Acta Mechanica 131, 169–175 (1998)

ACTA MECHANICA

Periodic unsteady flows of a non-Newtonian fluid

T. Hayat, S. Asghar, Islamabad, Pakistan, and **A. M. Siddiqui,** York, Pennsylvania

(Received September 25, 1997)

Summary. Exact analytic solutions for the flow of non-Newtonian fluid generated by periodic oscillations of a rigid plate are discussed. Some interesting flows caused by certain special oscillations are also obtained.

1 Introduction

There are very few cases in which the exact analytic solutions of Navier-Stokes equations can be obtained. These are even rare if the constitutive equations for the non-Newtonian fluid are considered. Although there are many models used to describe non-Newtonian behavior of the fluids, the fluids of differential type (cf. Truesdell and Noll [1]) have received special attention. In recent years, interest in the flows of non-Newtonian fluids has very much increased, and we refer to the papers by Sirvatsava [2], Rajeswari and Rathna [3], Beard and Walters [4], Mansutti et al. [5], Siddiqui and Kaloni [6], Massoudi and Ramezan [7], Benharbit and Siddiqui [8], Erdogan [9], Rajagopal et al. [10], [11]. In another paper, Rajagopal [12] gave solutions for an unsteady unidirectional flow of an incompressible second grade fluid. He analyzed the flow due to a rigid plate oscillating in the form of $U \cos \omega_o t$ and calculated the velocity field using separation of variables.

The aim of the present paper is to investigate the general class of flow problems due to arbitrary periodic oscillations of a rigid plate. For the periodic oscillations we can construct the Fourier transform directly from its Fourier series representation. The resulting Fourier transform for a periodic oscillation consists of a train of impulses in frequency, with the areas of the impulses proportional to the Fourier series coefficients. This will turn out to be a very important representation, as it will facilitate out treatment of the application of Fourier analysis techniques to problems of modulation. A general periodic oscillation $f(t)$ with period T_o is considered. The response of oscillations in the flow field can be built up using Fourier series representation and the temporal Fourier transform. The exact analytical solution for unsteady unidirectional flow due to periodic rigid plate oscillations is obtained. The flow field due to certain special values of oscillations is then derived as a special case of the periodic oscillations including the one considered by Rajagopal [12]. Thus, the technique used and the arbitrary nature of oscillation may be a significant contribution in the theory of non-Newtonian fluid (2nd grade fluid) of which very little is known.

2 Basic equations

The Cauchy stress tensor $\boldsymbol{T}$ for an incompressible homogeneous fluid of second grade has the following form [13], [14]:

$$\boldsymbol{T} = -p\mathbf{1} + \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1{}^2 \tag{1}$$

where $-p\mathbf{1}$ is the indeterminate spherical stress, μ is the coefficient of dynamic viscosity, and α_1 and α_2 are material moduli which are usually referred to as the normal stress coefficients. The kinematical (Rivlin-Ericksen) tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined through [1]

$$\boldsymbol{A}_1 = (\text{grad}\, \boldsymbol{\nu}) + (\text{grad}\, \boldsymbol{\nu})^T, \tag{2.1}$$

and

$$\boldsymbol{A}_2 = \frac{d}{dt}\boldsymbol{A}_1 + \boldsymbol{A}_1(\text{grad}\, \boldsymbol{\nu}) + (\text{grad}\, \boldsymbol{\nu})^T \boldsymbol{A}_1, \tag{2.2}$$

where d/dt denotes the material time derivative and $\boldsymbol{\nu}$ denotes the velocity. The signs of the coefficients α_1 and α_2 are the subject of some controversy and a thorough discussion of the same can be found in Dunn and Fosdick [15], Fosdick and Rajagopal [16] and the more critical review by Dunn and Rajagopal [17].

For fluid of the type (1) that is compatible with thermodynamics, in the sense that all motions of the fluid meet the Clausius-Duhem inequality, which is interpreted as the second law of thermodynamics, and the assumption that the specific Helmholtz free energy is a minimum when the fluid is in equilibrium, the following restrictions on the signs of the material moduli hold:

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \alpha_1 + \alpha_2 = 0. \tag{3}$$

By replacing the constitutive expression (1) into the balance of linear momentum

$$\text{div}\, \boldsymbol{T} + \varrho \boldsymbol{b} = \varrho \frac{d\boldsymbol{\nu}}{dt} \tag{4}$$

and using the fact that the fluid can undergo only isochoric motion since it is incompressible, i.e.

$$\text{div}\, \boldsymbol{\nu} = 0, \tag{5}$$

we have

$$\mu \Delta \boldsymbol{\nu} + \alpha_1 \Delta \frac{\partial \boldsymbol{\nu}}{\partial t} + \alpha_1 (\Delta w \times \boldsymbol{\nu}) + (\alpha_1 + \alpha_2) \\ \times \{\boldsymbol{A}_1 \Delta \boldsymbol{\nu} + 2\, \text{div}\, [(\text{grad}\, \boldsymbol{\nu})\, (\text{grad}\, \boldsymbol{\nu})^T]\} - \varrho (w \times \boldsymbol{\nu}) - \varrho \frac{\partial \boldsymbol{\nu}}{\partial t} = \text{grad}\, P, \tag{6}$$

where

$$P = p - \alpha_1 (\boldsymbol{\nu}.\Delta \boldsymbol{\nu}) - \frac{(2\alpha_1 + \alpha_2)}{4} |\boldsymbol{A}_1|^2 + \frac{\varrho}{2} |\boldsymbol{\nu}|^2 + \varrho \phi.$$

In the above equations we have used that the body force $\boldsymbol{b}$ is conservative and hence can be expressed as $\boldsymbol{b} = \text{grad}\, \phi$. In Eq. (6), Δ denotes the Laplacian, ∇ the gradient operator, $|\boldsymbol{A}_1|$ the trace norm of $\boldsymbol{A}_1$ and

$$\boldsymbol{w} = \text{curl}\, \boldsymbol{\nu}. \tag{7}$$

Since we are dealing with unidirectional flow,

$$\nu = u(y, t)\, \boldsymbol{i}, \tag{8}$$

where $\boldsymbol{i}$ denotes a unit vector in the x-coordinate direction.

Using Eq. (8) in Eq. (6) we get

$$\mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2\, \partial t} - \varrho \frac{\partial u}{\partial t} = \frac{\partial p}{\partial x}, \tag{9}$$

$$(2\alpha_1 + \alpha_2) \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right)^2 = \frac{\partial p}{\partial y}, \tag{10}$$

$$0 = \frac{\partial p}{\partial z}. \tag{11}$$

Furthermore, by defining

$$\hat{p} = p - (2\alpha_1 + \alpha_2) \left(\frac{\partial u}{\partial y} \right)^2 \tag{12}$$

we have from Eqs. (9) to (11)

$$\mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2\, \partial t} - \varrho \frac{\partial u}{\partial t} = \frac{\partial \hat{p}}{\partial x}, \tag{13.1}$$

$$0 = \frac{\partial \hat{p}}{\partial y} = \frac{\partial \hat{p}}{\partial z}. \tag{14.1, 2}$$

From Eqs. (13) and (14.1, 2) we obtain

$$\mu \frac{\partial^3 u}{\partial y^3} + \alpha_1 \frac{\partial^4 u}{\partial y^3\, \partial t} - \varrho \frac{\partial^2 u}{\partial y\, \partial t} = 0. \tag{15}$$

Now we suppose that the upper half of the (x, y)-plane be occupied by fluid, the rigid boundary being at $y = 0$. The plate is making periodic oscillations of the form $f(t)$ with period T_0. The rigid boundary is having the velocity $Uf(t)$. The Fourier series representation of $f(t)$ is given by

$$f(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{ik\omega_0 t}, \tag{16.1}$$

where

$$a_k = \frac{1}{T_0} \int f(t)\, e^{-ik\omega_0 t}\, dt \tag{16.2}$$

with non zero fundamental frequency $\omega_0 = 2\pi/T_0$. Equation (16.1) is referred to as the synthesis equation and Eq. (16.2) as the analysis equation. The coefficients $\{a_k\}$ are the Fourier series coefficients or the spectral coefficients of $f(t)$. In practice the fluid motion would be set up from rest, and, for some time after the initiation of the motion, the flow field contains "transients" determined by these initial conditions. It may be shown that the fluid velocity gradually becomes a harmonic function of t, with the same frequency as the velocity of the boundary, and only this periodic state will be considered here. The governing differential equation is (15), with the boundary condition

$$u(0,t) = U \sum_{k=-\infty}^{\infty} \alpha_k {}^{ik\omega_0 t}. \tag{16.3}$$

3 Solution of the boundary value problem

We attempt to find the solution using Fourier transform. So the temporal Fourier transform pair is defined as

$$\psi(y,\omega) = \int_{-\infty}^{\infty} u(y,t)\, e^{-i\omega t}\, dt\,, \tag{17}$$

$$u(y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(y,\omega)\, e^{i\omega t}\, d\omega\,, \tag{18}$$

ω being the temporal frequency. From Eqs. (15) and (17) we find

$$\frac{d^2}{dy^2}\psi(y,\omega) - (m+in)^2 \psi(y,\omega) = 0\,, \tag{19}$$

where

$$(m+in)^2 = \frac{\varrho\omega(i\mu + \omega\alpha_1)}{\mu^2 + (\omega\alpha_1)^2}\,. \tag{20}$$

Transforming the boundary condition (16.1) we get

$$\psi(0,\omega) = U \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_o)\,. \tag{21}$$

Note that Eq. (21) is of the form of a linear combination of impulses equally spaced in frequency. We see that Eq. (16.3) corresponds to the Fourier series representation of $u(0,t)$, and its Fourier transform given by Eq. (21) can be interpreted as a train of impulses occurring at the harmonically related frequencies. The area of the impulse at the kth harmonic frequency $k\omega_o$ is 2π times the kth Fourier series coefficient a_k.

The only solution that remains finite as $y \to \infty$ is

$$\psi(y,\omega) = Ae^{-(m+in)y}. \tag{22}$$

From Eqs. (21) and (22)

$$\psi(y,\omega) = U \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_o)\, e^{-(m+in)y}. \tag{23}$$

Substituting Eq. (23) in Eq. (18) and then solving the integral in the resulting expression after using the property of delta function, we arrive at

$$u(y,t) = U \sum_{k=-\infty}^{\infty} a_k\, e^{-m_k y + i(k\omega_o t - n_k y)}, \tag{24}$$

where from Eq. (20)

$$m_k^{\,2} = \frac{1}{2}\left(\frac{k\varrho\omega_0}{[\mu^2 + (\alpha_1 k\omega_o)^2]}\right)\{[\mu^2 + (\alpha_1 k\omega_o)^2]^{1/2} + k\alpha_1\omega_o\}\,, \tag{25.1}$$

$$n_k^{\,2} = \frac{1}{2}\left(\frac{k\varrho\omega_0}{[\mu^2 + (\alpha_1 k\omega_o)^2]}\right)\{[\mu^2 + (\alpha_1 k\omega_o)^2]^{1/2} - k\alpha_1\omega_o\}\,. \tag{25.2}$$

Equation (24) gives the complete analytic solution for the velocity field due to the rigid plate oscillating periodically in its own plane. As a special case of this oscillation, the flow field for different plate oscillations is obtained by an appropriate choice of the Fourier coefficients which give rise to different plate oscillations. The periodic oscillations and their corresponding Fourier coefficients are given in the following table:

Oscillations $f(t)$	Fourier coefficients a_k
(i) $e^{i\omega_0 t}$	$a_1 = 1$ and $a_k = 0 (k \neq 1)$
(ii) $\cos\,\omega_o t$	$a_1 = a_{-1} = \frac{1}{2}$ and $a_k = 0$, otherwise,
(iii) $\sin\,\omega_o t$	$a_1 = -a_{-1} = \frac{1}{2i}$ and $a_k = 0$, otherwise,
(iv) $\begin{cases} 1, \lvert t\rvert < T_1 \\ 0, T_1 < \lvert t\rvert < T_o/2 \end{cases}$	$a_o = 2T_1/T_o; a_k = \frac{\sin\,(k\omega_o T_1)}{k\pi}$, for all $k \neq 0$,
(v) $\sum_{k=-\infty}^{\infty} \delta(t - kT_o)$	$a_k = 1/T_o$ for all k.

The flow fields in above five cases can be easily obtained by using successively the appropriate Fourier coefficients in Eq. (24). The resulting flow fields for these cases are then given by

$$u_1(y,t) = Ue^{-m_1 y + i(\omega_0 t - n_1 y)}\,, \tag{26}$$

$$\begin{aligned} u_2(y,t) &= \frac{U}{2}\,\{e^{-m_1 y + i(\omega_0 t - n_1 y)} + e^{-m_{-1} y - i(\omega_0 t + n_{-1} y)}\}\,, \\ &= Ue^{-m_1 y}\cos\,(\omega_0 t - n_1 y)\,, \end{aligned} \tag{27}$$

$$\begin{aligned} u_3(y,t) &= \frac{-iU}{2}\,\{e^{-m_1 y + i(\omega_0 t - n_1 y)} - e^{-m_{-1} y - i(\omega_0 t + n_{-1} y)}\}\,, \\ &= Ue^{-m_1 y}\sin\,(\omega_0 t - n_1 y)\,, \end{aligned} \tag{28}$$

$$u_4(y,t) = \frac{U}{\pi}\sum_{k=-\infty}^{\infty}\frac{\sin\,(k\omega_0 T_1)}{k}\,e^{-m_k y + i(k\omega_0 t - n_k y)}\,, \qquad k \neq 0\,, \tag{29}$$

$$u_5(y,t) = \frac{U}{T_o}\sum_{k=-\infty}^{\infty} e^{-m_k^* y + i(2\pi kt/T_o - n_k^* y)}\,, \tag{30}$$

where

$$m_k^{2*} = m_k^{\,2}\big|_{\omega_0 = 2\pi/T_o}\,, \qquad n_k^{2*} = n_k^{\,2}\big|_{\omega_0 = 2\pi/T_o}\,, \qquad n_{-1} = -n_1\,.$$

We note that $u_2(y,t)$ given by (27) agrees with the result of Rajogopal [12] corresponding to the oscillations $U\cos\omega_0 t$.

5 Concluding remarks

We have solved a canonical boundary value problem for the flow field of a non-Newtonian fluid, due to a rigid plate oscillating in its own plane. It is noted that the velocity field in the case of a second grade fluid is governed by a third order partial differential equation as compared to second order Navier-Stokes equations for Newtonian fluids. The velocity profile (24) represents a wave of transversal vibrations propagating inwards from the boundary (in the y-direction) with the phase velocity $(k\omega_0/n_k)$ and rapidly diminishing amplitude. The damping is such as to make the amplitude of the oscillations fall off as $\exp\{-m_k y\}$.

The linear magnitude $(m + in)^{-1}$ is of great physical importance in all problems of oscillatory motion which do not involve changes of density, as indicating the extent to which the effects of viscosity penetrate into the fluid. It is worth mentioning that the formula (24) for an arbitrary harmonic component of the velocity of the rigid boundary can be used to build up the solution for a certain special periodic motion of the rigid boundary. Further, the results of flow for Newtonian fluid can be obtained as a special case of this problem by choosing $\alpha_1 = 0$.

Acknowledgement

A. M. Siddiqui gratefully acknowledges the support of the Pennsylvania State University R.D.G. for Commonwealth Campuses.

References

[1] Truesdell, C., Noll, W.: The non-linear field theories of mechanics (Handbuch der Physik, III/3). Berlin Heidelberg New York: Springer 1965.

[2] Srivatsava, A. C.: The flow of a non-Newtonian liquid near a stagnation point. ZAMP **9,** 80–84 (1958).

[3] Rajeswari, G., Rathna, S. L.: Flow of a particular class of non-Newtonian visco-elastic and visco-inelastic fluids near a stagnation point. ZAMP **13,** 43–57 (1962).

[4] Beard, D. W., Walters, K.: Elastico-viscous boundary layer flows. Proc. Camb. Phil. Soc. **60,** 667–674 (1964).

[5] Mansutti, D., Pontrelli, G., Rajagopal, K. R.: Non-similar flow of a non-Newtonian fluid past a wedge. Int. J. Eng. Sci. **31,** 637–647 (1993).

[6] Siddiqui, A. M., Kaloni, P. N.: Certain inverse solutions of a non-Newtonian fluid. Int. J. Non-Lin. Mech. **21,** 459–473 (1986).

[7] Massoudi, M., Ramezan, M.: Effect of injection or suction on the Falkner-Skin flows of second grade fluids. Int. J. Non-Lin. Mech. **24,** 221–227 (1989).

[8] Benharbit, A. M., Siddiqui, A. M.: Certain solutions of equations of the planar motion of a second grade fluid for steady and unsteady cases. Acta Mech. **94,** 85–96 (1992).

[9] Erdogan, M. E.: Plane surface suddenly set in motion in a non-Newtonian fluid. Acta Mech. **108,** 179–187 (1995).

[10] Rajagopal, K. R., Gupta, A. S., Wineman, A. S.: On a boundary layer theory for non-Newtonian fluids. Lett. Appl. Eng. Sci. **18,** 875–883 (1980).

[11] Rajagopal, K. R., Gupta, A. S., Na, T. Y.: A note on the Falkner-Skan flows of a non-Newtonian fluid. Int. J. Non-Lin. Mech. **18,** 313–320 (1983).

[12] Rajagopal, K. R.: A note on unsteady unidirectional flows of a non-Newtonian fluid. Int. J. Non-Lin. Mech. **17,** 369–373 (1982).

[13] Coleman, B. D., Noll, W.: An approximation theorem for functionals with application in continuum mechanics. Arch. Rat. Mech. Anal. **6,** 335–370 (1960).

[14] Rivlin, R. S., Ericksen, J. L.: Stress deformation relations for isotropic materials. J. Rat. Mech. Anal. **4,** 323–425 (1955).
[15] Dunn, J. E., Fosdick, R. L.: Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade. Arch. Rat. Mech. Mech. Anal. **56,** 191–252 (1974).
[16] Fosdick, R. L., Rajagopal, K. R.: Anomalous features in the model of second-order fluids. Arch. Rat. Mech. Mech. Anal. **70,** 145–152 (1979).
[17] Dunn, J. E., Rajagopal, K. R.: Fluids of differential type: critical review and thermodynamic analysis. Int. J. Eng. Sci. **33,** 689–729 (1995).

Authors' addresses: T. Hayat and S. Asghar, Department of Mathematics, Quaid-I-Azam University, Islamabad, Pakistan; A. M. Siddiqui, Department of Mathematics, Pennsylvania State University, York Campus, York, Pennsylvania 17403, U.S.A.