Acta Mechanica 30, 145—155 (1978)

ACTA MECHANICA

A Mathematical Theory of Shock-Wave Formation in Arterial Blood Flow

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With 1 Figure

(*Received June 10, 1977*)

Summary — Zusammenfassung

A Mathematical Theory of Shock-Wave Formation in Arterial Blood Flow. Theoretical and experimental evidence is available for the occurrence of shock waves in arterial blood flow. A sudden increase of pressure at the entrance of a fluid-filled semi-infinite elastic tube is considered as a model to investigate mathematically the possibility of shock-wave formation. By use of the method of wavefront expansions explicit results are obtained about the circumstances under which shocks may form and about the time and distance at which this may occur.

Eine mathematische Theorie für Stoßwellen bei Blutströmungen in Arterien. Es gibt theoretische und experimentelle Hinweise für das Auftreten von Stoßwellen bei Blutströmungen in Arterien. Ein Drucksprung am Eintritt eines halbunendlich langen, elastischen Rohres wird als Modell benutzt, um mathematisch die Möglichkeit der Bildung einer Stoßwelle zu untersuchen.

Durch Entwicklung in der Umgebung der Wellenfront ergeben sich explizite Ergebnisse über Ort und Zeit der Stoßbildung.

1. Introduction

The mathematical theory of blood flow in the larger arteries is usually based on an unsteady one-dimensional model in which the internal pressure and fluid velocity are averaged over the cross section of the artery. Since the resulting set of nonlinear first-order partial differential equations is hyperbolic and resembles strongly the equations governing unsteady one-dimensional gasdynamics, the natural question arises whether shock waves may develop in the solution. There is experimental evidence (cf. Anliker et al. [1]) that this may occur under physiological circumstances. Rudinger [2] considered inviscid flow through a semi-infinite uniform distensible tube which is initially undisturbed and prescribed a continuous but non-smooth pressure rise at the entrance of the tube. Using the method of characteristics he found an expression for the shock-formation distance.

In the present paper we shall generalize Rudinger's problem to flows with friction through non-uniform elastic tubes. We are interested in the circumstances under which a shock wave will form and at which time and at which distance from the entrance of the tube this will happen. It is to be expected that the non-uniformity of the tube and the viscosity of the fluid will have a considerable influence on the shock-wave formation. After the formulation of the mathematical model in Section 2, we consider first in Section 3 Rudinger's problem as a special case of the general problem. This problem which is solved within the framework of simple-wave solutions (cf. Whitham [3]) serves as a convenient introduction to the general case which is dealt with in Section 4. For the general problem no simple-wave solutions exist and we employ in Section 4 the useful method of wavefront expansions [3] to derive an ordinary differential equation for the evolution in time of the jump in the first derivatives of the pressure at the wavefront. This equation can be solved in closed form, but in general a numerical evaluation is needed to obtain the time instant at which the jump in the first derivatives of the pressure becomes infinite, that is, at which a shock wave forms. In Section 5, therefore, we consider some special cases for which explicit results can be obtained. It will turn out that for a tube which becomes narrower for increasing distance from the entrance the shock wave will form earlier and nearer to the entrance than for a uniform tube. Furthermore, friction has a delaying effect on shock-wave formation.

2. The One-Dimensional Theory

The usual one-dimensional model for blood flow in the larger arteries is based on a number of assumptions. The blood is considered as an incompressible fluid and the artery as a straight elastic tube of circular cross section. Since arteries are constrained longitudinally, only axisymmetric bulging motions of the (impermeable) tube wall are considered. Furthermore, the tube radius is usually much smaller than the typical axial wavelength of the flow, and thus radial accelerations and pressure forces in the fluid are neglected. Hence only axial fluid accelerations and forces are taken into account, and the fluid velocity and the pressure are averaged over the internal cross-section of the tube. The only independent variables in the mathematical model are thus the axial coordinate x and the time t.

We are now in a position to formulate the equations of motion for the fluid [1], [2]. First we have the continuity equation

$$A_t + (Av)_x = 0, \tag{2.1}$$

where A denotes the cross-sectional area and v the axial fluid velocity (averaged over the cross section). Next we have the momentum equation

$$v_t + vv_x + \frac{1}{\varrho} p_x = f(v, A), \tag{2.2}$$

where p is the pressure difference across the tube wall (the exterior pressure being taken constant), ϱ denotes the constant density of the fluid and $f(v, A)$ is a friction term (with $f(0, A) = 0$) which will be specified later. It should be remarked that for flows in which discontinuities (shocks) have developed the validity of these equations has to be questioned. Prior to the shock formation, however, Eqs. (2.1) and (2.2) are valid.

We shall consider the flow as a perturbation (not necessarily small) of the undisturbed state $v = 0$, $p = p_0 =$ constant, $A = A_0(x)$ given, p_0, A_0 positive, which corresponds to an inflated tube with non-uniform cross-section $A_0(x)$.

To complement Eqs. (2.1) and (2.2) we need a third equation. For an elastic tube one usually takes the relation between p and A which holds under static conditions:

$$A = A(p, x), \tag{2.3}$$

with $A(p_0, x) = A_0(x)$ and $A_p(p_0, x) > 0$ for $x \geqq 0$. This means that the tube wall reacts instantaneously on pressure changes in the fluid.

Eqs. (2.1)—(2.3) constitute a hyperbolic set of quasi-linear first-order partial differential equations very similar to those of one-dimensional unsteady gas-dynamics. Hence there is a possibility that discontinuities in the form of shock waves develop in the solution.

In the case of flow through a uniform tube ($A = A(p)$, $A(p_0) = A_0 =$ constant) without friction ($f = 0$) we may consider infinitesimally small perturbations $\tilde{A}$, $\tilde{p}$ and $\tilde{v}$ about the undisturbed state. When we substitute $A = A_0 + \tilde{A}$, $p = p_0 + \tilde{p}$, $v = \tilde{v}$ into Eqs. (2.1)—(2.3) and subsequently linearize, we find that $\tilde{A}$, $\tilde{p}$ and $\tilde{v}$ all satisfy a one-dimensional wave equation of the form

$$u_{tt} - c_0^2 u_{xx} = 0,$$

where

$$c_0 = \left\{\frac{A_0}{\varrho A'(p_0)}\right\}^{1/2}$$

is the infinitesimal-wave propagation velocity. For actual blood flow in the larger human arteries c_0 is of the order of magnitude of 3—10 m/s. The fluid velocity v, however, is very much smaller in general. For the general case we now define the "local wave velocity" $c(p, x)$ by the relation

$$c(p, x) = \left\{\frac{A(p, x)}{\varrho A_p(p, x)}\right\}^{1/2}. \tag{2.4}$$

It appears to be convenient to rewrite Eqs. (2.1)—(2.3) as two equations involving p and v only. We arrive at the following equations:

$$p_t + v p_x + \varrho c^2(p, x)\, v_x + \frac{A_x(p, x)}{A_p(p, x)}\, v = 0, \tag{2.5}$$

$$v_t + v v_x + \frac{1}{\varrho}\, p_x = f\{v, A(p, x)\}. \tag{2.6}$$

With appropriate initial and boundary conditions these equations form the basis for the calculations in the sequel.

3. Inviscid Flow Through a Uniform Elastic Tube

In the special case of inviscid flow ($f = 0$) through a uniform elastic tube we have $A = A(p)$, $A(p_0) = A_0 =$ constant, $c = c(p)$, $c(p_0) = c_0 =$ constant. Then Eqs. (2.5) and (2.6) simplify to

$$p_t + vp_x + \varrho c^2(p)\, v_x = 0, \tag{3.1}$$

$$v_t + vv_x + \frac{1}{\varrho}\, p_x = 0. \tag{3.2}$$

These equations have exact solutions in the form of "simple waves" which are easy to obtain.

We consider a semi-infinite tube ($x \geqq 0$) which is initially undisturbed. Thus,

$$p(x, 0) = p_0, \qquad v(x, 0) = 0. \tag{3.3a}$$

For $t \geqq 0$ we prescribe the pressure at $x = 0$:

$$p(0, t) = g(t), \tag{3.3b}$$

and for small t we assume g to be expandible as

$$g(t) = p_0 + rt + O(t^2), \qquad r > 0. \tag{3.3c}$$

So we consider a continuous but non-smooth increase of the pressure at $x = 0$. This may describe the situation at the entrance of the aorta where the aortic valve remains closed as long as the pressure inside the left ventricle of the heart is below p_0 and it opens suddenly when the ventricle pressure rapidly exceeds p_0.

Since the equations are hyperbolic, continuous solutions with discontinuous first derivatives are allowed. Discontinuities in first derivatives propagate along characteristics. In our problem this will occur along the wavefront characteristic C_0^+ emanating from the origin in the x,t-plane and pointing into the region $x, t > 0$, see Fig. 1. The strength of the discontinuity may increase when prop-

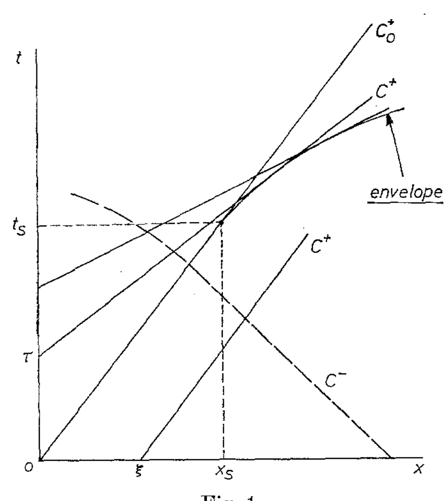

Fig. 1

agating along C_0^+ as time increases. If it becomes infinite at some point (x_s, t_s) on C_0^+, a shock has to be fitted in the solution for $x \geqq x_s$, $t \geqq t_s$ in an appropriate way in order to avoid a multivalued solution. Geometrically the point (x_s, t_s) is the point nearest to the origin on the envelope formed by the characteristics starting from points $(0, \tau)$ on the positive t-axis for small τ. We shall assume for simplicity that C_0^+ is the only characteristic on which a shock may develop. Therefore one might think of g as a smooth function with a single hump and an infinite tail (like, for example, $g(t) = p_0 + rte^{-t}$) so that the compressive part of the pressure disturbance is wholly restricted to the neighbourhood of the wave-front characteristic C_0^+. We are especially interested in the circumstances under which a shock wave may occur and in the point (x_s, t_s) on C_0^+ where it starts.

Taking a suitable linear combination of Eqs. (3.1) and (3.2) we obtain the two characteristic forms

$$p_t + (v \pm c)\, p_x \pm \varrho c[v_t + (v \pm c)\, v_x] = 0\,.$$

In other words, the two Riemann variables

$$v \pm \Phi(p) \overset{df}{=} v \pm \frac{1}{\varrho} \int \frac{dp}{c(p)} = \text{constant}$$

on the $C^{\pm}$-characteristics satisfying

$$\frac{dx}{dt} = v \pm c(p)\,.$$

Since $|v| \ll c$ for our problem, all C^--characteristics have negative slope. So all C^--characteristics in the region $x, t \geqq 0$ start from the positive x-axis where we have the undisturbed situation. Hence we have for all $x, t \geqq 0$:

$$v - \Phi(p) = -\Phi(p_0) = \text{constant}\,. \tag{3.4}$$

This means that we have a simple-wave solution in this case and the problem reduces to solving one first-order partial differential equation for p. This equation has the form of a kinematic wave equation (cf. Whitham [3]) and it is obtained by writing down the C^+-characteristic form and making use of relation (3.4):

$$p_t + [\Phi(p) - \Phi(p_0) + c(p)]\, p_x = 0\,, \qquad x, t \geqq 0\,. \tag{3.5}$$

This equation implies that $p =$ constant on the C^+-characteristics satisfying

$$\frac{dx}{dt} = \Phi(p) - \Phi(p_0) + c(p)\,.$$

So the C^+-characteristics are straight lines. In particular, C_0^+ is the line $x = c_0 t$.

The solution of Eq. (3.5) satisfying conditions (3.3) is given by

$$p = g(\tau)\,, \tag{3.6a}$$

$$x = [\Phi\{g(\tau)\} - \Phi(p_0) + c\{g(\tau)\}]\,(t - \tau)\,, \qquad \tau \geqq 0\,, \tag{3.6b}$$

in the region D_τ covered by the C^+-characteristics emanating from points $(0, \tau)$ on the positive t-axis, and by

$$p = p_0$$

in the region D_ξ covered by the C^+-characteristics $x - c_0 t = \xi$ emanating from points $(\xi, 0)$ on the positive x-axis.

For small τ the C^+-characteristics through $(0, \tau)$ are approximately given by

$$x = [c_0 + \gamma r c_0 \tau + O(\tau^2)]\,(t - \tau), \tag{3.7}$$

where the constant γ is defined as

$$\gamma = \frac{1}{c_0}\left\{\frac{1}{\varrho c_0} + c'(p_0)\right\}. \tag{3.8}$$

When $\gamma > 0$, the C^+-characteristics emanating from points $(0, \tau)$ on the positive t-axis will form an envelope located in the region $x \geqq c_0 t$. This means that the regions D_ξ and D_τ overlap, thus giving rise to a region of multivaluedness in D_ξ. The envelope is determined from Eq. (3.6b) and the equation obtained after differentiation of Eq. (3.6b) with respect to τ. As we are only interested in the location of the point (x_s, t_s) on C_0^+ where the envelope begins (notice that at (x_s, t_s) we have $\tau = 0$), it suffices to differentiate Eq. (3.7) with respect to τ. This yields

$$0 = \gamma r c_0 t - c_0 + O(\tau). \tag{3.9}$$

Taking $\tau = 0$ in Eqs. (3.7) and (3.9) we find

$$x_s = c_0 t_s = \frac{c_0^2}{r}\left\{\frac{1}{\varrho c_0} + c'(p_0)\right\}^{-1} = \frac{c_0}{\gamma r}, \tag{3.10}$$

which is in agreement with Rudinger [2].

There is some experimental evidence that $c'(p_0) \geqq 0$ for blood vessels ([1], [2]). Hence the occurrence of shock waves is to be expected in practice. Whether the predicted values (3.10) for x_s and t_s are physiologically realistic is open to doubt. A tapering of the tube and the inclusion of a friction term in the basic equations will probably influence the results greatly. This shall be analyzed in the next section.

Olsen and Shapiro [4] investigated both theoretically and experimentally flows through a uniform distensible tube made of a rubber-like material obeying the pressure-area relation

$$p = p_0 + \frac{1}{2}\varrho c_0^2\left(1 - \frac{A_0^2}{A^2}\right), \tag{3.11}$$

and found that steepening effects were absent. Now (3.11) implies

$$c(p) = c_0\left(1 - \frac{p - p_0}{\varrho c_0^2/2}\right)^{1/2},$$

and so $c'(p_0) = -(\varrho c_0)^{-1}$. Thus, $\gamma = 0$ for the material obeying relation (3.11) which confirms that no shock will form in this case.

4. A General Wavefront Expansion

In this section we shall consider the more general case of flow through a non-uniform elastic tube included the effect of friction. That is, we consider Eqs. (2.5) and (2.6) in the quarterplane $x, t \geqq 0$ under the same initial and boundary conditions (3.3) as for the uniform-tube case in the preceding section.

Globally we have the same situation: a jump in the first derivatives of the solution which propagates along the wavefront characteristic C_0^+ through the origin and which may become infinite at some point (x_s, t_s) on C_0^+, so that we have the formation of a shock at (x_s, t_s). In contrast with the simpler problem of Section 3, C_0^+ is not a straight line in general. The two characteristic forms of Eqs. (2.5) and (2.6) read

$$p_t + (v \pm c)\, p_x \pm \varrho c[v_t + (v \pm c)\, v_x] + \frac{A_x(p, x)}{A_p(p, x)}\, v \mp \varrho c f = 0, \qquad (4.1)$$

and hence the $C^\pm$-characteristics are to be determined from

$$\frac{dx}{dt} = v \pm c(p, x). \qquad (4.2)$$

The wavefront characteristic C_0^+ separates the undisturbed region in the x,t-plane from the disturbed one. Thus, along C_0^+ we have $v = 0$ and $p = p_0$. Introducing the representation $t = T(x)$ for C_0^+ we find from Eq. (4.2):

$$T(x) = \int_0^x \frac{d\xi}{c(p_0, \xi)}. \qquad (4.3)$$

Frequently we shall use the inverse representation $x = X(t)$ for C_0^+, so $X(T(x)) = x$.

An elementary exact solution like the simple-wave solution in Section 3 is not possible in the case of Eqs. (2.5) and (2.6). The additional terms in the characteristic forms (4.1) make it impossible to integrate out one of the Riemann variables. Hence we must resort to a different approach, namely the wavefront expansion (cf. Whitham [3]). This type of expansion yields an exact equation for the evolution in time of the jump in the first derivatives of p and v along the wavefront characteristic C_0^+.

The basic idea is as follows. If one introduces the variable τ by

$$\tau = t - T(x), \qquad (4.4)$$

the quarterplane $x, t \geqq 0$ is divided into an undisturbed region $\tau < 0$ (where $p = p_0$ and $v = 0$) and a disturbed region $\tau > 0$ separated by the wavefront characteristic C_0^+ upon which $\tau = 0$. In the immediate neighbourhood of C_0^+ we assume the following "wavefront expansions" for p and v:

$$\begin{aligned} \tau < 0:&\quad p = p_0, \quad v = 0, \\ \tau > 0:&\quad p = p_0 + \tau p_1(t) + \frac{1}{2}\tau^2 p_2(t) + \cdots, \quad p_1(0) = r > 0, \qquad (4.5) \\ &\quad v = \tau v_1(t) + \frac{1}{2}\tau^2 v_2(t) + \cdots. \end{aligned}$$

Note that p_1 and v_1 are a measure for the jump in the normal derivatives of p and v, respectively, along C_0^+ and that $p_1(0)$ is determined by the boundary condition (3.3b, c) at $x = 0$. Wavefront expansions like (4.5) provide a nice formal tool to calculate the evolution along characteristics of discontinuities in derivatives of solutions of hyperbolic equations.

Expansions (4.5) for $\tau > 0$ are inserted into the basic Eqs. (2.5) and (2.6) and, after expanding all terms in increasing powers of τ, the coefficients of successive powers of τ are equated to zero. Introducing the notations

$$c\{p_0, X(t)\} = c^{(0)}(t), \quad c_p\{p_0, X(t)\} = c_p^{(0)}(t), \quad c_x\{p_0, X(t)\} = c_x^{(0)}(t),$$

$$A_p\{p_0, X(t)\} = A_p^{(0)}(t), \quad A_x\{p_0, X(t)\} = A_x^{(0)}(t), \quad f_v[0, A\{p_0, X(t)\}] = f_v^{(0)}(t),$$

noting that differentiations transform according to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} \to -T'(x)\,\frac{\partial}{\partial \tau} = -\frac{1}{c(p_0, x)}\,\frac{\partial}{\partial \tau},$$

and keeping in mind that the argument x has to be expanded everywhere as

$$\begin{aligned} x = X(t - \tau) &= X(t) - \tau X'(t) + \cdots = X(t) - \tau c\{p_0, X(t)\} + \cdots \\ &= X(t) - \tau c^{(0)}(t) + \cdots, \end{aligned}$$

we find that the coefficients of τ^0 provide two equivalent equations relating p_1 and v_1:

$$p_1 - \varrho c^{(0)} v_1 = 0, \qquad v_1 - \frac{p_1}{\varrho c^{(0)}} = 0. \tag{4.6}$$

If the curve $x = X(t)$ had not been a characteristic, Eqs. (4.6) would have been independent and then the only possible solution would have been $p_1 = v_1 = 0$. This shows that jumps in first derivatives of the solution can only propagate along characteristics.

The coefficients of τ lead to the equations

$$p_1' + p_2 - \frac{p_1 v_1}{c^{(0)}} - \varrho c^{(0)} v_2 - 2\varrho c_p^{(0)} p_1 v_1 + \varrho c^{(0)} c_x^{(0)} v_1 + \frac{A_x^{(0)} v_1}{A_p^{(0)}} = 0, \tag{4.7a}$$

$$v_1' + v_2 - \frac{v_1^2}{c^{(0)}} - \frac{1}{\varrho c^{(0)}}\{c_x^{(0)} p_1 + p_2\} = v_1 f_v^{(0)}. \tag{4.7b}$$

When we consider Eqs. (4.7) as a set of two inhomogeneous linear algebraic equations for v_2 and p_2 we see that the coefficient determinant equals zero (as in Eqs. (4.6)), and hence v_2 and p_2 can be eliminated completely from Eqs. (4.7). Using (4.6) to eliminate further v_1, we obtain after some manipulation a single ordinary differential equation of Riccati type for p_1:

$$\frac{dp_1}{dt} + P(t)\, p_1^2 + Q(t)\, p_1 = 0, \tag{4.8}$$

where the coefficient functions P and Q are given by

$$P = -\frac{1}{c^{(0)}}\left\{\frac{1}{\varrho c^{(0)}} + c_p{}^{(0)}\right\}, \quad Q = \frac{1}{2}\left\{\frac{A_x{}^{(0)}}{\varrho c^{(0)} A_p{}^{(0)}} - c_x{}^{(0)} - f_v{}^{(0)}\right\}. \tag{4.9}$$

The initial condition for Eq. (4.8) is $p_1(0) = r$.

Eq. (4.8) can be written as a linear equation for $1/p_1$,

$$\frac{d}{dt}\left(\frac{1}{p_1}\right) - \frac{Q}{p_1} - P = 0,$$

and hence the solution of Eq. (4.8) satisfying the initial condition $p_1(0) = r$ is obtained in the form of the variation-of-constants formula

$$\frac{1}{p_1(t)} = \left[\frac{1}{r} + \int_0^t P(s)\left\{\exp \int_s^0 Q(s')\, ds'\right\} ds\right] \exp \int_0^t Q(s)\, ds. \tag{4.10}$$

In general a numerical evaluation of (4.10) will be needed to investigate whether p_1 becomes infinite for some value t_s of the time t. However, for a few special cases the solution of Eq. (4.8) can be evaluated easily and the conditions for shock formation together with the value of t_s can be obtained explicitly. This will be done in the next section.

5. Some Special Cases

5.1. Uniform Tube; Inviscid Flow

This is the problem of Section 3. We have $f = 0$ and $A_x(p,x) = 0$. So $c_x(p, x) = 0$ and $c^{(0)}(t) = c(p_0) = c_0 =$ constant. The wavefront characteristic $C_0{}^+$ is given by $x = X(t) = c_0 t$. The Riccati Eq. (4.8) for p_1 reads in this case:

$$\frac{dp_1}{dt} - \frac{1}{c_0}\left\{\frac{1}{\varrho c_0} + c'(p_0)\right\} p_1{}^2 = 0.$$

The solution of this equation which satisfies the initial condition $p_1(0) = r$ is given by

$$p_1(t) = \frac{r}{1 - \gamma r t}, \quad \gamma = \frac{1}{c_0}\left\{\frac{1}{\varrho c_0} + c'(p_0)\right\}. \tag{5.1}$$

We observe that p_1 becomes infinite in finite time if $\gamma > 0$, and the time at which this happens is $t_s = (\gamma r)^{-1}$. These results agree with those obtained in Section 3.

5.2. Exponentially Tapered Tube; Inviscid Flow

In this case we have $f = 0$ and $A(p, x) = e^{-\beta x}\tilde{A}(p)$ where β is a measure for the taper of the tube: $\beta > 0$ corresponds to a tube which becomes narrower as x increases, $\beta < 0$ to one which widens as x increases. The remarkable feature of this case is that the wave velocity c is independent of x:

$$c^2(p) = \frac{\tilde{A}(p)}{\varrho \tilde{A}'(p)}.$$

This means that the wavefront characteristic C_0^+ is again the straight line $x = X(t) = c_0 t$, $c_0 = c(p_0)$. Thus we have

$$\frac{A_x^{(0)}}{A_p^{(0)}} = -\beta \varrho c_0^2, \qquad c_x^{(0)} = 0.$$

The Riccati equation for p_1 becomes

$$\frac{dp_1}{dt} - \frac{1}{c_0}\left\{\frac{1}{\varrho c_0} + c'(p_0)\right\} p_1^2 - \frac{1}{2}\beta c_0 p_1 = 0$$

with solution given by

$$\frac{1}{p_1(t)} = \frac{1}{r} e^{-\frac{1}{2}\beta c_0 t} + \frac{2\gamma}{\beta c_0}\left(e^{-\frac{1}{2}\beta c_0 t} - 1\right).$$

If p_1 becomes infinite in finite time t_s then

$$t_s = \frac{x_s}{c_0} = \frac{2}{\beta c_0} \ln\left(1 + \frac{\beta c_0}{2\gamma r}\right). \tag{5.2}$$

From (5.2) it is seen that shock-wave formation occurs in finite time when

$$\gamma > 0, \qquad \frac{\beta c_0}{2\gamma r} > -1. \tag{5.3}$$

So shocks may develop even in a widening tube ($\beta < 0$) provided γ and r are sufficiently large positive. For small values of $|\beta c_0/(\gamma r)|$ the logarithm in (5.2) can be expanded in a series, and we find for t_s:

$$t_s = \frac{1}{\gamma r}\left[1 - \frac{\beta c_0}{4\gamma r} + O\left(\frac{\beta^2 c_0^2}{\gamma^2 r^2}\right)\right].$$

For $\beta \to 0$ we find again the results for the uniform case. We observe that a narrowing of the tube as x increases ($\beta > 0$) leads to a more rapid shock formation compared with the uniform case ($\beta = 0$).

5.3. Uniform Tube; Flow With Friction

The effect of friction can be incorporated in the model by a choice of the function $f(v, A)$ in the basic equations. We shall consider two possibilities:

laminar flow: $$f = -\frac{8\pi\mu v}{\varrho A}, \tag{5.4}$$

turbulent flow: $$f = -0.1360 \left(\frac{\mu}{\varrho}\right)^{1/4} \frac{|v|^{7/4}}{A^{5/8}} \operatorname{sgn} v, \tag{5.5}$$

where μ is the coefficient of viscosity of the fluid. The expressions (5.4) and (5.5) represent the well-known friction coefficients for steady flow in a circular pipe at low and high Reynolds number, respectively [1]. Of course one might object that we use steady-state results for a flow which is essentially non-steady. Nevertheless we believe that for the present problem relations (5.4) and (5.5) lead to a reasonable picture of the effect of friction on shock-wave formation.

In the case of turbulent flow we find

$$f_v^{(0)}(t) = f_v[0, A\{p_0, X(t)\}] = 0\,,$$

so within the framework of our theory turbulence has no influence on shock formation. For laminar flow, however, we have

$$f_v^{(0)}(t) = -\frac{8\pi\mu}{\varrho A_0},$$

and hence the Riccati equation for p_1 becomes

$$\frac{dp_1}{dt} - \frac{1}{c_0}\left\{\frac{1}{\varrho c_0} + c'(p_0)\right\} p_1^2 + \frac{4\pi\mu}{\varrho A_0}\, p_1 = 0\,.$$

We find for p_1:

$$\frac{1}{p_1(t)} = \frac{1}{r}\exp\left(\frac{4\pi\mu t}{\varrho A_0}\right) - \frac{\gamma\varrho A_0}{4\pi\mu}\left\{\exp\left(\frac{4\pi\mu t}{\varrho A_0}\right) - 1\right\}.$$

This implies that a shock will develop in finite time t_s,

$$t_s = \frac{x_s}{c_0} = -\frac{\varrho A_0}{4\pi\mu}\ln\left(1 - \frac{4\pi\mu}{\varrho A_0\gamma r}\right)$$

when

$$\frac{4\pi\mu}{\varrho A_0\gamma r} < 1\,, \qquad \gamma > 0\,.$$

For small values of $\mu/(\varrho A_0\gamma r)$ we have

$$t_s = \frac{1}{\gamma r}\left[1 + \frac{2\pi\mu}{\varrho A_0\gamma r} + O\left(\frac{\mu^2}{\varrho^2 A_0^2\gamma^2 r^2}\right)\right],$$

which shows that friction delays shock-wave formation.

References

[1] Anliker, M., Rockwell, R. L., Ogden, E.: Nonlinear analysis of flow pulses and shock waves in arteries, Pt. I & II. ZAMP **22**, 217–246, 563–581 (1971).

[2] Rudinger, G.: Shock waves in mathematical models of the aorta. ASME J. Appl. Mech. **37**, 34–37 (1970).

[3] Whitham, G. B.: Linear and nonlinear waves. Wiley-Interscience. 1974.

[4] Olsen, J. H., Shapiro, A. S.: Large-amplitude unsteady flow in liquid-filled elastic tubes. J. Fluid Mech. **29**, 513–538 (1967).

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