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Rapid Couette Flow of Cohesionless Granular Materials

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With 4 Figures

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Summary

Presented is an analysis on the Couette flow of cohesionless granular materials between two co-axial rotating cylinders. The constitutive equations employed have been postulated on the basis of available experimental and theoretical results which take into account the particle collisions as well as dynamic pressures induced by the trace of the unsemble phase average of the square of flow fluctuations. These constitutive equations loosely resemble the Reiner-Rivlin fluid behavior, and predict normal stress effects.

New non-Newtonian effects in striking manners have been predicted in the cases of outer cylinder rotating-inner cylinder fixed as well as outer cylinder fixed-inner cylinder rotating. The theoretical predictions for the free surface profile for these two cases agree with our experimental observations and point to the validity of the proposed constitutive equations. All our results are based on no-slip conditions on the boundary surfaces. Furthermore, the results obtained are different from the classical results obtained for the Couette flow of simple non-Newtonian fluids.

Introduction

Two idealized flow regimes exist in cohesionless granular materials. The first flow regime which could loosely be termed the slow flow regime or the initial flow regime corresponds to a case in which the interparticle interactions normally arise mainly due to interparticle Coulomb type friction and sliding. In this regime the effect of particle collisions are negligible and one may assume that the constitutive equations are rate independent. Calculations based on this assumption for the steady plastic flows and evolution of slip planes have been carried out by many authors such as Drucker and Prager [1], Shield [2], [3], Drucker, Gibson, and Henkel [4], Jenike and Shield [5], Jenike [6], de Josselin de Jong [7], [8], [9], Shunsuke [10], Rowe [11], Spencer [12], Horne [13], [14], [15], Spencer and Kingston [16], Mandl and Luque [17], Drescher and De Jong [18], Mroz and Drescher [19], Nikolaevskii [20], Drescher [21], Wilde [22], Michalowski and Mroz [23], Mehrabadi and Cowin [24], and Nemat-Nasser and Shokouh [25].

In the second flow regime, what is called the rapid flow regime, on the contrary to the first slow plastic flow regime, the effect of Coulomb type interparticle friction and interaction is negligible compared to the interparticle forces that arise due to the exchange of momentum in particle collisions. In this regime

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the nature of constitutive equations are entirely different from the plastic type constitutive equations corresponding to the first regime. Surprisingly very few works have been reported on these types of rapid flows. Historically, Hagen [26] was the first to study the flow of sand in tubes. Bagnold [27], [28] has studies the Couette flow of gravity-free suspension of solid particles in a Newtonian fluid and has defined what is now known as inertia dominated regions in flowing granular materials. Brown and Richards [29] have extensively discussed the various developments on the mechanics and flow of granular materials up to the year 1970. Savage [30] has extensively discussed the literature on various gravity flows of cohesionless granular materials and has mentioned the difficulties encountered in using what is called the Goodman-Cowin continuum theory of granular materials [31], [32], [33] to such flows. Jenkins and Cowin [34] have entensively discussed the various theories for flowing granular material and have elaborated on the need for acceptable constitutive relationships for the rapid flow of granular materials. Of interest are the works of McTigue [35], Blinowski [36], Ogawa [37], and Kanatani [38], in which new theories for flowing granular materials have been proposed and seem to support Bagnold's observations and results.

In the next section we shall elaborate on such theories and we propose a constitutive equation for the rapid flow of cohesionless granular materials that would be consistent with the accepted observations and analytical results obtained for such rapid flows. Having proposed our model we shall apply the model to the problem of rapid Couette flow of cohesionless granular materials between two co-axial rotating cylinders. The results obtained indicate that he behavior of such materials in Couette flow is analogous to the behavior of a class of non-Newtonian fluids possessing a Reiner-Rivlin type of constitutive equations and infact certain Weissenberg effects [43] can be predicted. However, the free surface profiles are fundamentally different in nature from the ones considered in [40], [42], [43]. In order to test the validity of the proposed constitutive equations we conduct two simple Couette flow experiments with iodized table salt to show some evidence supporting the proposed mathematical model.

Proposed Constitutive Equations

Jenkins and Cowin [34] have extensively discussed the various forms of constitutive equations that may be suitable for the rapid flow of cohesionless granular materials. However, their work is not quite conclusive and infact they conclude that in a theory for rapidly flowing granular materials the stress should typically have two parts. One part depends explicitly upon the deformation rate tensor D_{ij}, i.e.,

$$D_{ij} = \frac{1}{2} (V_{i,j} + V_{j,i}), \tag{1}$$

where V_i is the velocity vector of the granules and a comma here denotes covariants differentiation with respect to a fixed curvilinear coordinate system X^i. The second part of the stress need not vanish with D_{ij}. Bagnold [27] had concluded that for the inertia dominated regions in rapid simple shear flows of granular materials in absence of gravitational effects the shear and normal stresses are proportional to the square of the rate of deformation D_{ij}. McTigue [35] has employed analytical tools for the classical "billiard ball" problem in treating the shear flow of cohesionless granular materials. His analysis employs a collision frequency function to calculate the net momentum exchange in particle-particle interaction to essentially arrive at results similar to Bagnold's [27]. He concludes that the gravity free parts of stresses are given by

$$\tau_{12} = \frac{64\gamma a^2}{35\pi} \nu^2 \left|\frac{du_2}{dX_1}\right| \frac{du_2}{dX_1}, \tag{2}$$

$$\tau_{11} = \frac{24\gamma a^2}{35} \nu^2 \left(\frac{du_2}{dX_1}\right)^2, \tag{3}$$

where X_2 denotes the direction of motion while X_1 is perpendicular to the direction of motion, γ is the particle density, a is its radius and ν is the solid volume fraction.

McTigue [35] has proposed that constitutive equations compatible to the Reiner-Rivlin type constitutive equations (Reiner [39], Rivlin [40]), may be suitable for the general case. However, he does not give clear interpretation of the pressure term that appears in his proposed equation. Kanatani [38] has presented a micropolar continuum theory for the flow of granular materials and in particular the fast flows. He has shown that particle velocity fluctuations play a dominant role in such fast flows and, further, that for inclined gravity flows

$$\tau_{12} = \frac{3\sqrt{15}}{200} C(\varrho) \left(\frac{dv_2}{dx_1}\right)^2, \tag{4}$$

$$\tau_{11} = \frac{\sqrt{6r}}{40\mu_a} C(\varrho) \left(\frac{dv_2}{dx_1}\right)^2, \tag{5}$$

where r is occupation radius, $C(\varrho)$ is a given function of density, μ is a kinetic friction coefficient, and a is the particle radius. Blinowski [36] has employed statistical methods developed for the turbulent flow of fluids to describe the granular media rapid flow irregularities and particle fluctuations. From his analysis it is clear that a tensor $K_{ij} \equiv \langle V_i' V_j' \rangle$, i.e. statitical ensemble phase average of the square of particle velocity fluctuations V_i', plays an important role in the constitutive equations. In fact as shown by Jenkins and Cowin [34] the trace of K_{ij}, i.e., K_{ii} can be shown to represent a pressure like term in the equation for the stress tensor. Ogawa [37][1], employing a two-temperature theory for granular materials has obtained constitutive equations which depend on the average of the square of the magnitude of the fluctuations in velocity, this being considered as a second temperature in the theory. Although, still no clear cut form for the constitutive equations for the rapid flow of granular materials seem to exist, however, it would seem useful to propose one, based on the available experimental and theoretical results, and to check whether it agrees with the experimentally observed results for a particular rapid flow problem. Thus,

[1] Also see Ogawa, Omemura, Oshima [44]

we intend to treat the rapid Couette flow of granular materials. For this purpose we propose the following constitutive equations

$$\tau_{ij} = -p\delta_{ij} + \alpha_1 \nu^2 \, |I_2|^{1/2} \, D_{ij} + \alpha_2 \nu^2 D_{ik} D_{kj}, \tag{6}$$

$$I_2 = \frac{1}{2} \, (D_{mm} D_{nn} - D_{mn} D_{nm}), \tag{7}$$

where p is a dynamic pressure term that is assumed to be due to the statistical trace average of the square of particle velocity fluctuations, i.e., $\langle V_i' V_i' \rangle$ as well as pure hydrostatic effects due to gravitation, α_1 and α_2 are constants, and ν is the solid volume traction, i.e., $\varrho = \gamma \nu$, where ϱ is the bulk density and γ is the grain density. As can be seen the constitutive equations are compatible with the Reiner-Rivlin type constitutive equations except for the dependence on solid volume fraction ν as well as the role of the ensemble phase trace average $\langle V_i' V_i' \rangle$ in the pressure term p. Equation (6) is also compatible with the experimental results of Bagnold [27] although in his experiment the effect of gravity was absent. We shall elaborate on this point in the next section.

Couette Flow of Cohesionless Granular Materials

Using a cylindrical polar coordinate system, we seek a solution to

$$\tau_{ij;j} + \varrho f_i = \varrho \, \frac{dV_i}{dt}, \tag{8}$$

where $f_i \equiv (0, 0, -g)$, g being the gravitational acceleration, and

$$V_1 = V_r = V_3 = V_z = 0, \qquad V_2 = V_\theta = r\omega(r), \qquad R_i \leqq r \leqq R_0. \tag{9}$$

This corresponds to a steady Couette flow between two rotating co-axial cylinder. In order for the considered solutions to be valid the gap between the coaxial cylinders, i.e., $R_0 - R_i$ should be sufficiently small to prevent axial, radial, and secondary flows. In this case

$$D_{ij} = \frac{1}{2} \begin{bmatrix} 0 & r\omega' & 0 \\ r\omega' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{ij} D_{jk} = \frac{1}{4} \begin{bmatrix} r^2\omega'^2 & 0 & 0 \\ 0 & r^2\omega'^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{10}$$

where $\omega' = d\omega/dr$.

Based on the above expressions the components of the stress tensor reduce to

$$\tau_{11} = \tau_{22} = -p + \frac{1}{4} \, \alpha_2 \nu^2 r^2 \omega'^2 \tag{11}$$

$$\tau_{12} = \frac{1}{4} \, \alpha_1 \nu^2 \, |r\omega'| \, r\omega', \tag{12}$$

$$\tau_{13} = \tau_{23} = 0, \qquad \tau_{33} = -p. \tag{13}$$

Note that from the above equations there exist a relationship between the physical components of the normal and shear stress, i.e.,

$$P = \tau_{11} + p = \frac{1}{4}\,\alpha_2 \nu^2 r^2 \omega'^2, \tag{14}$$

$$T = \tau_{12} = \frac{1}{4}\,\alpha_1 \nu^2\,|r\omega'|\,r\omega', \tag{15}$$

which are in agreement with both Bagnold's experimental results [27] as well as McTigues's results [35], and Kanatani's results [37]. The governing Eq. (8) now reduce to

$$-\varrho r \omega^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{11}) - \frac{1}{r}\,\tau_{22}, \tag{16}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\tau_{12}) = 0, \tag{17}$$

$$\frac{\partial p}{\partial z} = -\varrho g. \tag{18}$$

From Eq. (18) we obtain

$$p = -\varrho g z + f(r). \tag{19}$$

Note that we have treated ϱ as a constant which means that in such steady flows the bluk density is assumed to remain constant. As explained by Shahinpoor [41], in this case, the bulk density corresponds to the critical density, i.e., density for loose random packing. This is due to the fact that in such rapid flows all "Voronoi Cells" have equal chance for being created and annihilated, thus giving rise to uniform distribution for "Voronoi Cells" or characteristic void spaces[2]. This constancy of ϱ or its solid counterpart ν in steady Couette flow is a crucial point in the present analysis.

From Eq. (17) we obtain that

$$\tau_{12} = A r^{-2}, \tag{20}$$

where A is a constant. From Eqs. (12) and (20) we obtain the following nonlinear differential equation for determining $\omega(r)$:

$$A r^{-2} = \frac{1}{4}\,\alpha_1 \nu_{cr}^2\,|r\omega'|\,r\omega', \tag{21}$$

where ν_{cr} is the critical solid volume fraction[2]. From Eqs. (11) and (16) we obtain the following differential equation

$$-\varrho r \omega^2 = -\frac{\partial p}{\partial r} + \frac{1}{4}\,\alpha_2 \nu_{cr}^2 \frac{d}{dr}(r^2\omega'^2). \tag{22}$$

From Eq. (19) we obtain that

$$\frac{\partial p}{\partial r} = f', \qquad \frac{dp}{dr} = -\varrho g \frac{dz}{dr} + f'. \qquad (23.1,\ 23.2)$$

[2] For random aggregates of equal spheres the critical void ratio $e = (1 - \nu)/\nu$ equals $e_{cr} \approx 0.64$ under 20 psi overburden pressure (Shahinpoor [41]).

From Eqs. (22) and (23), we obtain

$$f' = \varrho r \omega^2 + \frac{1}{4} \alpha_2 \nu_{cr}^2 \frac{d}{dr} (r^2 \omega'^2). \tag{24}$$

Thus,

$$f(r) = \varrho \int r\omega^2 \, dr + \frac{1}{4} \alpha_2 \nu_{cr}^2 r^2 \omega'^2 + B, \tag{25}$$

where B is another constant of integration. We are now in a position to analyse two distinct cases. First, the case corresponding to the inner cylinder fixed — outer cylinder rotating and Second, the case corresponding to the inner cylinder rotating outer cylinder fixed. In both cases we assume no slip conditions for the velocity V_θ on the boundary walls. This can be achieved in experiments if the contacting walls are made from rubber like materials or if they are covered with a thin layer of a transparent glue.

Case 1 — Inner Cylinder Fixed — Outer Cylinder Rotating at a Constant Speed ω_0

In this case $\omega' > 0$ and thus from Eq. (21) we find that

$$\omega(r) = -2 \sqrt{A/\alpha_1} \, \nu_{cr}^{-1} r^{-1} + C, \tag{26}$$

where C is another constant of integration. Surprisingly the above solution is different from the one obtained for the Couette flow of non-Newtonian fluids by Serrin [42]. Applying the no-slip boundary conditions

$$\omega(R_i) = 0, \qquad \omega(R_0) = \omega_0, \tag{27}$$

where R_i and R_0 correspond to the inner and outer radii, we obtain from Eq. (26) that

$$C = 2 \sqrt{A/\alpha_1} \, \nu_{cr}^{-1} R_i^{-1}, \qquad A = \frac{\alpha_1 {\omega_0}^2 \nu_{cr}^2}{4} \left(\frac{1}{R_i} - \frac{1}{R_0} \right)^{-1}. \tag{28.1, 28.2}$$

Thus,

$$\omega(r) = \omega_0 \left(\frac{1}{R_i} - \frac{1}{R_0} \right)^{-1} \left(\frac{1}{R_i} - \frac{1}{r} \right). \tag{29.1}$$

From Eqs. (28.1, 28.2) we note that

$$C = \omega_0 \nu_{cr}^{-1} R_i^{-1} \left(\frac{1}{R_i} - \frac{1}{R_0} \right)^{-1} > 0, \qquad \frac{A}{\alpha_1} > 0. \tag{29.2}$$

Shape of the Free Surface: Case 1

Considering Eq. (23.2) and simplifing that for the free surface, for which $p =$ constant, we obtain

$$\frac{dz}{dr} = \frac{f'}{\varrho_{cr} g}. \tag{30}$$

Substituting for $f'(r)$ from Eq. (24) in the above equation yields

$$\frac{dz}{dr} = \left[\frac{\varrho_{cr} r\omega^2 + \frac{1}{4}\alpha_2 \nu_{cr}^2 \frac{d}{dr}(r^2\omega'^2)}{\varrho_{cr} g}\right] \tag{31.1}$$

$$\varrho_{cr} g z = \varrho_{cr} \int r\omega^2 \, dr + \frac{1}{4}\alpha_2 \nu_{cr}^2 r^2 \omega'^2 + B. \tag{32}$$

This equation can further be simplified to

$$\varrho_{cr} g z = \varrho_{cr}\omega_0^2 \left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2} \left(\frac{r^2}{2R_1^2} + \ln r - \frac{2r}{R_i}\right) + \frac{1}{4}\alpha_2 \nu_{cr}^2 \omega_0^2 \left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2} r^{-2} + B. \tag{33}$$

Thus, generally, in this case one winds up with a nonlinear free surface profile which could be tested against experimentally obtained surface profiles for various granular materials to check the validity of the proposed constitutive equations.

Of interest is also the slope of the free surface $\frac{dz}{dr}$ which is given, generally, by Eq. (31.1) and, specifically, for this case by

$$\frac{dz}{dr} = \varrho_{cr}^{-1} g^{-1} \left[\varrho_{cr}\omega_0^2 \left(\frac{r}{R_i^2} + \frac{1}{r} - \frac{2}{R_i}\right) - \frac{1}{r^3}\left(\frac{1}{2}\alpha_2 \nu_{cr}^2 \omega_0^2\right)\right] \left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2}. \tag{34}$$

At $r = R_i$ we obtain from Eq. (34) that

$$\left.\frac{dz}{dr}\right|_{r=R_1} = -\frac{1}{2}\varrho_{cr}^{-1} g^{-1} \alpha_2 \nu_{cr}^2 \omega_0^2 \left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2} R_i^{-3}. \tag{35}$$

Thus, depending on the sign of α_2 the fluid tends to climb up or down the inner cylinder, accordingly, if $\alpha_2 > 0$ or $\alpha_2 < 0$, respectively.

Suitable experiments of this kind can be performed to determine the sign and the value of α_2 for any particular granular material. Our experimental result with iodized salt (Figs. 1 and 2) indicate that $\alpha_2 > 0$, and in fact $\alpha \approx 0.087 \frac{\text{Lb sec}^2}{\text{ft}^2}$.

In fact one can easily show that generally for the slope of the free surface to be zero at some distance from the center, α_2 must be a positive constant. This can be proved by setting the general expression (31.1) equal to zero and calculating a r at which $\frac{dz}{dr}$ is equal to zero.

We here present some specific results for this case. For a particle size distribution between 100 to 300 microns, the average diameter was about 189 microns with a standard deviation of about 9%. We randomly created a loose packing of these particles to obtain a $\nu_{cr} \approx 0.5$ according to [41]. The particles weight density was measured to be $\gamma_s \approx 119$ Lb/ft^3. For an $\omega_0 = 2\pi$ radians per second, and $R_i = 1''$, $R_0 = 3''$, we measured $\theta_{1(i)} \approx -11°$ and $\theta_{1(0)} \approx 17°$. In order to check the validity of our theory we calculated the value of α_2 from expression (35) and plugged in expression (34) to find the corresponding $\theta_{1(0)}$ to see whether it

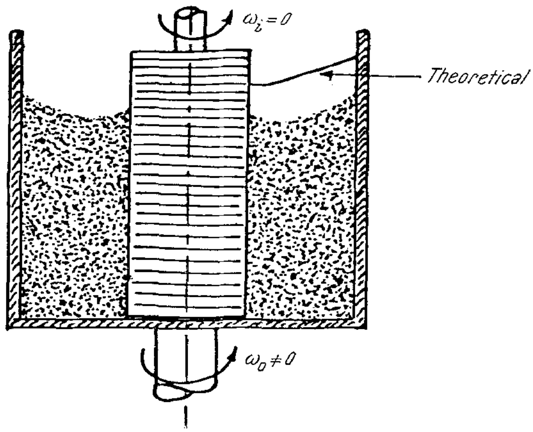

Fig. 1. The shape of the free surface corresponding to case I, i.e., inner cylinder fixed outer cylinder rotating, $\omega_0 = 2\pi$ rad/sec., $R_i = 1''$, $R_0 = 3''$

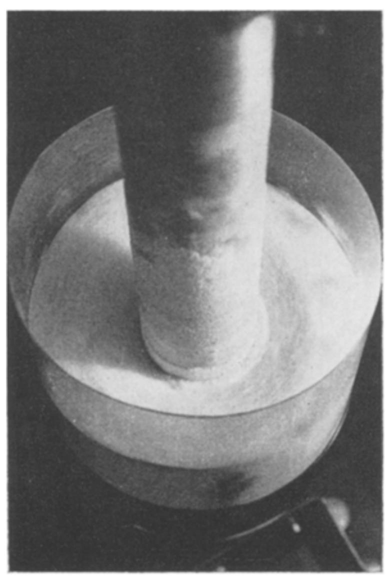

Fig. 2. Photograph of the experimental set up corresponding to case 1

matches the experimentally measured value of 17°. Thus

$$\left.\frac{dz}{dr}\right|_{r=R_i} \approx -\tan 11^\circ \approx -0.19438 \approx -\frac{1}{2}\gamma_s^{-1}\nu_{cr}\alpha_2\omega_0^2\left(\frac{1}{R_i}-\frac{1}{R_0}\right)^{-2} R_i^{-3}$$

or

$$+0.19438 \approx \frac{0.5\times 4\times \pi^2\times 12}{2\times 119}\left(1-\frac{1}{3}\right)^{-2}\alpha_2 \approx 2.2393\alpha_2$$

or

$$\boxed{\alpha_2 \approx 0.0868\,\mathrm{Lb\,sec^2/ft^2}}$$

Substituting this value of α_2 in expression (34) we find that

$$\left.\frac{dz}{dr}\right|_{r=R_0} = g^{-1}\omega_0^2 \left[\left(\frac{R_0}{R_i^2} + \frac{1}{R_0} - \frac{2}{R_i}\right) - \frac{1}{2}\,\varrho_s^{-1}\nu_{cr}R_0^{-3}\alpha_2\right]\left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2} \approx 0.2993$$

or

$$\tan\theta_{1(0)} \approx 0.2993 \Rightarrow \boxed{\theta_{1(0)} \approx 16.7^\circ}$$

which checks with the experimental results. The location for the minimum z to occur (Fig. 1) is obtained by setting $\frac{dz}{dr}$ equal to zero with α_2 being 0.087 Lb sec²/ft² and is found to be at $r \approx 1.0715''$ which is different from the experimentally found location as shown in Fig. 1.

We note that the above predictions are entirely different from the ones obtained by Serrin (42) for the Couette flow of a simple non-Newtonian fluid corresponding to this case.

It is interesting to note that the slope at $r = R_0$ depends on the sign of the following expression

$$A^* \equiv \varrho_{cr}\left(\frac{R_0}{R_i} - 1\right)^2 - \frac{1}{2}\,\alpha_2\nu_{cr}^2R_0^{-2}. \tag{36}$$

Obviously, for sufficiently large R_0/R_i the above expression could be positive as is the case for our experiments (Figs. 1 and 2). However for smaller values of R_0/R_i the above expression could be negative unless

$$\alpha_2 < 2\varrho_{cr}\nu_{cr}^{-2}\left(\frac{R_0}{R_i} - 1\right)^2 R_0^2. \tag{37}$$

Case 2 — Inner Cylinder Rotating — Outer Cylinder Fixed

Since in this case $\omega' < 0$, we find from Eq. (21) that

$$Ar^{-2} = -\frac{1}{4}\,\alpha_1\nu_{cr}^2r^2\omega'^2. \tag{38}$$

Thus, this can be integrated to yield

$$\omega(r) = -2\sqrt{(-A/\alpha_1)}\,\nu_{cr}^{-1}r^{-1} + C, \tag{39}$$

where C is a constant of integration. Again this solution is different from the one obtained by Serrin [42] for a simple non-Newtonian fluid corresponding to this special case. Applying the no-slip boundary conditions

$$\omega(R_i) = \omega_i, \qquad \omega(R_0) = 0, \tag{40}$$

we obtain from Eq. (39) that

$$C = 2\sqrt{-A/\alpha_1}\,\nu_{cr}^{-1}R_0^{-1}, \qquad A = \frac{-\alpha_1\omega_i^2\nu_{cr}^2}{4}\left(\frac{1}{R_0} - \frac{1}{R_i}\right)^{-2}. \tag{41.1, 41.2}$$

Thus, in this case:

$$\omega(r) = \omega_i\left(\frac{1}{R_0} - \frac{1}{R_i}\right)^{-1}\left(\frac{1}{R_0} - \frac{1}{r}\right). \tag{42}$$

From Eqs. (41.1, 41.2) we note that

$$C = \omega_i \nu_{cr}^{-1} R_0^{-1} \left(\frac{1}{R_0} - \frac{1}{R_i}\right)^{-2} < 0, \qquad \left(\frac{-A}{\alpha_i}\right) > 0. \tag{43}$$

Shape of the Free Surface: Case 2

Considering Eq. (23.2) again and simplifying it for the free surface, for which $p =$ constant, we obtain

$$\frac{dz}{dr} = \varrho_{cr}^{-1} g^{-1} \left[\varrho_{cr} r \omega^2 + \frac{1}{4} \alpha_2 \nu_{cr}^2 \frac{d}{dr} (r^2 \omega'^2)\right]. \tag{31.2}$$

From Eqs (31.2) and (42) we obtain that

$$\begin{aligned} z = \varrho_{cr}^{-1} g^{-1} \Bigg[& \varrho_{cr} \omega_i^2 \left(\frac{1}{R_0} - \frac{1}{R_i}\right)^{-2} \left(\frac{r^2}{2R_0^2} + \ln r - \frac{2r}{R_0}\right) \\ & + \frac{1}{4} \alpha_2^2 \nu_{cr}^2 \omega_i^2 \left(\frac{1}{R_0} - \frac{1}{R_i}\right)^{-2} r^{-2} + B \Bigg]. \end{aligned} \tag{44}$$

Of interest again is the slope of the free surface, i.e., $\frac{dz}{dr}$ which is given exactly by

$$\begin{aligned} \frac{dz}{dr} = \varrho_{cr}^{-1} g^{-1} \Bigg[& \varrho_{cr} \omega_i^2 \left(\frac{1}{R_0} - \frac{1}{R_i}\right)^{-2} \left(\frac{r}{R_0^2} + \frac{1}{r} - \frac{2}{R_0}\right) \\ & - \frac{1}{2} \alpha_2 \nu_{cr}^2 \omega_i^2 \left(\frac{1}{R_0} - \frac{1}{R_i}\right)^{-2} r^{-3} \Bigg]. \end{aligned} \tag{45}$$

At $r = R_0$ we obtain from Eq. (45) that

$$\left.\frac{dz}{dr}\right|_{r=R_0} = -\frac{1}{2} \alpha_2 \nu_{cr}^2 \omega_i^2 \left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2} R_0^{-3} g^{-1} \varrho_{cr}^{-1}, \tag{46}$$

and again depending on the sign of α_2 the fluid tends to climb up or down the outer cylinder wall, accordingly, if $\alpha_2 > 0$ or $\alpha_2 < 0$, respectively. Our experimental results with iodized salt (Figs. 3 and 4) indicate again that $\alpha_2 = 0.087 \frac{\text{Lb sec}^2}{\text{ft}^2}$. Again we here present some specific results concerning this second case. We use the same particles for which $\nu_{cr} \approx 0.5$, $\gamma_s \approx 119$ Lb/ft³. For an $\omega_i \approx 20\pi$ radians/second, and $R_i = 1''$, $R_0 = 3''$, we measured $\theta_{2(i)} \approx 43°$, and $\theta_{2(0)} \approx -36°$. In order to check the validity of our theory for this second case we calculate the value of α_2 from expression (46) and substituted it in expression (45) evaluated at $r = R_i$ to find the corresponding $\theta_{2(i)}$ so see whether it matches the experimentally measured value of $\theta_{2(i)}$. Thus

$$\left.\frac{dz}{dr}\right|_{r=R_0} = \tan 36° = -0.72654 = -\frac{1}{2} \gamma_s^{-1} \nu_{cr} \alpha_2 \omega_i^2 \left(\frac{1}{R_i} - \frac{1}{R_0}\right)^{-2} R_0^{-3}$$

or

$$0.72654 = \frac{0.5 \times 0.5 \times 400\pi^2 \times 9 \times 12}{4 \times 27 \times 119} \alpha_2 = 8.2938 \alpha_2$$

or

$$\boxed{\alpha_2 \approx 0.0871 \text{ Lb sec}^2/\text{ft}^2}$$

which is close to the value found from the experimental results of the first case. Substituting the value of $\theta_{2(i)}$ in expression (45) we find that

$$\left.\frac{dz}{dr}\right|_{r=R_i} = g^{-1}\omega_i^2\left[\left(\frac{R_i}{R_0^2}+\frac{1}{R_i}-\frac{2}{R_0}\right)-\frac{1}{2}\,\varrho_s^{-1}\nu_{cr}\alpha_2R_i^{-3}\right]\left(\frac{1}{R_i}-\frac{1}{R_0}\right)^{-2}$$

$$\approx \tan 43^\circ \approx 0.9325$$

or

$$10.216982 - 223.932\alpha_2 = 0.9325 \Rightarrow \boxed{\alpha_2 = 0.0415\,\frac{\text{Lb sec}^2}{\text{ft}^2}}$$

which is not consistent with our previous results. However the order of magnitude of α_2 is the same as before. Again the location for the maximum z to occur (Fig. 2)

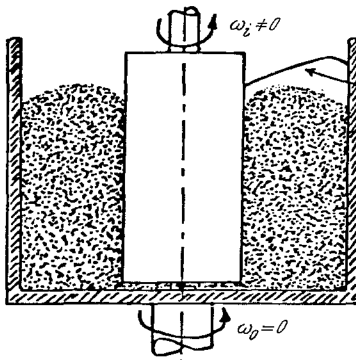

Fig. 3. The shape of the free surface corresponding to case 2, i. e., Inner cylindnr rotating — outer cylinder fixed, $\omega_i = 20\pi$ rad/sec, $R_i = 1''$, $R_0 = 3''$

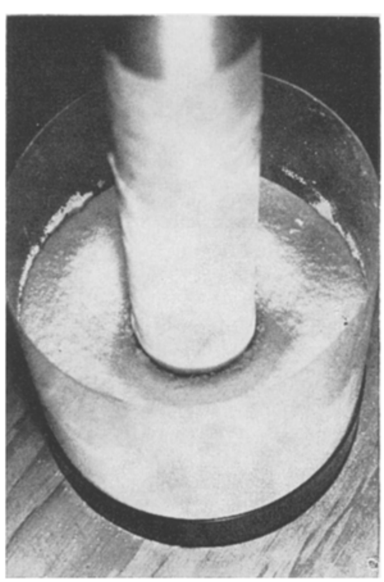

Fig. 4. Photograph of the experimental set up corresponding to case 2

is obtained by setting $\frac{dz}{dr}$ equal to zero with α_2 being $0.087 \frac{\text{Lb sec}^2}{\text{ft}^2}$ and is found to be at $r \approx 2.88''$ which is different from the experimentally found location as shown in Fig. 3.

Again we note that the above predictions are different from the ones obtained by Serrin [42] for the Couette flow of a simple non-Newtonian fluid corresponding to this case. The slope at $r = R_i$ depends on the sign of the following expression

$$B^* = \varrho_{cr} \left(\frac{R_i}{R_0} - 1\right)^2 - \frac{1}{2} \alpha_2 \nu_{cr}^2 R_i^{-2}. \tag{47}$$

Therefore, for sufficiently small R_i/R_0 the above expression could be positive as is the case for our experiment (Figs. 3 and 4).

For larger values of R_i/R_0 the above expression (47) can be negative unless

$$\alpha_2 < 2\varrho_{cr}\nu_{cr}^{-2} \left(\frac{R_i}{R_0} - 1\right)^2 R_i^2. \tag{48}$$

Further experimental results are needed for various granular materials to establish the validity of the above conclusions.

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