velocity $\vec{v}_O$ from $\vec{v}_{A_k}$ for any arbitrary point $O \in (\mathscr{C})$ non-coplanar with points $A_k$. This may be used for finding relative velocities in order to apply *Theorem 3*.

### 5.2 *Angular acceleration tensor*

Also for the angular acceleration tensor $\boldsymbol{\Psi}$ we have to know only the absolute velocities and absolute accelerations of three non-collinear points. In order to utilise *Theorem 3*, the absolute acceleration $\vec{a}_O$ of a point $O \in (\mathscr{C})$ non-coplanar to $A_k$ has to be established.

By means of notations (52) and (54), we get from relation (2):

$$\vec{a}_{A_k} - \vec{a}_O = (\tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}})\,\vec{r}_k \,. \tag{94}$$

Taking the scalar product with $\vec{r}_k$ yields:

$$\vec{a}_{A_k} \cdot \vec{r}_k - \vec{a}_O \cdot \vec{r}_k = -(\vec{\omega} \times \vec{r}_k)^2 \,, \quad \text{no summation.} \tag{95}$$

Taking into account (90), relation (95) becomes:

$$\vec{a}_O \cdot \vec{r}_k = \vec{a}_{A_k} \cdot \vec{r}_k + (\vec{v}_{A_k} - \vec{v}_O)^2 \,, \quad \text{no summation,} \tag{96}$$

where the absolute velocity $\vec{v}_O$ is given by (93).

In basis $\mathscr{B}^*$ the acceleration vector reads:

$$\vec{a}_O = (\vec{a}_O \cdot \vec{r}_k)\,\vec{r}^{\,k} \,, \tag{97}$$

yielding the final result:

$$\vec{a}_O = [\vec{a}_{A_k} \cdot \vec{r}_k + (\vec{v}_{A_k} - \vec{v}_O)^2]\,\vec{r}^{\,k} \,. \tag{98}$$

According to (54) accelerations $\vec{a}_k$ may now be computed and used in *Theorem 3* to find tensor $\boldsymbol{\Psi}$. By means of (93) and (98) for $\vec{v}_O$ and $\vec{a}_O$, the compatibility conditions (61) and (62) become:

$$(\vec{v}_{A_i} - \vec{v}_{A_j}) \cdot (\vec{\varrho}_{A_i} - \vec{\varrho}_{A_j}) = 0 \,, \tag{99}$$

$$(\vec{a}_{A_i} - \vec{a}_{A_j}) \cdot (\vec{\varrho}_{A_i} - \vec{\varrho}_{A_j}) + (\vec{v}_{A_i} - \vec{v}_{A_j})^2 = 0 \,. \tag{100}$$

### *5.3. Computation of tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ from relative motion data*

Finally it will be shown that tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ can be computed based on the relative velocities and accelerations of only two points $A_k$, $k = \overline{1,2}$, with respect to a point $O \in (\mathscr{C})$ non-collinear to $A_1$ and $A_2$.

Let us consider a third point $A_3$, located relative to point $O$ by the position vector

$$\vec{r}_3 = \alpha(\vec{r}_1 \times \vec{r}_2)\,, \tag{101}$$

where $\alpha \in \mathbb{R}_+^*$ is a dimensional constant, e.g. $\alpha = 1/|\vec{r}_1|$. By differentiation we find:

$$\vec{v}_3 = \alpha(\vec{v}_1 \times \vec{r}_2 + \vec{r}_1 \times \vec{v}_2)\,, \tag{102}$$

$$\vec{a}_3 = \alpha(\vec{a}_1 \times \vec{r}_2 + 2\vec{v}_1 \times \vec{v}_2 + \vec{r}_1 \times \vec{a}_2)\,. \tag{103}$$

Vectors $\{\vec{r}_1, \vec{r}_2, \alpha\vec{r}_1 \times \vec{r}_2\}$ make up a basis in the free vectors' space. The reciprocal basis is made up from vectors:

$$\vec{r}^{\,1} = \frac{\vec{r}_2 \times (\vec{r}_1 \times \vec{r}_2)}{(\vec{r}_1 \times \vec{r}_2)^2}\,, \tag{104}$$

$$\vec{r}^{\,2} = \frac{(\vec{r}_1 \times \vec{r}_2) \times \vec{r}_1}{(\vec{r}_1 \times \vec{r}_2)^2}\,, \tag{105}$$

$$\vec{r}^{\,3} = \frac{\vec{r}_1 \times \vec{r}_2}{\alpha(\vec{r}_1 \times \vec{r}_2)^2}\,. \tag{106}$$

Using relations (102)–(106) in *Theorem 3* yields the following expressions for tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$:

$$\mathbf{\Phi} = \vec{v}_1 \otimes \frac{\vec{r}_2 \times (\vec{r}_1 \times \vec{r}_2)}{(\vec{r}_1 \times \vec{r}_2)^2} + \vec{v}_2 \otimes \frac{(\vec{r}_1 \times \vec{r}_2) \times \vec{r}_1}{(\vec{r}_1 \times \vec{r}_2)^2} + (\vec{v}_1 \times \vec{r}_2 + \vec{r}_1 \times \vec{v}_2) \otimes \frac{\vec{r}_1 \times \vec{r}_2}{(\vec{r}_1 \times \vec{r}_2)^2}\,, \tag{107}$$

$$\mathbf{\Psi} = \vec{a}_1 \otimes \frac{\vec{r}_2 \times (\vec{r}_1 \times \vec{r}_2)}{(\vec{r}_1 \times \vec{r}_2)^2} + \vec{a}_2 \otimes \frac{(\vec{r}_1 \times \vec{r}_2) \times \vec{r}_1}{(\vec{r}_1 \times \vec{r}_2)^2} + (\vec{a}_1 \times \vec{r}_2 + 2\vec{v}_1 \times \vec{v}_2 + \vec{r}_1 \times \vec{a}_2) \otimes \frac{\vec{r}_1 \times \vec{r}_2}{(\vec{r}_1 \times \vec{r}_2)^2}\,. \tag{108}$$

Analogously we find from (88) and (89) the instantaneous angular velocity and acceleration:

$$\vec{\omega} = \mathbf{T}_1\vec{v}_1 + \mathbf{T}_2\vec{v}_2\,, \tag{109}$$

$$\vec{\varepsilon} = \mathbf{T}_1\vec{a}_1 + \mathbf{T}_2\vec{a}_2 + \mathbf{T}_3(\vec{v}_1 \times \vec{v}_2)\,, \tag{110}$$

where tensors $\mathbf{T}_k$ are given by:

$$\mathbf{T}_1 = \frac{(\vec{r}_1 \times \vec{r}_2) \otimes \vec{r}_2 - 2\vec{r}_2 \otimes (\vec{r}_1 \times \vec{r}_2)}{2(\vec{r}_1 \times \vec{r}_2)^2} = \frac{(\vec{r}_1 \times \vec{r}_2) \otimes \vec{r}_2 - 2[(\vec{r}_1 \times \vec{r}_2) \otimes \vec{r}_2]^T}{2(\vec{r}_1 \times \vec{r}_2)^2}\,, \tag{111}$$

$$\mathbf{T}_2 = \frac{(\vec{r}_2 \times \vec{r}_1) \otimes \vec{r}_1 - 2\vec{r}_1 \otimes (\vec{r}_2 \times \vec{r}_1)}{2(\vec{r}_1 \times \vec{r}_2)^2} = \frac{(\vec{r}_2 \times \vec{r}_1) \otimes \vec{r}_1 - 2[(\vec{r}_2 \times \vec{r}_1) \otimes \vec{r}_1]^T}{2(\vec{r}_1 \times \vec{r}_2)^2}\,, \tag{112}$$

$$\mathbf{T}_3 = \frac{\vec{r}_2 \otimes \vec{r}_1 - \vec{r}_1 \otimes \vec{r}_2}{(\vec{r}_1 \times \vec{r}_2)^2} = \frac{\vec{r}_2 \otimes \vec{r}_1 - [\vec{r}_2 \otimes \vec{r}_1]^T}{(\vec{r}_1 \times \vec{r}_2)^2} = \frac{\widetilde{\mathbf{r}_1\vec{r}_2}}{(\vec{r}_1 \times \vec{r}_2)^2}\,. \tag{113}$$

In writing expression (113) of tensor $\mathbf{T}_3$, relation (A.7) has been used.

## 6 Algorithms for automatic computation

The vectorial characteristics (velocities, accelerations) measured in different points of a rigid body in motion are customarily given through their components expressed in a right oriented orthonormal basis $\{\vec{i}_1, \vec{i}_2, \vec{i}_3\}$ attached to the reference frame $\{\mathscr{R}\}$.

Denote by $\mathbf{a} = [a_1 \quad a_2 \quad a_3]^T$ the column matrix associated to vector $\vec{a} = a_1\vec{i}_1 + a_2\vec{i}_2 + a_3\vec{i}_3$ and by

$$\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

the skew-symmetric matrix associated with it. Every vectorial operation used in the previous paragraphs corresponds to a matrix operation as follows:

- the scalar product $\vec{a} \cdot \vec{b}$ corresponds to the matrix product $\mathbf{a}^T\mathbf{b}$;
- the vector product $\vec{a} \times \vec{b}$ corresponds to the matrix product $\tilde{\mathbf{a}}\,\mathbf{b}$;
- the tensor product $\vec{a} \otimes \vec{b}$ is represented by the square matrix $\mathbf{a}\,\mathbf{b}^T$;
- the scalar triple product $\langle \vec{a}, \vec{b}, \vec{c} \rangle = \vec{a} \cdot (\vec{b} \times \vec{c})$ is equal to $\mathbf{a}^T\tilde{\mathbf{b}}\,\mathbf{c} = \det[\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}]$.

Under these hypotheses, square matrices are associated to tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$.

The relations established in the previous paragraphs set up the theoretical basis allowing, under certain assumptions, to determine the angular velocity tensor and the angular acceleration tensor of a rigid body in general motion. Further on, three algorithms for the computation of the square matrices associated to tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are presented.

### *6.1. Algorithm A*

In the case that the relative velocities $\vec{v}_k$ and relative accelerations $\vec{a}_k$ of three non-collinear points relative to a non-coplanar point $O$ are known, the following procedure results from *Theorem 3:*

(i) Denote by $\mathbf{r}_k$, $\mathbf{v}_k$ and $\mathbf{a}_k$ the column matrices associated to the relative vectors $\vec{r}_k$, $\vec{v}_k$ and $\vec{a}_k$. They may be summarized as columns of the matrices:

$$\mathbf{R} = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3], \tag{114}$$

$$\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3], \tag{115}$$

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]. \tag{116}$$

(ii) Check up compatibility conditions (61) and (62) written in matrix form as:

$$\mathbf{R}^T\mathbf{V} + \mathbf{V}^T\mathbf{R} = \mathbf{O}_3, \tag{117}$$

$$\mathbf{R}^T\mathbf{A} + 2\mathbf{V}^T\mathbf{V} + \mathbf{A}^T\mathbf{R} = \mathbf{O}_3. \tag{118}$$

Relations (117) and (118) express the fact that matrices $\mathbf{R}^T\mathbf{V}$ and $\mathbf{R}^T\mathbf{A} + \mathbf{V}^T\mathbf{V}$ are skew-symmetric.

(iii) Compute the square matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ associated to the angular velocity tensor and the angular acceleration tensor according to (58) and (59):

$$\mathbf{\Phi}\mathbf{R} = \mathbf{V}, \tag{119}$$

$$\mathbf{\Psi}\mathbf{R} = \mathbf{A}. \tag{120}$$

Since points $O$ and $A_k$, $k = \overline{1,3}$, are non-coplanar, matrix $\mathbf{R}$ is non-singular. Denoting by $\mathbf{R}^{-1}$ the inverse of matrix $\mathbf{R}$, we get

$$\mathbf{\Phi} = \mathbf{V}\mathbf{R}^{-1}, \tag{121}$$

$$\mathbf{\Psi} = \mathbf{A}\mathbf{R}^{-1}. \tag{122}$$

Relations (121) and (122) represent the matrix replica of *Theorem 3.*

(iv) Compute the column matrices corresponding to instantaneous angular velocity vector $\vec{\omega}$ and to instantaneous angular acceleration vector $\vec{\varepsilon}$ according to (84) and (85):

$$\boldsymbol{\omega} = \text{vect}\,(\mathbf{V}\mathbf{R}^{-1}), \tag{123}$$

$$\boldsymbol{\varepsilon} = \text{vect}\,(\mathbf{A}\mathbf{R}^{-1}). \tag{124}$$

### 6.2. *Algorithm B*

In case that the absolute velocities and absolute accelerations of three non-collinear points of the rigid body are known, another algorithm can be applied. Denote by $\boldsymbol{\varrho}_{A_k}$ the column matrices associated to the position vectors of points $A_k$, $k = \overline{1,3}$, in the considered reference frame. Let $\mathbf{v}_{A_k}$ and $\mathbf{a}_{A_k}$ be the column matrices associated with the absolute velocities and absolute accelerations of these points. Then the following procedure may be used:

(i) Choose an arbitrary point $O \in (\mathscr{C})$ non-coplanar with points $A_k$ , given by its column matrix $\boldsymbol{\varrho}_O$. Then compute the column matrices

$$\mathbf{r}_k = \boldsymbol{\varrho}_{A_k} - \boldsymbol{\varrho}_O\,, \quad k = \overline{1,3}\,, \tag{125}$$

making up the square matrix

$$\mathbf{R} = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3]\,. \tag{126}$$

(ii) Compute the square matrix $\mathbf{R}^*$ having as columns the column matrices $\mathbf{r}^k$, corresponding to the reciprocal vectors $\vec{r}^k$ :

$$\mathbf{R}^* = (\mathbf{R}^{-1})^T = [\mathbf{r}^1 \quad \mathbf{r}^2 \quad \mathbf{r}^3]\,. \tag{127}$$

(iii) Compute the column matrices corresponding to the absolute velocity and absolute acceleration of point $O$ based on the matrix replica of relations (93) and (98):

$$\mathbf{v}_O = (\mathbf{v}_{A_k}^T \mathbf{r}_k)\,\mathbf{r}^k, \tag{128}$$

$$\mathbf{a}_O = [\mathbf{a}_{A_k}^T \mathbf{r}_k + (\mathbf{v}_{A_k} - \mathbf{v}_O)^T(\mathbf{v}_{A_k} - \mathbf{v}_O)]\,\mathbf{r}^k. \tag{129}$$

(iv) Build up matrices:

$$\mathbf{V} = [\mathbf{v}_{A_1} - \mathbf{v}_O \quad \mathbf{v}_{A_2} - \mathbf{v}_O \quad \mathbf{v}_{A_3} - \mathbf{v}_O]\,, \tag{130}$$

$$\mathbf{A} = [\mathbf{a}_{A_1} - \mathbf{a}_O \quad \mathbf{a}_{A_2} - \mathbf{a}_O \quad \mathbf{a}_{A_3} - \mathbf{a}_O]\,. \tag{131}$$

(v) Check up the compatibility conditions (117) and (118).

(vi) Apply relations (121) and (122) in order to compute matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$.

### 6.3. *Algorithm C*

In the last case, where the relative velocities and relative accelerations of two points $A_k$, $k=\overline{1,2}$, about a point $O \in (\mathscr{C})$ are known, the following procedure applies:

(i) Compute the column matrices:

$$\mathbf{r}_k = \boldsymbol{\varrho}_{A_k} - \boldsymbol{\varrho}_O\,, \quad k=\overline{1,2}\,. \tag{132}$$

(ii) Check up compatibility conditions (61) and (62) written as:

$$\mathbf{r}_1{}^T\mathbf{v}_2 + \mathbf{r}_2{}^T\mathbf{v}_1 = 0\,, \tag{133}$$

$$\mathbf{r}_1{}^T\mathbf{a}_2 + 2\,\mathbf{v}_1{}^T\mathbf{v}_2 + \mathbf{r}_2{}^T\mathbf{a}_1 = 0\,. \tag{134}$$

(iii) Compute matrices:

$$\mathbf{n} = \tilde{\mathbf{r}}_1\mathbf{r}_2\,, \tag{135}$$

$$\mathbf{m} = \mathbf{n}\mathbf{r}_2{}^T\,, \tag{136}$$

$$\mathbf{p} = -\mathbf{n}\mathbf{r}_1{}^T\,. \tag{137}$$

(iv) According to (111)–(113), compute matrices:

$$\mathbf{T}_1 = \frac{\mathbf{m} - 2\mathbf{m}^T}{2\mathbf{n}^T\mathbf{n}}\,, \tag{138}$$

$$\mathbf{T}_2 = \frac{\mathbf{p} - 2\mathbf{p}^T}{2\mathbf{n}^T\mathbf{n}}\,, \tag{139}$$

$$\mathbf{T}_3 = \frac{\tilde{\mathbf{n}}}{\mathbf{n}^T\mathbf{n}}\,. \tag{140}$$

(v) Compute the column matrices $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ corresponding to the instantaneous angular velocity vector $\vec{\omega}$ and to the instantaneous angular acceleration $\vec{\varepsilon}$ according to (109) and (110):

$$\boldsymbol{\omega} = \mathbf{T}_1\mathbf{v}_1 + \mathbf{T}_2\mathbf{v}_2\,, \tag{141}$$

$$\boldsymbol{\varepsilon} = \mathbf{T}_1\mathbf{a}_1 + \mathbf{T}_2\mathbf{a}_2 + \mathbf{T}_3\tilde{\mathbf{v}}_1\mathbf{v}_2\,. \tag{142}$$

(vi) Finally, tensors (matrices) $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are given by relations (3) and (4):

$$\mathbf{\Phi} = \tilde{\boldsymbol{\omega}}\,, \tag{143}$$

$$\mathbf{\Psi} = \tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}}\,, \tag{144}$$

or, taking (107) and (108) into account:

$$\mathbf{\Phi} = \frac{\mathbf{v}_1(\tilde{\mathbf{r}}_2\mathbf{n})^T + \mathbf{v}_2(\tilde{\mathbf{n}}\mathbf{r}_1)^T + (\tilde{\mathbf{r}}_1\mathbf{v}_2 - \tilde{\mathbf{r}}_2\mathbf{v}_1)\,\mathbf{n}^T}{\mathbf{n}^T\mathbf{n}}\,, \tag{145}$$

$$\mathbf{\Psi} = \frac{\mathbf{a}_1(\tilde{\mathbf{r}}_2\mathbf{n})^T + \mathbf{a}_2(\tilde{\mathbf{n}}\mathbf{r}_1)^T + (\tilde{\mathbf{r}}_1\mathbf{a}_2 + 2\tilde{\mathbf{v}}_1\mathbf{v}_2 - \tilde{\mathbf{r}}_2\mathbf{a}_1)\,\mathbf{n}^T}{\mathbf{n}^T\mathbf{n}}\,. \tag{146}$$

## 7 Numerical examples

*7.1. In a first example, let $A_k$ be three non-collinear points of a rigid body located in a reference frame by the column matrices of their position vectors*

$$\boldsymbol{\varrho}_{A_1} = [0 \quad 0 \quad 0]^T, \qquad \boldsymbol{\varrho}_{A_2} = [0 \quad 1 \quad 0]^T, \qquad \boldsymbol{\varrho}_{A_3} = [0 \quad 1 \quad 1]^T. \tag{147}$$

Let $O \in (\mathscr{C})$ be a point of the rigid body given by $\boldsymbol{\varrho}_O = [1 \quad 0 \quad 1]^T$. Assume the relative velocities and relative accelerations of points $A_k$, $k = \overline{1,3}$, about point $O$ to be given as:

$$\mathbf{v}_1 = [1 \quad 0 \quad -1]^T, \qquad \mathbf{v}_2 = [0 \quad 0 \quad 0]^T, \qquad \mathbf{v}_3 = [-1 \quad -1 \quad 0]^T, \tag{148}$$

$$\mathbf{a}_1 = [1 \quad 0 \quad 1]^T, \qquad \mathbf{a}_2 = [1 \quad -2 \quad -3]^T, \qquad \mathbf{a}_3 = [2 \quad 0 \quad -5]^T. \tag{149}$$

Compute angular velocity tensor $\boldsymbol{\Phi}$, angular acceleration tensor $\boldsymbol{\Psi}$, instantaneous angular velocity vector $\boldsymbol{\omega}$ and instantaneous angular acceleration vector $\boldsymbol{\varepsilon}$.

According to *Algorithm A*, we first compute matrices:

$$\mathbf{r}_1 = [-1 \quad 0 \quad -1]^T, \qquad \mathbf{r}_2 = [-1 \quad 1 \quad -1]^T, \qquad \mathbf{r}_3 = [-1 \quad 1 \quad 0]^T. \tag{150}$$

Next, we build up matrices $\mathbf{R}$, $\mathbf{V}$ and $\mathbf{A}$ given by relations (114)–(116):

$$\mathbf{R} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 1 & -3 & -5 \end{bmatrix}. \tag{151}$$

The check (117), (118) shows the skew-symmetry of:

$$\mathbf{R}^T\mathbf{V} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{R}^T\mathbf{A} + \mathbf{V}^T\mathbf{V} = \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}. \tag{152}$$

Matrix $\mathbf{R}$ given by (151) is non-singular resulting in its inverse:

$$\mathbf{R}^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}. \tag{153}$$

Finally, from (121)–(124) we find:

$$\boldsymbol{\Phi} = \mathbf{V}\mathbf{R}^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{\omega} = \operatorname{vect} \boldsymbol{\Phi} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \tag{154}$$

$$\boldsymbol{\Psi} = \mathbf{A}\mathbf{R}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ -2 & -2 & 2 \\ 1 & -4 & -2 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{\varepsilon} = \operatorname{vect} \boldsymbol{\Psi} = \operatorname{vect} \frac{1}{2}[\boldsymbol{\Psi} - \boldsymbol{\Psi}^T] = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}. \tag{155}$$

*7.2. In the second example, the same points (147) are used, but now let us assume that the absolute velocities and accelerations are given:*

$$\mathbf{v}_{A_1} = [1 \quad 1 \quad 0]^T, \qquad \mathbf{v}_{A_2} = [0 \quad 1 \quad 1]^T, \qquad \mathbf{v}_{A_3} = [-1 \quad 0 \quad 1]^T, \tag{156}$$

$$\mathbf{a}_{A_1} = [0 \quad 1 \quad 6]^T, \qquad \mathbf{a}_{A_2} = [1 \quad -1 \quad 2]^T, \qquad \mathbf{a}_{A_3} = [1 \quad 1 \quad 0]^T. \tag{157}$$

According to *Algorithm B* we arbitrarily choose a point $O$ non-coplanar with $A_k$ as:

$$\boldsymbol{\varrho}_O = [1 \quad 0 \quad 1]^T. \tag{158}$$

In the same way as in relation (150), matrix $\mathbf{R}$ is determined being identical to (151). The transpose of its inverse (153), i.e.:

$$\mathbf{R}^* = (\mathbf{R}^{-1})^T = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \tag{159}$$

defines the reciprocal basis by its columns:

$$\mathbf{r}^1 = [-1 \quad -1 \quad 0]^T, \qquad \mathbf{r}^2 = [1 \quad 1 \quad -1]^T, \quad \mathbf{r}^3 = [-1 \quad 0 \quad 1]^T. \tag{160}$$

From (128) and (129) the velocity $\mathbf{v}_O$ and acceleration $\mathbf{a}_O$ are computed using (150) and (160):

$$\mathbf{v}_O = [0 \quad 1 \quad 1]^T, \qquad \mathbf{a}_O = [-1 \quad 1 \quad 5]^T. \tag{161}$$

The matrices (130) and (131) are the same as in the first example, see (151). Thus, the remainder of the procedure is identical to *Algorithm A*.

*7.3. In our last example, assume that only the relative velocities* $\mathbf{v}_1, \mathbf{v}_2$ *and relative accelerations* $\mathbf{a}_1, \mathbf{a}_2$ *of* (148) *and* (149) *are given for two points* $A_1, A_2$ *located by the relative positions* $\mathbf{r}_1, \mathbf{r}_2$ *of* (150)

In this case *Algorithm C* is applied, where we first have to compute $\mathbf{n}$, $\mathbf{m}$ and $\mathbf{p}$ by means of (135)–(137):

$$\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{m} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}. \tag{162}$$

Next, matrices $\mathbf{T}_k$ are computed by means of (138)–(140):

$$\mathbf{T}_1 = \frac{1}{4}\begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 2 \\ 3 & -1 & -1 \end{bmatrix}, \quad \mathbf{T}_2 = \frac{1}{4}\begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad \mathbf{T}_3 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{163}$$

The column matrices $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ are computed by means of (141) and (142), i.e.:

$$\boldsymbol{\omega} = [1 \quad -1 \quad 1]^T, \qquad \boldsymbol{\varepsilon} = [-3 \quad 0 \quad -1]^T. \tag{164}$$

Finally, the resulting matrices $\boldsymbol{\Phi} = \tilde{\boldsymbol{\omega}}$ and $\boldsymbol{\Psi} = \tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}}$ are computed identically to (154), (155).

## 8 Conclusions

In this paper, the properties of the angular velocity tensor $\mathbf{\Phi} = \tilde{\boldsymbol{\omega}}$ and the angular acceleration tensor $\mathbf{\Psi} = \tilde{\boldsymbol{\omega}}^2 + \tilde{\boldsymbol{\varepsilon}}$ are systematically studied based on tensor algebra. This approach benefits from the intrinsic character of the obtained relations. Thus, the vectors emerging in the expressions of these tensors in *Theorem 3* (relations (56) and (57)) may be expressed either in an inertial frame or in a body-fixed frame.

Our paper considers a tensor as a $\mathbb{R}$-linear mapping of the free vectors set $\mathscr{V}_3$ into $\mathscr{V}_3$. A tensor may be represented by a matrix only after a basis in the linear space $\mathscr{V}_3$ has been selected. Preferring an orthonormal basis (the most frequently used practice [4], [11]) simplifies the calculus, but this particular choice may conceal some essential properties. This is probably why, to the knowledge of the authors, the expressions of the adjugate angular velocity tensor and adjugate angular acceleration tensor (and implicitly the vectorial invariant $\vec{I}_2$ (40)) have not been defined in any rigid body kinematics treatise yet.

The computation of the two tensors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ by point velocity and point acceleration data is an important issue in robotics, especially in parallel manipulator kinematics [3]. The algorithms depicted in this paper differ from the ones given by Angeles [1]–[3] and possess some specific features.

Besides its intrinsic character formerly mentioned, *Algorithm B* presented here is frame invariant. It comprises the evaluation of the velocity and acceleration of an arbitrary point on the rigid body non-coplanar with the measurement points as a computation step. Thus, the matrix $\mathbf{R}$ (114) is always non-singular. In the procedure proposed by Angeles, which is not frame invariant, inconvenient singularities may occur. Angeles describes in [2] the way these singularities may be surmounted through a change of the reference frame.

From our point of view, *Algorithm C* seems to be the most suitable one for automatic computation since no inverse matrix calculation is needed. The computation of tensors $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$ from (138)–(140) depends only on the measurement points, and once they are computed, they may be used for more measurements.

The relations (109) and (110) allow to expeditiously compute both the instantaneous angular velocity and instantaneous angular acceleration of a rigid body under general motion based on the movement laws of three non-collinear points of the rigid body. This situation is frequently encountered in robotics [3]. This computation does not require previous evaluation of either Euler angles, Euler-Rodrigues parameters, rotation quaternions or the rotation matrix, as e.g. in [6], [9], [10].

Finally, the relation (57) from *Theorem 3* reveals that the field of accelerations of a rigid body under general motion is completely determined by the accelerations of four non-coplanar points of the body, without any prior information on their velocities. This property may be extended in the case of the $n$-th order accelerations, $n \in \mathbf{N}$, $n \geq 2$. Moreover, from (4) and (57) we get:

$$\tilde{\boldsymbol{\omega}}^2 = \frac{1}{2}[\vec{a}_i \otimes \vec{r}^{\,i} + \vec{r}^{\,i} \otimes \vec{a}_i]\,, \tag{165}$$

and the relation (74) gives:

$$\vec{\omega}^2 = -\frac{1}{2}\vec{a}_i \cdot \vec{r}^{\,i}\,. \tag{166}$$

Thus, the angular velocity tensor can be determined up to a sign without any velocity measurements.

## Appendix

Let $\mathscr{V}_3$ be the free vectors set as subset of the three dimensional Euclidean space $\mathscr{E}_3$. An $\mathbb{R}$-linear mapping is called *a tensor*:

$$\begin{aligned} &\mathbf{\Phi} : \mathscr{V}_3 \to \mathscr{V}_3 \,, \\ &\mathbf{\Phi}(\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2) = \lambda_1 \mathbf{\Phi} \vec{v}_1 + \lambda_2 \mathbf{\Phi} \vec{v}_2 \qquad \forall \lambda_1, \lambda_2 \in \mathbb{R} \qquad \forall \vec{v}_1, \vec{v}_2 \in \mathscr{V}_3 \,. \end{aligned} \tag{A.1}$$

The *identity tensor* is denoted by $\mathbf{I}$ and the *zero tensor* is denoted by $\mathbf{O}$. For a given tensor $\mathbf{\Phi}$, the *transposed tensor* is denoted by $\mathbf{\Phi}^T$.

Given two vectors $\vec{a}, \vec{b} \in \mathscr{V}_3$, $\vec{a} \otimes \vec{b}$ denotes a tensor defined by:

$$\begin{aligned} &\vec{a} \otimes \vec{b} : \mathscr{V}_3 \to \mathscr{V}_3 \,, \\ &(\vec{a} \otimes \vec{b})\, \vec{v} = (\vec{v} \cdot \vec{b})\, \vec{a} \qquad \forall \vec{v} \in \mathscr{V}_3 \,. \end{aligned} \tag{A.2}$$

Tensor $\vec{a} \otimes \vec{b}$ is called *tensor (dyadic) product* of vectors $\vec{a}$ and $\vec{b}$. Its transposed tensor is $(\vec{a} \otimes \vec{b})^T = \vec{b} \otimes \vec{a}$. In general, $\vec{a} \otimes \vec{b} \neq \vec{b} \otimes \vec{a}$.

Given a vector $\vec{a} \in \mathscr{V}_3$, the skew-symmetric tensor associated with vector $\vec{a}$ is denoted by $\tilde{\mathbf{a}}$ and is defined by:

$$\begin{aligned} &\tilde{\mathbf{a}} : \mathscr{V}_3 \to \mathscr{V}_3 \,, \\ &\tilde{\mathbf{a}} \vec{v} = \vec{a} \times \vec{v} \qquad \forall \vec{v} \in \mathscr{V}_3 \,. \end{aligned} \tag{A.3}$$

The correspondence is denoted by $\vec{a} = \operatorname{vect} \tilde{\mathbf{a}}$. For an arbitrary tensor $\mathbf{\Phi}$, the *axial vector* of $\mathbf{\Phi}$ is defined by:

$$\operatorname{vect} \mathbf{\Phi} = \operatorname{vect} \frac{1}{2} \left[ \mathbf{\Phi} - \mathbf{\Phi}^T \right] . \tag{A.4}$$

The following lemma can be proven:

*Lemma 1:*

For every choice of vectors $\vec{\omega}, \vec{\varepsilon}, \vec{r}_1, \vec{r}_2 \in \mathscr{V}_3$, the following identities hold:

$$\vec{\omega} \otimes \vec{\omega} = \tilde{\boldsymbol{\omega}}^2 + \vec{\omega}^2 \mathbf{I} \,, \tag{A.5}$$

$$\tilde{\boldsymbol{\varepsilon}} \cdot (\vec{\omega} \otimes \vec{\omega}) + (\vec{\omega} \otimes \vec{\omega}) \cdot \tilde{\boldsymbol{\varepsilon}} = -\widetilde{\tilde{\boldsymbol{\omega}}^2 \vec{\varepsilon}} \,, \tag{A.6}$$

$$\widetilde{\tilde{\mathbf{r}}_1 \vec{r}_2} = \vec{r}_2 \otimes \vec{r}_1 - \vec{r}_1 \otimes \vec{r}_2 \,. \tag{A.7}$$

*Proof:*

Using the rule for double vector product computation, we successively obtain:

$$\begin{aligned} \tilde{\boldsymbol{\omega}}^2 \vec{v} &= \vec{\omega} \times (\vec{\omega} \times \vec{v}) = (\vec{\omega} \cdot \vec{v})\, \vec{\omega} - \vec{\omega}^2 \vec{v} \\ &= (\vec{\omega} \otimes \vec{\omega})\, \vec{v} - \vec{\omega}^2 \vec{v} = (\vec{\omega} \otimes \vec{\omega} - \vec{\omega}^2 \mathbf{I})\, \vec{v} \qquad \forall \vec{v} \in \mathscr{V}_3 \,, \end{aligned} \tag{A.8}$$

which proves (A.5).

Next, let us evaluate the following expression:

$$[\tilde{\boldsymbol{\varepsilon}} \cdot (\vec{\omega} \otimes \vec{\omega}) + (\vec{\omega} \otimes \vec{\omega}) \cdot \tilde{\boldsymbol{\varepsilon}}]\,\vec{v} = \tilde{\boldsymbol{\varepsilon}}[(\vec{\omega} \otimes \vec{\omega})\,\vec{v}] + (\vec{\omega} \otimes \vec{\omega})[\tilde{\boldsymbol{\varepsilon}}\vec{v}]$$
$$= \tilde{\boldsymbol{\varepsilon}}[(\vec{v} \cdot \vec{\omega})\,\vec{\omega}] + (\vec{\omega} \otimes \vec{\omega})\,(\vec{\varepsilon} \times \vec{v}) = (\vec{v} \cdot \vec{\omega})\,(\vec{\varepsilon} \times \vec{\omega}) + [(\vec{\varepsilon} \times \vec{v}) \cdot \vec{\omega}]\,\vec{\omega}$$
$$= -(\vec{v} \cdot \vec{\omega})\,(\vec{\omega} \times \vec{\varepsilon}) + [(\vec{\omega} \times \vec{\varepsilon}) \cdot \vec{v}]\,\vec{\omega} = \vec{v} \times [\vec{\omega} \times (\vec{\omega} \times \vec{\varepsilon})] = -(\widetilde{\boldsymbol{\omega}^2\vec{\varepsilon}})\,\vec{v} \qquad \forall \vec{v} \in \mathscr{V}_3 . \tag{A.9}$$

From this we get (A.6).

By means of definitions (A.2) and (A.3) we find:

$$(\widetilde{\mathbf{r}_1\vec{r}_2})\,\vec{v} = (\vec{r}_1 \times \vec{r}_2) \times \vec{v} = (\vec{r}_1 \cdot \vec{v})\,\vec{r}_2 - (\vec{r}_2 \cdot \vec{v})\,\vec{r}_1 = (\vec{r}_2 \otimes \vec{r}_1)\,\vec{v} - (\vec{r}_1 \otimes \vec{r}_2)\,\vec{v}$$
$$= (\vec{r}_2 \otimes \vec{r}_1 - \vec{r}_1 \otimes \vec{r}_2)\,\vec{v} \qquad \forall \vec{v} \in \mathscr{V}_3 , \tag{A.10}$$

proving (A.7). □

Let $\mathscr{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be a basis of the free vectors set $\mathscr{V}_3$. Let $\mathscr{B}^* = \{\vec{e}^{\,1}, \vec{e}^{\,2}, \vec{e}^{\,3}\}$ be the reciprocal basis determined by [5]:

$$\vec{e}^{\,i} = \varepsilon^{ijk} \frac{\vec{e}_j \times \vec{e}_k}{2\langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle} , \tag{A.11}$$

where $\varepsilon^{ijk}$ represents Ricci's permutation symbol, and $\langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle = \vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)$ denotes the scalar triple product of vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$. Einstein's rule for mute indexes summation has been used. All indexes run from 1 to 3.

For an arbitrary tensor we can prove [5]:

*Lemma 2:*

A tensor $\boldsymbol{\Phi} : \mathscr{V}_3 \to \mathscr{V}_3$ is uniquely determined by the values of application of $\boldsymbol{\Phi}$ to the elements of basis $\mathscr{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ :

$$\boldsymbol{\Phi} = (\boldsymbol{\Phi}\vec{e}_i) \otimes \vec{e}^{\,i} . \tag{A.12}$$

□

The scalar invariants of tensor $\boldsymbol{\Phi}$ are computed from condition:

$$\boldsymbol{\Phi}\vec{v} = \lambda\vec{v}, \qquad \lambda \in \mathbb{R}, \qquad \vec{v} \neq \vec{0} \tag{A.13}$$

that determines the characteristic equation of tensor $\boldsymbol{\Phi}$:

$$p(\lambda) = \det(\boldsymbol{\Phi} - \lambda\mathbf{I}) = \mathbf{0} . \tag{A.14}$$

Using the expression: $\mathbf{I} = \vec{e}_i \otimes \vec{e}^{\,i}$ for the identity tensor, we find from (A.14) by means of *Lemma 2:*

$$\det\{[\boldsymbol{\Phi}\vec{e}_i - \lambda\vec{e}_i] \otimes \vec{e}^{\,i}\} = 0 . \tag{A.15}$$

This proves that vectors $\boldsymbol{\Phi}\vec{e}_i - \lambda\vec{e}_i$, $i = \overline{1,3}$, are linearly dependent. Therefore their scalar triple product is zero:

$$\langle \boldsymbol{\Phi}\vec{e}_1 - \lambda\vec{e}_1 , \quad \boldsymbol{\Phi}\vec{e}_2 - \lambda\vec{e}_2 , \quad \boldsymbol{\Phi}\vec{e}_3 - \lambda\vec{e}_3 \rangle = 0 . \tag{A.16}$$

By means of the properties of the scalar triple product, we get from (A.16) the characteristic equation:

$$\lambda^3 - \Phi_I\lambda^2 + \Phi_{II}\lambda - \Phi_{III} = 0 , \tag{A.17}$$

where $\Phi_I$, $\Phi_{II}$ and $\Phi_{III}$ denote the scalar invariants of tensor $\mathbf{\Phi}$:

$$\Phi_I = \mathrm{trace}\,\mathbf{\Phi} = (\mathbf{\Phi}\vec{e}_i)\cdot\vec{e}^{\,i}\,, \tag{A.18}$$

$$\Phi_{II} = \frac{\langle\vec{e}_1, \mathbf{\Phi}\vec{e}_2, \mathbf{\Phi}\vec{e}_3\rangle + \langle\mathbf{\Phi}\vec{e}_1, \vec{e}_2, \mathbf{\Phi}\vec{e}_3\rangle + \langle\mathbf{\Phi}\vec{e}_1, \mathbf{\Phi}\vec{e}_2, \vec{e}_3\rangle}{\langle\vec{e}_1, \vec{e}_2, \vec{e}_3\rangle}\,, \tag{A.19}$$

$$\Phi_{III} = \det\mathbf{\Phi} = \frac{\langle\mathbf{\Phi}\vec{e}_1, \mathbf{\Phi}\vec{e}_2, \mathbf{\Phi}\vec{e}_3\rangle}{\langle\vec{e}_1, \vec{e}_2, \vec{e}_3\rangle}\,. \tag{A.20}$$

Cayley-Hamilton's theorem [3] states that any tensor satisfies its characteristic equation, therefore:

$$\mathbf{\Phi}^3 - \Phi_I\mathbf{\Phi}^2 + \Phi_{II}\mathbf{\Phi} - \Phi_{III}\mathbf{I} = \mathbf{O}\,. \tag{A.21}$$

The vectorial invariant of tensor $\mathbf{\Phi}$ can be computed according to (A.4). From *Lemma 2* with relation $\mathrm{vect}\,(\vec{a}\otimes\vec{b}) = (\vec{b}\times\vec{a})/2$ we find [3]:

$$\mathrm{vect}\,\mathbf{\Phi} = \frac{1}{2}\vec{e}^{\,i}\times(\mathbf{\Phi}\vec{e}^{\,i})\,. \tag{A.22}$$

The adjugate tensor of $\mathbf{\Phi}$, denoted by $\mathbf{\Phi}^*$, is uniquely determined by:

$$\mathbf{\Phi}\cdot\mathbf{\Phi}^* = (\det\mathbf{\Phi})\,\mathbf{I}\,. \tag{A.23}$$

An explicit expression for $\mathbf{\Phi}^*$ is:

$$\mathbf{\Phi}^* = \vec{e}_1\otimes\frac{(\mathbf{\Phi}\vec{e}_2)\times(\mathbf{\Phi}\vec{e}_3)}{\langle\vec{e}_1, \vec{e}_2, \vec{e}_3\rangle} + \vec{e}_2\otimes\frac{(\mathbf{\Phi}\vec{e}_3)\times(\mathbf{\Phi}\vec{e}_1)}{\langle\vec{e}_1, \vec{e}_2, \vec{e}_3\rangle} + \vec{e}_3\otimes\frac{(\mathbf{\Phi}\vec{e}_1)\times(\mathbf{\Phi}\vec{e}_2)}{\langle\vec{e}_1, \vec{e}_2, \vec{e}_3\rangle}\,, \tag{A.24}$$

which can be tested by (A.23) using (A.12) and (A.20).

In case of $\det\mathbf{\Phi}\neq 0$, tensor $\mathbf{\Phi}$ is non-singular and:

$$\mathbf{\Phi}^{-1} = \frac{1}{\det\mathbf{\Phi}}\mathbf{\Phi}^*\,. \tag{A.25}$$

By means of (A.20) and (A.24), we get from (A.25):

$$\mathbf{\Phi}^{-1} = \vec{e}_1\otimes\frac{(\mathbf{\Phi}\vec{e}_2)\times(\mathbf{\Phi}\vec{e}_3)}{\langle\mathbf{\Phi}\vec{e}_1, \mathbf{\Phi}\vec{e}_2, \mathbf{\Phi}\vec{e}_3\rangle} + \vec{e}_2\otimes\frac{(\mathbf{\Phi}\vec{e}_3)\times(\mathbf{\Phi}\vec{e}_1)}{\langle\mathbf{\Phi}\vec{e}_1, \mathbf{\Phi}\vec{e}_2, \mathbf{\Phi}\vec{e}_3\rangle} + \vec{e}_3\otimes\frac{(\mathbf{\Phi}\vec{e}_1)\times(\mathbf{\Phi}\vec{e}_2)}{\langle\mathbf{\Phi}\vec{e}_1, \mathbf{\Phi}\vec{e}_2, \mathbf{\Phi}\vec{e}_3\rangle}\,. \tag{A.26}$$

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