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The interaction between a penny-shaped crack and a spherical inhomogeneity in an infinite solid under uniaxial tension

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Summary. This paper presents an approximate three-dimensional analytical solution to the elastic stress field of a penny-shaped crack and a spherical inhomogeneity embedded in an infinite and isotropic matrix. The body is subjected to an uniaxial tension applied at infinity. The inhomogeneity is also isotropic but has different elastic moduli from the matrix. The interaction between the crack and the inhomogeneity is treated by the superposition principle of elasticity theory and Eshelby's equivalent inclusion method. The stress intensity factor at the boundary of the penny-shaped crack and the stress field inside the inhomogeneity are evaluated in the form of a series which involves the ratio of the radii of the spherical inhomogeneity and the crack to the distance between the centers of inhomogeneity and crack. Numerical calculations are carried out and show the variation of the stress intensity factor with the configuration and the elastic properties of the matrix and the inhomogeneity.

1 Introduction

With the increasing use of composite materials and the accompanying need to understand their fracture behavior in more detail, a great deal of efforts has been devoted to studying the interaction between the crack and the inhomogeneity in an infinite medium. Because of the mathematical difficulties most of the studies have dealt with two-dimensional problems. The interaction between a crack and a circular inclusion in a sheet under tension was studied by Tamate [1] and an exact solution was obtained. Hsu and Shivakumar [2] investigated the interaction between an elastic circular inclusion and two symmetrically placed collinear cracks. The problem of a crack located between two rigid inclusions was studied by Sendeckyj [3]. All these problems, however, were limited to two-dimensional and were treated by the complex variable method. A few papers concerned with the interaction of a crack and a cavity (a special kind of inclusion) in three-dimensional space have been published. Srivastava and Mahajan [4] determined the stress distribution in an infinite solid containing a spherical cavity and an external crack. Hiral and Satake [5] solved the problem involving a penny-shaped crack located between two spherical cavities in an infinite solid. These results were obtained mainly by numerical computation and the stress intensity factor at the penny-shaped crack boundary was shown graphically.

The objective of this paper is to determine the stress field and the stress intensity factor for a penny-shaped crack located near a spherical inhomogeneity in an infinite isotropic matrix with elastic moduli λ_0 and μ_0. The inhomogeneity is also isotropic with the radius a and elastic moduli λ_1, μ_1. The center of the spherical inhomogeneity is located in the same

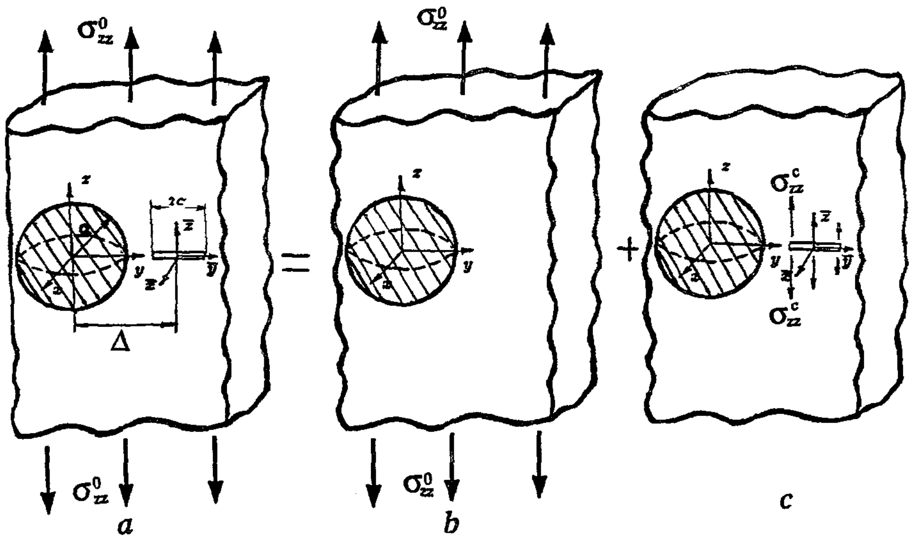

Fig. 1. The theoretical model

plane as the penny-shaped crack which has the radius c. The matrix is under uniaxial tension in the z-direction applied at infinity, i.e. σ^0_{zz} = constant at infinity (see Fig. 1a). Two coordinate systems (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$ are used, of which the origins are situated at the centers of the sphere and the penny-shaped crack, respectively. These two coordinate systems are related by

$$\bar{x} = x, \qquad \bar{y} = y - \Delta, \qquad \bar{z} = z,$$

where Δ is the distance between the centers of the inhomogeneity and the crack which is located in $z = 0$ plane. A dimensionless quantity δ is introduced to be equal to max $(a, c)/\Delta$. In this paper the stress field inside the spherical inhomogeneity is evaluated by dropping the terms with the fifth or higher order of δ. The stress intensity factor at the boundary of the crack is obtained by neglecting the terms with the sixth or higher order of δ. As a result, the stress inside the spherical inhomogeneity is a linear function of the coordinates. Moschovidis and Mura [6] showed a numerical result on the problem of two ellipsoidal inhomogeneities in an infinite solid. They determined that the interaction between the inhomogeneities becomes negligible if a/Δ is less than 1/4. Numerical examples of our solutions indicate a similar result on the interaction.

2 Superposition process

Since the problem is symmetric about $x-y$ plane, the shear stresses on the plane $z = 0$ vanish. The superposition principle of the elasticity theory is adopted to determine the stress distribution of the problem. When the crack is removed, the stress distribution at the crack domain in the z-direction is supposed to be σ^c_{zz}. The original problem shown in Fig. 1a is decomposed into the sum of Fig. 1b and Fig. 1c. We are to find the stress σ^c_{zz} which is applied on the top and the bottom surfaces of the crack in Fig. 1c.

The stress field σ^{in}_{ij} inside the inhomogeneity in Fig. 1b is determined by Eshelby [7] as

$$\sigma^{in}_{xx} = \sigma^{in}_{yy} = \frac{1}{3}\left[\frac{\varkappa_1}{\alpha_0(\varkappa_1 - \varkappa_0) + \varkappa_0} - \frac{\mu_1}{\beta_0(\mu_1 - \mu_0) + \mu_0}\right]\sigma^0_{zz},$$

$$\sigma_{zz}^{in} = \left[\frac{2}{3}\frac{\mu_1}{\beta_0(\mu_1 - \mu_0) + \mu_0} + \frac{1}{3}\frac{\varkappa_1}{\alpha_0(\varkappa_1 - \varkappa_0) + \varkappa_0}\right]\sigma_{zz}^0,$$

where $\varkappa_1$, μ_1 and $\varkappa_0$, μ_0 are the bulk and shear moduli of the inhomogeneity and the matrix, respectively, and

$$\alpha_0 = \frac{1}{3}\frac{1 + \nu_0}{1 - \nu_0}, \qquad \beta_0 = \frac{2}{15}\frac{4 - 5\nu_0}{1 - \nu_0},$$

in which ν_0 is Poisson's ratio of the matrix.

The stress distribution outside the inhomogeneity in Fig. 1b is rather complex. An alternative method proposed by Tanaka and Mura [8] to evaluate the elastic stress distribution for points exterior to the inhomogeneity is adopted. The process of solving the problem is illustrated in Fig. 2. The figure shows that the problem of a spherical inhomogeneity in a matrix under σ_{ij} applied at infinity is the sum of the problem of a spherical void in the matrix under $\sigma_{ij} - \sigma_{ij}^{in}$ and the problem of an infinite body under σ_{ij}^{in}. The stress distribution in Fig. 2b due to $\sigma_{ij} - \sigma_{ij}^{in}$ applied at infinity is given by

$$\sigma_{zz} = \sigma_{zz}^0 - \sigma_{zz}^{in}, \qquad \sigma_{xx} = \sigma_{yy} = -\sigma_{xx}^{in}.$$

We decompose this stress distribution into a uniaxial tension in z-direction as $\sigma_{zz}^1 = \sigma_{zz}^0 - \sigma_{zz}^{in} + \sigma_{xx}^{in}$ and a hydrostatic pressure $\sigma_{zz}^2 = \sigma_{xx}^2 = \sigma_{yy}^2 = -\sigma_{xx}^{in}$. By using Timoshenko's elastic solution [9] of the problem of a spherical void in an infinite body under uniaxial tension and by adding the stress field of the hydrostatic pressure to the elastic solution, we obtain the stress σ_{zz} on the $z = 0$ plane for Fig. 2b as

$$\sigma_{zz}(z = 0) = \frac{(\beta_0 - 1)(\mu_1 - \mu_0)}{\beta_0(\mu_1 - \mu_0) + \mu_0}\left[1 + \frac{4 - 5\nu_0}{2(7 - 5\nu_0)}\left(\frac{a}{r}\right)^3 + \frac{9}{2(7 - 5\nu_0)}\left(\frac{a}{r}\right)^5\right]\sigma_{zz}^0$$
$$+ \frac{1}{3}\left[\frac{\mu_1}{\beta_0(\mu_1 - \mu_0) + \mu_0} - \frac{\varkappa_1}{\alpha_0(\varkappa_1 - \varkappa_0) + \varkappa_0}\right]\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]\sigma_{zz}^0,$$

where $r = \sqrt{x^2 + y^2}$.

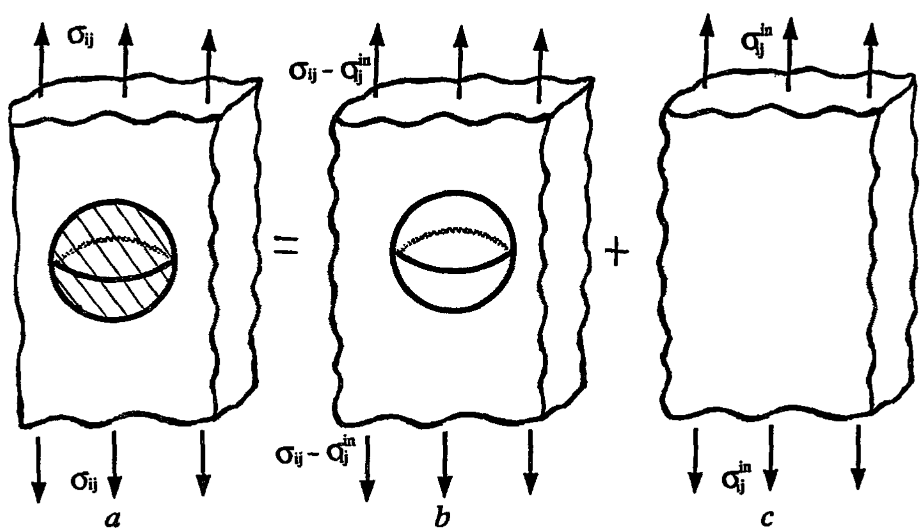

Fig. 2. Tanaka-Mura's alternative method

By combining Fig. 2b and Fig. 2c, σ_{zz} at $z = 0$ in Fig. 2a or Fig. 1b outside the inhomogeneity is obtained as

$$\sigma_{zz}^c(z=0) = \sigma_{zz}^0 \left\{1 + \left[\frac{1}{6}\left(\frac{\mu_1}{\beta_0(\mu_1-\mu_0)+\mu_0} - \frac{\varkappa_1}{\alpha_0(\varkappa_1-\varkappa_0)+\varkappa_0}\right) + \frac{4-5\nu_0}{2(7-5\nu_0)}\frac{(\beta_0-1)(\mu_1-\mu_0)}{\beta_0(\mu_1-\mu_0)+\mu_0}\right]\left(\frac{a}{r}\right)^3 + \frac{(\beta_0-1)(\mu_1-\mu_0)}{\beta_0(\mu_1-\mu_0)+\mu_0}\left(\frac{a}{r}\right)^5\right\}.$$

This is the stress σ_{zz}^c applied to the top and the bottom surfaces of the crack in Fig. 1c. We now change the coordinate system from (x, y, z) to $(\bar{x}, \bar{y}, \bar{z})$, so that $r = \sqrt{\bar{x}^2 + (\bar{y}+\Delta)^2}$. When the coordinates of a point $(\bar{x}, \bar{y})$ are on the surface of the crack, i.e., $\bar{x}^2 + \bar{y}^2 = \bar{r}^2 \leqq c^2$, we may express

$$\left(\frac{a}{r}\right)^3 = \left(\frac{a}{\Delta}\right)^3\left(1 + \frac{\bar{r}^2}{\Delta^2} + 2\frac{\bar{y}}{\Delta}\right)^{-3/2} = \left(\frac{a}{\Delta}\right)^3\left(1 - 3\frac{\bar{y}}{\Delta} - \frac{3}{2}\frac{\bar{r}^2}{\Delta^2} + \frac{15}{2}\frac{\bar{y}^2}{\Delta^2}\right) + O(\delta^6),$$

$$\left(\frac{a}{r}\right)^5 = \left(\frac{a}{\Delta}\right)^5 + O(\delta^6).$$

So the stress σ_{zz}^c on the crack surface is rewritten as

$$\sigma_{zz}^c(z=0) = \sigma_{zz}^0\left[1 + A\left(\frac{a}{\Delta}\right)^3\left(1 - 3\frac{\bar{y}}{\Delta} - \frac{3}{2}\frac{\bar{r}^2}{\Delta^2} + \frac{15}{2}\frac{\bar{y}^2}{\Delta^2}\right) + B\left(\frac{a}{\Delta}\right)^5\right] + O(\delta)^6, \quad (1)$$

where

$$A = \frac{1}{6}\left(\frac{\mu_1}{\beta_0(\mu_1-\mu_0)+\mu_0} - \frac{\varkappa_1}{\alpha_0(\varkappa_1-\varkappa_0)+\varkappa_0}\right) + \frac{4-5\nu_0}{2(7-5\nu_0)}\frac{(\beta_0-1)(\mu_1-\mu_0)}{\beta_0(\mu_1-\mu_0)+\mu_0}, \quad (2)$$

$$B = \frac{(\beta_0-1)(\mu_1-\mu_0)}{\beta_0(\mu_1-\mu_0)+\mu_0}. \quad (3)$$

A, B are constants which are related to the elastic moduli of the matrix and the inhomogeneity.

3 The equivalent inclusion method

We now determine the solution of Fig. 1c by adopting Eshelby's equivalent inclusion method [7]. The crack and the inhomogeneity are treated to have the same elastic moduli as the matrix by introducing so called eigenstrains [10] ε_{ij}^{*c} and ε_{ij}^{*s} inside the crack and the domain of the spherical inhomogeneity, respectively. The eigenstrains for the sphere and the crack are to satisfy the following equations:

$$c_{ijkl}^1(u_{k,l}^0 + u_{k,l}^{pt}) = c_{ijkl}^0(u_{k,l}^0 + u_{k,l}^{pt} - \varepsilon_{kl}^{*s}),$$

$$\text{or} \quad c_{ijkl}^1(\varepsilon_{kl}^0 + \varepsilon_{kl}^{pt}) = c_{ijkl}^0(\varepsilon_{k,l}^0 + \varepsilon_{kl}^{pt} - \varepsilon_{kl}^{*s}) \quad \text{for sphere}, \quad (4)$$

$$0 = c_{ijkl}^0(\bar{u}_{k,l}^0 + \bar{u}_{k,l}^{pt} - \varepsilon_{kl}^{*c}),$$

$$\text{or} \quad 0 = c_{ijkl}^0(\bar{\varepsilon}_{kl}^0 + \bar{\varepsilon}_{kl}^{pt} - \varepsilon_{kl}^{*c}) \quad \text{for crack}, \quad (5)$$

where c_{ijkl}^1 and c_{ijkl}^0 are the elastic moduli of the inhomogeneity and the matrix, respectively. $u_k{}^0$ is the displacement inside the spherical inclusion domain when the inhomogeneity is replaced by the same material as the matrix while the crack still exists. $\bar{u}_k{}^0$ is the displacement inside the crack site when the crack is filled by the same material as the matrix while the inhomogeneity still exists. The real displacements inside the spherical inclusion and the crack are $u_k{}^0 + u_k{}^{pt}$ and $\bar{u}_k{}^0 + \bar{u}_k{}^{pt}$, respectively. Because of the interaction between the crack and the sphere, $u^0_{k,l}$ and $\bar{u}^0_{k,l}$ are no longer uniform. Consequently, ε^{*c} and ε^{*s} are not uniform either. They are a function of the position vector.

Suppose the eigenstrain ε^{*c}_{ij} inside the crack is determined, when the spherical inhomogeneity is not present, the displacement field caused by ε^{*c}_{ij} in the matrix is given by a surface integral:

$$u_i{}^0(\bar{\boldsymbol{r}}) = \oint_{S_c} G_{ij}(\bar{\boldsymbol{r}} - \bar{\boldsymbol{r}}')\, c^0_{jklm}\varepsilon^{*c}_{lm} n_k \, ds' , \tag{6}$$

where $G_{ij}(\bar{\boldsymbol{r}} - \bar{\boldsymbol{r}}')$ is Green's tensor function and S_c is the surface of the crack.

Green's function in an infinite body may be written for our problem as

$$G_{ij}(\overline{\boldsymbol{R}}) = \frac{A_1}{\overline{R}}\,\delta_{ij} + \frac{A_2}{\overline{R}^3}\,\overline{R}_i\overline{R}_j,$$

where $\overline{\boldsymbol{R}} = \bar{\boldsymbol{r}} - \bar{\boldsymbol{r}}'$, $A_1 = \dfrac{3 - 4\nu_0}{16\pi\mu_0(1 - \nu_0)}$, and $A_2 = \dfrac{1}{16\pi\mu_0(1 - \nu_0)}$. We let the domain for the penny-shaped crack be

$$\frac{\bar{x}'^2}{c^2} + \frac{\bar{y}'^2}{c^2} + \frac{\bar{z}'^2}{a_3{}^2} \leqq 1 \quad \text{and} \quad a_3 \to 0 .$$

Equation (6) may thus be written as

$$\begin{aligned} u_i{}^0(\bar{\boldsymbol{r}}) = \lim_{a_3\to 0} \iint_\Omega \Bigg\{ \Bigg[& \frac{1}{\sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2 + \left(a_3\sqrt{1 - (\bar{x}'^2 + \bar{y}'^2)/c^2} - \bar{z}\right)^2}} \\ & - \frac{1}{\sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2 + \left(a_3\sqrt{1 - (\bar{x}'^2 + \bar{y}'^2)/c^2} + \bar{z}\right)^2}} \Bigg] A_1\delta ij \\ & + \Bigg[\left(\frac{1}{\sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2 + \left(a_3\sqrt{1 - (\bar{x}'^2 + \bar{y}'^2)/c^2} - \bar{z}\right)^2}} \right)^3 \\ & - \left(\frac{1}{\sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2 + \left(a_3\sqrt{1 - (\bar{x}'^2 + \bar{y}'^2)/c^2} + \bar{z}\right)^2}} \right)^3 \Bigg] A_2\overline{R}_i\overline{R}_j \Bigg\} c^0_{j3lm}\varepsilon^{*c}_{lm}\, d\bar{x}'\, d\bar{y}' \\ = \lim_{a_3\to 0} \iint_\Omega z \Bigg\{ & \frac{2A_1\delta_{ij}\sqrt{1 - (\bar{x}'^2 + \bar{y}'^2)/c^2}}{[(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2 + \bar{z}^2]^{3/2}} \\ & + \frac{6\sqrt{1 - (\bar{x}'^2 + \bar{y}'^2)/c^2}}{[(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2 + z^2]^{5/2}}\, A_2\overline{R}_i\overline{R}_j \Bigg\}\, a_3 c^0_{j3lm}\varepsilon^{*c}_{lm}\, d\bar{x}'\, d\bar{y}' , \end{aligned} \tag{7}$$

where Ω is the domain: $\bar{x}'^2 + \bar{y}'^2 \leqq c^2$.

The displacement $u_i{}^0(\bar{\boldsymbol{r}})$ in Eq. (7) is under coordinate system $(\bar{x}, \bar{y}, \bar{z})$. When $\bar{\boldsymbol{r}}$ is inside the spherical domain, we change the coordinate system to (x, y, z) as

$$\bar{x} = x, \qquad \bar{y} = y - \Delta, \qquad \bar{z} = z, \quad \text{and} \quad x^2 + y^2 + z^2 \leqq a^2 .$$

Note that the integration is still on the original coordinate system and $\bar{x}'^2 + \bar{y}'^2 \leqq c^2$, so we have

$$\begin{aligned} \frac{1}{[(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2 + \bar{z}^2]^{3/2}} &= \frac{1}{\Delta^3}\left(1 - 2\,\frac{y}{\Delta} + 2\,\frac{\bar{y}'}{\Delta} + \frac{\bar{r}'^2 + r^2}{\Delta^2} - 2\,\frac{x\bar{x}' + y\bar{y}'}{\Delta^2}\right)^{-3/2} \\ &= \frac{1}{\Delta^3}\left(1 + 3\,\frac{y}{\Delta} - 3\,\frac{\bar{y}'}{\Delta}\right) + O(\delta^5), \end{aligned} \tag{8}$$

$$\frac{\bar{R}_i\bar{R}_j}{[(\bar{x}'-\bar{x})^2+(\bar{y}'-\bar{y})^2+\bar{z}^2]^{3/2}}=\frac{R_iR_j}{\Delta^5}+O(\delta^5), \qquad R_i=r_i-\bar{r}_i'-\Delta\delta_{i2}. \tag{9}$$

If i or j is equal to 2, $\frac{R_iR_j}{\Delta^5}$ in Eq. (9) has a term with the order of δ^4. Substituting Eq. (8) and (9) into Eq. (7), we obtain

$$u_i^0(\mathbf{r})=\lim_{a_3\to 0}\iint_\Omega\left[2A_1\delta_{ij}\left(1+3\frac{y}{\Delta}-3\frac{\bar{y}'}{\Delta}\right)+6A_2\frac{R_iR_j}{\Delta^2}\right]\frac{z}{\Delta^3}\sqrt{1-\frac{\bar{x}'^2+\bar{y}'^2}{c^2}}$$
$$\times a_3c^0_{j3lm}\varepsilon^{*c}_{lm}\,d\bar{x}'\,d\bar{y}'+O(\delta^5). \tag{10}$$

Here $\mathbf{r}$ is inside the spherical domain. All the terms with the order δ^5 or higher are dropped in the above equation. Since there is a term $\frac{Z}{\Delta^3}$ appearing in the integration of Eq. (10) which has the order of δ^3, we only need to maintain the zeroth and first order terms of δ in ε^{*c}_{lm} to get $u_i^0(\mathbf{r})$. ε^{*c}_{lm} is determined by σ^c_{zz} and the effect caused by ε^{*s}_{lm}. Later we will show that the effect of ε^{*s}_{lm} on ε^{*c}_{lm} has the order of δ^6 and thus neglected in determining $u_i^0(\mathbf{r})$. From Eq. (1), $\sigma^c_{zz}=\sigma^0_{zz}+O(\delta^3)$, which means that only σ^0_{zz} can offer ε^{*c}_{lm} with the zeroth order tem of δ.

According to Mura [10], the eigenstrain caused by a constant stress σ^0_{zz} exerted on the top and the bottom surfaces of a penny-shaped crack is given by

$$\lim_{a_3\to 0}a_3\varepsilon^{*c}_{33}=\text{constant}=\varepsilon^*=\frac{2(1-\nu_0)\,c}{\mu_0\pi}\,\sigma^0_{zz}\quad\text{and other}\quad\varepsilon^{*c}_{ij}=0.$$

Substituting this result into Eq. (10), we obtain

$$u_i^0(\mathbf{r})=\iint_\Omega\left[2A_1\delta_{ij}\left(1+3\frac{y}{\Delta}-3\frac{y'}{\Delta}\right)+6A_2\frac{R_iR_j}{\Delta^2}\right]\frac{z}{\Delta^3}\sqrt{1-\frac{\bar{x}'^2+\bar{y}'^2}{c^2}}$$
$$\times(\lambda_0+2\mu_0)\,\varepsilon^*\delta_{j3}\,d\bar{x}'\,d\bar{y}'+O(\delta^5)$$
$$=\frac{1-\nu_0}{\pi\mu_0(1-2\nu_0)}\left(\frac{c}{\Delta}\right)^3\sigma^0_{zz}\left[\left(1-\frac{4}{3}\nu_0\right)\left(1+3\frac{y}{\Delta}\right)z\delta_{i3}-\frac{z^2}{\Delta}\delta_{i2}\right]+O(\delta^5). \tag{11}$$

From the above equation, we can easily find that $u^0_{k,l}(\mathbf{r})$ is a linear function of the coordinates. Sendeckyi [11] and Moschovidis [12] showed that ε^{*s}_{kl} determined by Eq. (4) is also a linear function of the coordinates as

$$\varepsilon^{*s}_{kl}=B_{kl}+B_{klm}x_m, \tag{12}$$

where B_{kl} and B_{klm} are constant tensors. $u^{pt}_{i,k}$ in Eq. (4) is related to ε^{*s}_{kl} by

$$u^{pt}_{i,k}=-\frac{1}{35(1-\nu_0)}\left[(B_{lli}+2B_{ill})\,x_k+(B_{jll}+B_{llj})\,x_j\delta_{ik}+(B_{kll}+B_{llk})\,x_i\right.$$
$$\left.+(B_{jik}+2B_{jki}+B_{kij})\,x_j\right]+\frac{\nu_0}{5(1-\nu_0)}\left(B_{mmi}x_k+B_{mmk}x_i+B_{mml}x_l\delta_{ik}\right)$$
$$+\frac{2}{5}\left(B_{ill}x_k+B_{ilk}x_l+B_{ikl}x_l\right)+S_{ijkl}B_{jl}, \tag{13}$$

in which S_{ijkl} is Eshelby's tensor for a sphere. After substituting Eqs. (11), (12) and (13) into Eq. (4), we compare the constant terms on both sides of Eq. (4) and find that

$$B_{11} = B_{22} = \frac{1}{3}\left[\frac{\mu_1 - \mu_0}{(\mu_1 - \mu_0)\,\beta_0 + \mu_0} - \frac{\varkappa_1 - \varkappa_0}{(\varkappa_1 - \varkappa_0)\,\alpha_0 + \varkappa_0}\right] A_3 \left(\frac{c}{\Delta}\right)^3 \sigma_{zz}^0 + O(\delta^5), \tag{14}$$

$$B_{33} = -\left[\frac{2}{3}\,\frac{\mu_1 - \mu_0}{(\mu_1 - \mu_0)\,\beta_0 + \mu_0} + \frac{1}{3}\,\frac{\varkappa_1 - \varkappa_0}{(\varkappa_1 - \varkappa_0)\,\alpha_0 + \varkappa_0}\right] A_3 \left(\frac{c}{\Delta}\right)^3 \sigma_{zz}^0 + O(\delta^5). \tag{15}$$

Here $A_3 = \dfrac{(1 - \nu_0)\,(3 - 4\nu_0)}{3\pi\mu_0(1 - 2\nu_0)}$, and other $B_{ij} = 0$. Comparing the coefficients of the linear terms on both sides of Eq. (2), after the same substitution, we can determine B_{kli}. It is found that only B_{112}, B_{211}, B_{222}, B_{233} and B_{332} are not zero. The general expressions for the coefficients B_{kli} are too long to be included in this paper. If we are to take $\nu_1 = \nu_0 = 1/3$, and let $\eta = \mu_1/\mu_0$, we can get a simplified expression as

$$\begin{bmatrix} 1 + \frac{57}{70}(\eta - 1) & \frac{9}{14}(\eta - 1) & 2 + \frac{73}{35}(\eta - 1) & \frac{9}{14} & 1 + \frac{57}{70}(\eta - 1) \\ 2 + \frac{37}{35}(\eta - 1) & \frac{9}{14}(\eta - 1) & 1 + \frac{15}{14}(\eta - 1) & \frac{13}{35}(\eta - 1) & 1 + \frac{7}{10}(\eta - 1) \\ 1 + \frac{7}{10}(\eta - 1) & \frac{13}{35}(\eta - 1) & 1 + \frac{15}{14}(\eta - 1) & \frac{9}{14}(\eta - 1) & 2 + \frac{37}{35}(\eta - 1) \\ \frac{27}{70}(\eta - 1) & 2 + \frac{44}{35}(\eta - 1) & \frac{27}{70}(\eta - 1) & 0 & 0 \\ 0 & 0 & \frac{27}{70}(\eta - 1) & 2 + \frac{27}{70}(\eta - 1) & \frac{27}{70}(\eta - 1) \end{bmatrix}$$

$$\times \begin{bmatrix} B_{112} \\ B_{211} \\ B_{222} \\ B_{233} \\ B_{332} \end{bmatrix} = A_4 \sigma_{zz}^0 \begin{bmatrix} 5 \\ 5 \\ 10 \\ 0 \\ -1 \end{bmatrix}$$

in which

$$A_4 = \frac{2(1 - \eta)}{3\pi\mu_0\Delta}\left(\frac{c}{\Delta}\right)^3.$$

This result will be used when we calculate the numerical examples. Once $u_{i,k}^{pt}$ is determined through Eq. (13) after B_{ij} and B_{ijk} are obtained, the real stress field inside the domain of the spherical inhomogeneity can be calculated easily by

$$\sigma_{ij}^s = c_{ijkl}^1(\varepsilon_{kl}^0 + \varepsilon_{kl}^{pt}) = c_{ijkl}^1(u_{i,j}^0 + u_{i,j}^{pt}),$$

in which $u_i{}^0$ is given in Eq. (11). We will discuss in detail the stress field inside the inhomogeneity in Section 5.

We now consider the effect of the inhomogeneity on the crack since we have gotten ε_{kl}^{*s} as in Eq. (12). By the same equivalent inclusion method we used earlier, first we determine the strain $\bar\varepsilon_{ij}^0$ in the crack domain when the crack is filled by the same material as the matrix. The strain $\bar\varepsilon_{ij}^0$ may be divided into two component parts as

$$\bar\varepsilon_{ij}^0 = \bar\varepsilon_{ij}^{01} + \bar\varepsilon_{ij}^{02},$$

where $\bar{\varepsilon}_{ij}^{01}$ is due to the stress σ_{zz}^c exerted on the crack surface and $\bar{\varepsilon}_{ij}^{02}$ is caused by the inhomogeneity. The two component strains are given by

$$\bar{\varepsilon}_{33}^{01} = \frac{\sigma_{zz}^c}{E_0}, \qquad \bar{\varepsilon}_{11}^{01} = \bar{\varepsilon}_{22}^{01} = -\frac{\nu_0}{E_0}\,\sigma_{zz}^c, \qquad \bar{\varepsilon}_{ij}^{02} = \frac{1}{2}\left[\bar{u}_{i,j}^{02} + \bar{u}_{j,i}^{01}\right],$$

where

$$\bar{u}_i^{02}(\boldsymbol{r}) = \oint_{S_s'} G_{ij}(\boldsymbol{r} - \boldsymbol{r}')\, c_{jklm}^0 \varepsilon_{lm}^{*s} n_k\, ds'. \tag{16}$$

S_s' is the surface of the spherical inclusion: $x'^2 + y'^2 + z'^2 = a^2$, and $G_{ij}(\boldsymbol{r} - \boldsymbol{r}')$ is Green's tensor function. From the expressions of B_{ij} and B_{ijk} we know that ε_{lm}^{*s} has the order of δ^3. For the solution of the crack we keep the accuracy to the order of δ^5. So we need to keep the terms containing the second order of δ in $G_{ij}(\boldsymbol{r} - \boldsymbol{r}')$ to obtain $\bar{\varepsilon}_{ij}^{02}$ through Eq. (16). When $\boldsymbol{r}$ is inside the crack domain, we change the coordinates from (x, y, z) to $(\bar{x}, \bar{y}, \bar{z})$ as

$$x = \bar{x}, \qquad y = \bar{y} + \Delta, \qquad z = \bar{z}, \quad \text{and} \quad \frac{\bar{x}^2 + \bar{y}^2}{c^2} + \frac{\bar{z}^2}{a_3{}^2} \leqq 1 \quad \text{while} \quad a_3 \to 0.$$

So we have

$$\frac{1}{R} = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{1}{\Delta}\left(1 + \frac{\bar{y}}{\Delta} - 2\,\frac{y'}{\Delta}\right) + O(\delta^3),$$

$$\frac{R_iR_j}{R^3} = \frac{R_iR_j}{\Delta} + O(\delta^3) = \frac{1}{\Delta^2}\left[\delta_{i2}(\bar{r}_j - r_j') + \delta_{j2}(\bar{r}_i - r_i')\right] + O(\delta^3),$$

and Green's tensor function is changed into

$$\begin{aligned} G_{ij}(\boldsymbol{r} - \boldsymbol{r}') &= \frac{A_1}{R}\,\delta_{ij} + \frac{A_2}{R^3}\,R_iR_j \\ &= \frac{A_1}{\Delta}\left(1 + \frac{\bar{y}}{\Delta} - 2\,\frac{\bar{y}'}{\Delta}\right)\delta_{ij} + \frac{A_2}{\Delta^2}\left[\delta_{i2}(\bar{r}_j - r'_j) + \delta_{j2}(\bar{r}_i - r_i')\right] + O(\delta^3). \end{aligned}$$

Substituting this expression and Eq. (12) into Eq. (16), we have

$$\bar{u}_i^{02} = \oint_{S_s'} \left\{\frac{A_1\delta_{ip}}{\Delta}\left(1 + \frac{\bar{y}}{\Delta} - 2\,\frac{y'}{\Delta}\right) + \frac{A_2}{\Delta^2}\left[\delta_{i2}(\bar{r}_j - r_j') + \delta_{j2}(\bar{r}_i - r_i')\right]\right\} c_{pklm}^0 B_{lm} n_k\, ds' + O(\delta^6),$$

now we can easily get

$$\begin{aligned} \bar{u}_{i,j}^{02} &= \oint_{S_s'} \left[\frac{A_1}{\Delta^2}\,\delta_{j2}\delta_{ip} + \frac{A_2}{\Delta^2}\,(\delta_{i2}\delta_{pj} + \delta_{p2}\delta_{ij})\right] c_{jklm}^0 B_{lm} n_k\, ds' + O(\delta^6) \\ &= (A_1\delta_{j2}\delta_{ip} + A_2\delta_{i2}\delta_{pj} + A_2\delta_{p2}\delta_{ij})\, C_{jklm}^0 B_{lm}\,\frac{1}{\Delta^2}\oint_{S_s'} n_k\, ds' + O(\delta^6) = O(\delta^6). \end{aligned} \tag{17}$$

The above result shows that after the superposition process, the effect of the inhomogeneity on the crack in Fig. 1c has the order of $O(\delta^6)$ and can be neglected. This conclusion is used in Section 4 to determine the stress intensity factor on the boundary of the crack.

4 The stress intensity factor of the crack

As we discussed above, the effect of the spherical inhomogeneity on the crack in Fig. 1c can be neglected. Therefore the stress intensity factor on the crack boundary is determined by σ_{zz}^c which is given in Eq. (1). The stress σ_{zz}^c is a quadratic function of the coordinates and we divide it into three parts,

$$\sigma_{zz}^c = \sigma_{zz}^{c0} - \sigma_{zz}^{c1} + \sigma_{zz}^{c2},$$

where σ_{zz}^{c0} is the symmetric part of σ_{zz}^c about $\bar{z}$-axis:

$$\sigma_{zz}^{c0} = \sigma_{zz}^0 \left[1 + A \left(\frac{a}{\Delta}\right)^3 \left(1 - \frac{3}{2} \frac{\bar{r}^2}{\Delta^2}\right) + B \left(\frac{a}{\Delta}\right)^5\right].$$

The stress intensity factor $K_I{}^{c0}$ caused by σ_{zz}^{c0} is

$$K_I{}^{c0} = 2 \sqrt{\frac{c}{\pi}}\, \sigma_{zz}^0 \left[1 + A \left(\frac{a}{\Delta}\right)^3 \left(1 - \frac{c^2}{\Delta^2}\right) + B \left(\frac{a}{\Delta}\right)^5\right] \tag{18}$$

σ_{zz}^{c1} is the linear part of σ_{zz}^c and expressed by

$$\sigma_{zz}^{c1} = 3A\sigma_{zz}^0 \left(\frac{a}{\Delta}\right)^3 \frac{\bar{y}}{\Delta}.$$

Following Mura [10], the stress intensity factor $K_I{}^{c1}$ caused by σ_{zz}^{c1} is

$$K_I{}^{c1} = 2 \sqrt{\frac{c}{\pi}}\, A\sigma_{zz}^0 \left(\frac{a}{\Delta}\right)^3 \frac{\bar{y}}{\Delta}. \tag{19}$$

The stress σ_{zz}^{c2} is the quadratic part of σ_{zz}^c and is given by

$$\sigma_{zz}^{c2} = \frac{15}{2} A \left(\frac{a}{\Delta}\right)^3 \sigma_{zz}^0 \left(\frac{\bar{y}}{\Delta}\right)^2.$$

We change the expression of σ_{zz}^{c2} into

$$\sigma_{zz}^{c2} = \frac{15}{4} A \left(\frac{a}{\Delta}\right)^3 \sigma_{zz}^0 \frac{1}{\Delta^2} [(\bar{x}^2 + \bar{y}^2) - (\bar{x}^2 - \bar{y}^2)]$$
$$= \frac{15}{4} A \left(\frac{a}{\Delta}\right)^3 \sigma_{zz}^0 \left(\frac{\bar{r}}{\Delta}\right)^2 - \frac{15}{4} A \left(\frac{a}{\Delta}\right)^3 \sigma_{zz}^0 \frac{1}{\Delta^2} (\bar{x}^2 - \bar{y}^2).$$

According to Kassir and Sih [13], the second term in σ_{zz}^{c2} gives rise to the stress intensity factor

$$K_I{}' = -\frac{24}{5} A \sqrt{\frac{c}{\pi}} \left(\frac{a}{\Delta}\right)^3 \sigma_{zz}^0 \frac{1}{\Delta^2} (\bar{x}^2 - \bar{y}^2).$$

The first term in σ_{zz}^{c2} is symmetric about the crack and gives rise to the stress intensity factor

$$K_I{}'' = 5A \sqrt{\frac{c}{\pi}} \left(\frac{a}{\Delta}\right)^3 \left(\frac{c}{\Delta}\right)^2 \sigma_{zz}^0.$$

So the stress intensity factor caused by σ_{zz}^{c2} is

$$K_I{}^{c2} = K_I{}' + K_I{}'' = A \sqrt{\frac{c}{\Delta}} \left(\frac{a}{\Delta}\right)^3 \sigma_{zz}^0 \left[5 \left(\frac{c}{\Delta}\right)^2 - \frac{24}{5} \frac{1}{\Delta^2} (\bar{x}^2 - \bar{y}^2)\right].$$

But, since $\bar{x}^2 + \bar{y}^2 = c^2$, we may rewrite the above equation as

$$K_I{}^{c2} = \frac{1}{5} A \sqrt{\frac{c}{\pi}} \left(\frac{a}{\Delta}\right)^3 \sigma_{zz}^0 \left[\left(\frac{c}{\Delta}\right)^2 + 48 \left(\frac{\bar{y}}{\Delta}\right)^2\right]. \tag{20}$$

The stress intensity factor K_I caused by σ_{zz}^c is given by

$$K_I = K_I^{c0} - K_I^{c1} + K_I^{c2}. \tag{21}$$

Substituting Eqs. (18), (19) and (20) into Eq. (21), we obtain the stress intensity factor on the crack boundary as

$$K_I = 2\sqrt{\frac{c}{\pi}}\,\sigma_{zz}^0\left[1 + A\left(\frac{a}{\Delta}\right)^3 - \frac{4}{5}A\left(\frac{a}{\Delta}\right)^3\left(\frac{c}{\Delta}\right)^2 + B\left(\frac{a}{\Delta}\right)^5 - A\left(\frac{a}{\Delta}\right)^3\frac{\bar{y}}{\Delta} + \frac{48}{5}A\left(\frac{a}{\Delta}\right)^3\left(\frac{\bar{y}}{\Delta}\right)^2\right] + O(\delta^6) \tag{22}$$

where A, B are constants given in Eq. (2) and Eq. (3). Since the stress intensity factor in Fig. 1b is zero, the stress intensity factor for our original problem Fig. 1a is given by Eq. (22).

5 The stress field inside the inhomogeneity

In Section 3, we determined the stress field inside the spherical inhomogeneity in Fig. 1c to be:

$$\sigma_{ij}^s = \sigma_{ijkl}^0(u_{i,j}^0 + u_{i,j}^{pt} - \varepsilon_{ij}^{*s})$$

where u_i^0, ε_{ij}^{*s} and $u_{i,j}^{pt}$ are given by Eqs. (11), (12) and (13). The general expression of σ_{ij}^s is too long to be written down here. If we assume $\nu_1 = \nu_0 = 1/3$ for the purpose of obtaining numerical results and $\lambda_1/\lambda_0 = \mu_1/\mu_0 = \eta$, the stress field σ_{ij}^s thus becomes

$$\sigma_{21}^s = \frac{20}{3\pi}\left(\frac{c}{\Delta}\right)^3\sigma_{zz}^0 S_4\frac{x}{\Delta}, \qquad \sigma_{23}^s = \frac{20}{3\pi}\left(\frac{c}{\Delta}\right)^3\sigma_{zz}^0 S_5\frac{z}{\Delta}, \qquad \sigma_{13}^s = 0,$$

$$\sigma_{11}^s = \frac{20}{3\pi}\sigma_{zz}^0\left(S_1\frac{y}{\Delta} + S_{01}\right),$$

$$\sigma_{22}^s = \frac{20}{3\pi}\sigma_{zz}^0\left(S_2\frac{y}{\Delta} + S_{01}\right),$$

$$\sigma_{33}^s = \frac{20}{3\pi}\sigma_{zz}^0\left(S_3\frac{y}{\Delta} + S_{02}\right),$$

where S_i (i from 1 to 5) and S_{01}, S_{02} are constants related only to η by

$$S_1 = 3 + \frac{1}{70}(-66\bar{B}_{112} + 5\bar{B}_{222} - 21\bar{B}_{332} + 101\bar{B}_{211} + 82\bar{B}_{233}),$$

$$S_2 = 3 + \frac{1}{70}(13\bar{B}_{112} + 32\bar{B}_{222} + 13\bar{B}_{332} + 157\bar{B}_{211} + 157\bar{B}_{233}),$$

$$S_3 = 6 + \frac{1}{70}(-21\bar{B}_{112} + 5\bar{B}_{222} - 66\bar{B}_{332} + 82\bar{B}_{211} + 101\bar{B}_{233}),$$

$$S_4 = \frac{1}{70}(14\bar{B}_{112} + 14\bar{B}_{222} - 4\bar{B}_{332} - 24\bar{B}_{211} + 10\bar{B}_{233}),$$

$$S_5 = -\frac{1}{5} + \frac{1}{70}(4\bar{B}_{112} + 14\bar{B}_{222} + 14\bar{B}_{332} + 10\bar{B}_{211} - 24\bar{B}_{233}),$$

$$S_{01} = \frac{16}{5}\frac{\eta - 1}{2\eta + 1} - \frac{5(\eta - 1)}{7\eta + 8}, \qquad S_{02} = 1 + \frac{16}{5}\frac{\eta - 1}{2\eta + 1} + 10\frac{\eta - 1}{7\eta + 8},$$

in which $\bar{B}_{112}$, $\bar{B}_{211}$, $\bar{B}_{222}$, $\bar{B}_{233}$ and $\bar{B}_{332}$ are given by

$$\begin{bmatrix} 1+\frac{57}{70}(\eta-1) & \frac{9}{14}(\eta-1) & 2+\frac{73}{35}(\eta-1) & \frac{9}{14} & 1+\frac{57}{70}(\eta-1) \\ 2+\frac{37}{35}(\eta-1) & \frac{9}{14}(\eta-1) & 1+\frac{15}{14}(\eta-1) & \frac{13}{35}(\eta-1) & 1+\frac{7}{10}(\eta-1) \\ 1+\frac{7}{10}(\eta-1) & \frac{13}{35}(\eta-1) & 1+\frac{15}{14}(\eta-1) & \frac{9}{14}(\eta-1) & 2+\frac{37}{35}(\eta-1) \\ \frac{27}{70}(\eta-1) & 2+\frac{44}{35}(\eta-1) & \frac{27}{70}(\eta-1) & 0 & 0 \\ 0 & 0 & \frac{27}{70}(\eta-1) & 2+\frac{27}{70}(\eta-1) & \frac{27}{70}(\eta-1) \end{bmatrix}$$

$$\times \begin{bmatrix} \bar{B}_{112} \\ \bar{B}_{211} \\ \bar{B}_{222} \\ \bar{B}_{233} \\ \bar{B}_{332} \end{bmatrix} = (1-\eta) \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ -\frac{1}{5} \end{bmatrix}.$$

By adding the stress field inside the domain of the inhomogeneity shown on Fig. 1b and the stress expressions given above, we obtain the stress field inside the spherical inhomogeneity for our original problem of Fig. 1a as

$$\sigma_{11}^{s} = \sigma_{zz}^{0}\left[\frac{\eta}{2\eta+1} - \frac{5\eta}{7\eta+8} + \frac{20}{3\pi}\left(\frac{c}{\Delta}\right)^{3}\left(S_1\frac{y}{\Delta} + S_{01}\right)\right], \tag{23}$$

$$\sigma_{22}^{s} = \sigma_{zz}^{0}\left[\frac{\eta}{2\eta+1} - \frac{5\eta}{7\eta+8} + \frac{20}{3\pi}\left(\frac{c}{\Delta}\right)^{3}\left(S_2\frac{y}{\Delta} + S_{01}\right)\right], \tag{24}$$

$$\sigma_{33}^{s} = \sigma_{zz}^{0}\left[\frac{\eta}{2\eta+1} + \frac{10\eta}{7\eta+8} + \frac{20}{3\pi}\left(\frac{c}{\Delta}\right)^{3}\left(S_3\frac{y}{\Delta} + S_{02}\right)\right], \tag{25}$$

$$\sigma_{21}^{s} = \frac{20}{3\pi}\left(\frac{c}{\Delta}\right)^{3}\sigma_{zz}^{0}S_4\frac{x}{\Delta}, \qquad \sigma_{23}^{s} = \frac{20}{3\pi}\left(\frac{c}{\Delta}\right)^{3}S_5\frac{z}{\Delta}, \qquad \sigma_{13}^{s} = 0. \tag{26}$$

6 Numerical examples

For all our numerical examples, we let the radius c of the penny-shaped crack be one-half of the radius of the spherical inhomogeneity, and assume $\nu_1 = \nu_0 = 1/3$, $\eta = \lambda_1/\lambda_0 = \mu_1/\mu_0$. The stress field at infinity is $\sigma_{zz}^{0} =$ constant.

When the crack touches the inhomogeneity, the interaction between them becomes the largest. Figure 3 shows the variation of K_I/K_I^0 with the position factor $\bar{y}/\Delta$ when $\Delta = a + c = 3c$, i.e. the crack touches the inhomogeneity. K_I^0 is the stress intensity factor on the crack boundary without the inhomogeneity. $\bar{y}$ varies from $-c$ to c. When the inhomogeneity is rigid, that is $\eta = \infty$, K_I on the crack boundary decreases more than 20% on average compared with K_I^0. If $\eta = 3$, K_I decreases about 10% compared with K_I^0. If $\eta = 1/3$, that is, the inhomogeneity is softer than the matrix, K_I increases more than 10%. If the inhomogeneity is a cavity, that is, $\eta = 0$, the K_I value undergoes the greatest change with the maximum of 32.5% at $\bar{y}/\Delta = -c$.

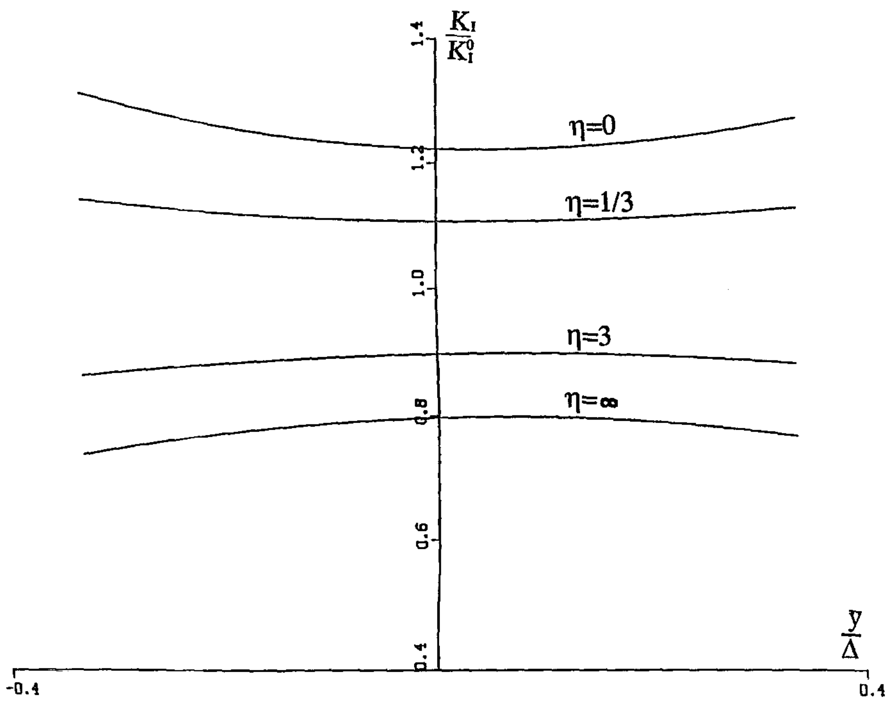

Fig. 3. The SIF at the boundary of the crack when the crack touches the inhomogeneity

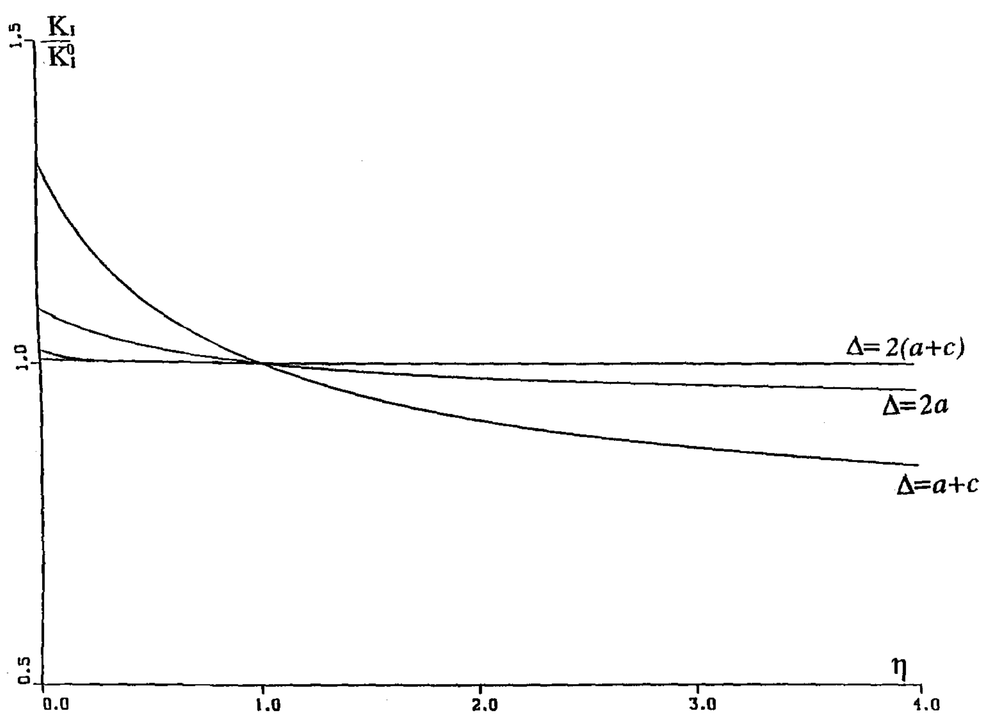

Fig. 4. The SIF at the left tip of the crack

As Fig. 3 shows, the greatest increase or decrease of $K_I/K_I{}^0$ occurs at the left crack tip ($\bar{y} = -c$). $K_I/K_I{}^0$ at this point fluctuates with η and Δ as shown in Fig. 4. For $\Delta = a + c$, that is, when the crack touches the inhomogeneity, $K_I/K_I{}^0$ varies from 1.33 for $\eta = 0$ to 0.82 for $\eta = 4$. When Δ increases, the interaction between the crack and the inhomogeneity decreases. When $\Delta = 2(a + c)$, $K_I/K_I{}^0$ remains almost constant at 1 for all values of η. In other words, the interaction between the crack and the inhomogeneity is negligible when $\Delta = 2(a + c)$. A similar behavior was observed by Moschovidis and Mura [6] with two cavities problems.

Without the crack, the stress field inside the inhomogeneity is constant. It is, however, a linear function of the coordinate y with the crack given by Eqs. (23) through (26). Because the stress exerted at infinity is only σ_{zz}^0, σ_{11}^s and σ_{22}^s are much smaller than σ_{33}^s, the maximum value of σ_{33}^s is reached at point $(0, a, 0)$, which is the nearest point to the crack. Table 1 shows the variation of $\sigma_{33}^s/\sigma_{33}^{in}$ with η and Δ. Note that σ_{33}^{in} is the stress inside the inhomogeneity without the crack.

Table 1 shows that the effect of the crack on the inhomogeneity is basically determined by the distance Δ.

Because of the crack, our problem is not symmetric about the Z-axis. Shear stress σ_{21}^s and σ_{23}^s exist inside the inhomogeneity. Both of them have the order of δ^4 and are much smaller than σ_{zz}^0.

Some special expressions of σ_{ij}^s are given in the following:

$$\eta = \infty, \qquad \sigma_{11}^s = \sigma_{zz}^0 \left(-0.1447 + 0.2874 \frac{y}{\Delta}\right),$$

$$\sigma_{22}^s = \sigma_{zz}^0 \left(-0.1447 + 0.2760 \frac{y}{\Delta}\right),$$

$$\sigma_{33}^s = \sigma_{zz}^0 \left(2.2452 + 0.7191 \frac{y}{\Delta}\right),$$

$$\sigma_{21}^s = 0.01594\sigma_{zz}^0 \frac{x}{\Delta}, \qquad \sigma_{23}^s = -0.07386\sigma_{zz}^0 \frac{z}{\Delta}, \qquad \sigma_{13}^s = 0.$$

$$\eta = 3, \qquad \sigma_{11}^s = \sigma_{zz}^0 \left(-0.04391 + 0.2633 \frac{y}{\Delta}\right),$$

$$\sigma_{22}^s = \sigma_{zz}^0 \left(-0.04391 + 0.2337 \frac{y}{\Delta}\right),$$

$$\sigma_{33}^s = \sigma_{zz}^0 \left(1.668 + 0.5634 \frac{y}{\Delta}\right),$$

$$\sigma_{21}^s = -2.336 \times 10^{-4}\sigma_{zz}^0 \frac{x}{\Delta}, \qquad \sigma_{23}^s = 0.03395\sigma_{zz}^0 \frac{z}{\Delta}, \qquad \sigma_{13}^s = 0.$$

$$\eta = \frac{1}{3}, \qquad \sigma_{11}^s = \sigma_{zz}^0 \left(-0.03654 + 0.2768 \frac{y}{\Delta}\right),$$

$$\sigma_{22}^s = \sigma_{zz}^0 \left(-0.03654 + 0.3998 \frac{y}{\Delta}\right),$$

$$\sigma_{33}^s = \sigma_{zz}^0 \left(0.4499 + 0.4715 \frac{y}{\Delta}\right),$$

$$\sigma_{21}^s = 0.01838\sigma_{zz}^0 \frac{x}{\Delta}, \qquad \sigma_{23}^s = 9.274 \times 10^{-4}\sigma_{zz}^0 \frac{z}{\Delta} \qquad \sigma_{13}^s = 0.$$

Table 1. The variation of $\sigma_{33}^{s}/\sigma_{33}^{in}$ with η and Δ

Δ	η		
	∞	3	1/3
$a + c$	1.4127	1.3951	1.4623
$2a$	1.1479	1.1398	1.1316
$2(a + c)$	1.0360	1.0344	1.0202

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