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Dynamic Expansion of a Compressible Hyperelastic Spherical Shell

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With 5 Figures

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Summary — Zusammenfassung

Dynamic Expansion of a Compressible Hyperelastic Spherical Shell. This paper is concerned with the finite spherically symmetric motion of a compressible hyperelastic spherical shell, subjected to a spatially uniform step function application of pressure at its inner surface. A method, given in a previous paper [1], for the determination of the field of characteristics, for expansion of a spherical cavity in an unbounded solid is adapted to consider the spherical shell problem. Results are presented graphically for a particular strain energy function and are compared with results obtained for an incompressible material and from linear elasticity theory.

Dynamische Aufweitung einer kompressiblen hyperelastischen Kugelschale. Diese Arbeit befaßt sich mit der endlichen, kugelförmig-symmetrischen Bewegung einer kompressiblen hyperelastischen Kugelschale, die auf der Innenseite durch eine räumlich gleichförmige Druck-Sprungfunktion belastet wird. Eine Methode, wie sie in einer früheren Arbeit [1] zur Bestimmung des Charakteristikenfeldes für die Aufweitung eines kugelförmigen Hohlraumes in einem unendlich ausgedehnten Festkörper gegeben ist, wird für die Behandlung des Kugelschalenproblems erweitert. Die Resultate werden graphisch für eine besondere Verzerrungsenergiefunktion angegeben und mit Resultaten für inkompressibles Material und Resultaten der linearen Elastizitätstheorie verglichen.

I. Introduction

The finite spherically symmetric motion of a spherical shell is considered in this paper. It is assumed that the shell is composed of an isotropic compressible hyperelastic material, homogeneous in the undeformed state, and that it is expanded by a spatially uniform step function application of pressure at the inner surface. In previous papers [1], [2] the authors considered the expansion of a spherical cavity in an unbounded compressible hyperelastic solid. The governing equations in Lagrangian form for the method of characteristics and a numerical procedure for the determination of the field of characteristics and shock path in the characteristic plane are given in [1] for a spatially uniform step function application of pressure at the cavity wall. An approximate method based on a discrete model is proposed in [2] and this method appears to have some computational advantages over the method of characteristics for certain problems of spherically symmetric motion of an unbounded hyperelastic medium. However, it was found that the method of characteristics is more easily adaptable to consider the motion of a spherical shell and an extension of the method of characteristics described in [1] is given in this paper.

It was shown in [1] that a shock is initiated at the cavity wall when a step function application of pressure is applied if the strain energy function satisfies certain physically reasonable conditions. When a step function application of pressure is applied at the inner surface of a spherical shell the response before the shock reaches the outer surface, is the same as for the expansion of a spherical cavity in an unbounded medium. After the shock is reflected from the outer surface of the shell it propagates radially inwards into deformed material in motion and undergoes further reflections from the inner and outer surfaces.

The corresponding problem in linear elasticity theory has been considered by Rose, Chou and Chou [3] who presented numerical results obtained by the method of characteristics. These results show that the motion of a thick walled compressible Hookean spherical shell due to a step function application of internal pressure is not periodic. The spherically symmetric motion of an incompressible elastic spherical shell has been considered by Guo and Solecki [4], Eringen and Suhubi [5] and for a thin walled shell, by Wang [6]. It follows from the analyses given in [4], [5], and [6] that if a radial oscillation results from a step function application of internal pressure this oscillation is periodic. The response of an incompressible spherical shell to a sudden application of internal pressure is, in a sense, qualitatively different from that of a compressible shell since the effect of the pressure is felt instantaneously throughout the shell whereas for a compressible shell a shock wave propagates back and forth. Nevertheless, it is reasonable to expect that the response of a compressible shell should approximate that for an incompressible shell if the compressibility is small. Numerical results presented in this paper confirm this expectation.

II. Governing Equations

Notation used in this paper is the same as in [1] with the radial coordinate of a point in the sphere denoted by (R, r) in the (undeformed, deformed) state and the motion is given by

$$r = r(R, t).$$

The strain energy per unit undeformed volume is expressed as a function

$$W = W(\delta, \lambda)$$

of the radial stretch $\delta = dr/dR$ and the circumferential stretch $\lambda = r/R$ and the nominal radial and circumferential stresses are given by

$$P = W_\delta/\lambda^2 \quad \text{and} \quad Q = W_\lambda/(2\delta\lambda)$$

respectively, where the subscripts denote partial differentiation. Governing equations in Lagrangian form are given in [1] and are summarized as follows,

$$\frac{\partial P}{\partial R} + \frac{2(P - Q)}{R} = \varrho_0 \frac{\partial v}{\partial t} \tag{1}$$

which is the equation of motion and

$$\frac{\partial P}{\partial t} = W_{\delta\delta} \frac{\partial v}{\partial R} + W_{\delta\lambda} \frac{v}{R}, \tag{2}$$

$$2 \frac{\partial Q}{\partial t} = W_{\delta\lambda} \frac{\partial v}{\partial R} + W_{\lambda\lambda} \frac{v}{R}. \tag{3}$$

In Eqs. (1), (2) and (3) ϱ_0 is the density in the undeformed state, t is time and $v = \partial r/\partial t$ is the particle velocity. Eqs. (1), (2) and (3) are a set of totally hyperbolic quasi-linear first order partial differential equations with dependent variables P, Q and v and independent variables R and t. The characteristics in the (R, t) plane are given by the differential equations

$$\frac{dR}{dt} = \zeta^+, \quad \frac{dR}{dt} = \zeta^- \quad \text{and} \quad \frac{dR}{dt} = \zeta^0$$

where

$$\zeta^+ = C(\lambda, \delta), \qquad \zeta^- = -C(\lambda, \delta), \qquad \zeta^0 = 0$$

and

$$C = \sqrt{(W_{\delta\delta}/\varrho_0)}.$$

It follows that the ζ^0 characteristics are straight lines parallel to the t axis but the ζ^+ and ζ^- characteristics are not in general straight lines and must be determined as part of the solution.

It is convenient to introduce the following non-dimensional quantities $\overline{R} = R/A$, $\overline{W} = W/\mu$, $\overline{P} = P/\mu$, $\overline{Q} = Q/\mu$, $\overline{p} = p/\mu$,

$$\overline{v} = v\sqrt{\frac{\varrho_0}{\mu}}, \quad \overline{C} = C\sqrt{\frac{\varrho_0}{\mu}} = \overline{W}_{\delta\delta}, \quad \overline{t} = \frac{t}{A}\sqrt{\frac{\mu}{\varrho_0}}, \quad n = B/A$$

where A and B are the inner and outer radii respectively in the undeformed state, μ is the modulus of rigidity for infinitesimal deformation from the undeformed state and p is the spatially uniform pressure applied at $R = A$. The non-dimensional relations along the characteristics are

$$\left\{\frac{\overline{v}}{\overline{R}} \overline{W}_{\delta\lambda} \mp 2\left(\frac{\overline{P} - \overline{Q}}{\overline{R}}\right) \overline{C}\right\} d\overline{R} + \overline{C}^2 \, d\overline{v} + \overline{C} \, d\overline{P} = 0 \quad \text{along} \quad \zeta^\pm \tag{4}$$

and $$\left\{d\overline{P} - \overline{W}_{\lambda\delta} \frac{\overline{v}}{\overline{R}} \, d\overline{t}\right\} \overline{W}_{\lambda\delta} - \left\{2d\overline{Q} - \overline{W}_{\lambda\lambda} \frac{\overline{v}}{\overline{R}} \, dt\right\} \overline{C}^2 = 0 \quad \text{along} \quad \zeta^0. \tag{5}$$

Relations along the characteristics, given by Chou and Koenig [7], for the corresponding spherically symmetric problem in linear elasticity can be obtained as a limiting case of Eqs. (4) and (5) by neglecting terms $0(e^2)$ where $e^2 = (\delta - 1)^2 + 2(\lambda - 1)^2$. The ζ^+ and ζ^- characteristics are straight lines in the linear elasticity problem since $\overline{C}$ is constant and, unlike the finite deformation non-linear problem, the field of characteristics does not have to be determined as part of the solution.

III. Strong Discontinuities

In the problem considered a shock moves radially outwards into undeformed material at rest after the sudden application of pressure occurs at $R = A$. When this shock reaches the outer surface $R = B$ it is reflected and after this first reflection moves into deformed material in motion. Consequently the relations across a shock or strong discontinuity propagating into deformed material in motion must be considered. The stresses P and Q, particle velocity v and the radial stretch δ are discontinuous across a shock but the circumferential stretch λ is continuous since the displacement is continuous. The isentropic approximation is adopted consequently only two relations across the shock are required, the momentum equation

$$-[P]N = \varrho_0 V[v] \tag{6}$$

and the compatibility equation

$$[v] + V[\delta - 1]N = 0, \tag{7}$$

where V is the speed (Lagrangian) of propagation of the shock and $N = 1$ for outward propagation and $N = -1$ for inward propagation. In Eqs. (6) and (7) the square brackets have the significance $[A] = A_+ - A_-$ where A_+ and A_- are the values of A just behind and ahead of the shock respectively. It follows from Eqs. (6) and (7) that

$$V = \left\{\frac{[P]}{\varrho_0[\delta]}\right\}^{1/2}$$

or in non-dimensional form

$$\bar{V} = V\sqrt{\frac{\varrho_0}{\mu}} = \left\{\frac{[\bar{P}]}{[\delta]}\right\}^{1/2}.$$

In the linear elasticity problem $\bar{V}$ is constant and $\bar{V} = \bar{C}$ so that the path, in the $(\bar{R}, \bar{t})$ plane of a strong discontinuity, that is a discontinuity of radial stress, coincides with a ζ^+ or ζ^- characteristic.

Numerical results are obtained for a spatially uniform application of pressure at $\bar{R} = 1$ given in non-dimensional form by

$$\bar{p}(\bar{t}) = \bar{q}H(\bar{t}), \tag{8}$$

where $H(\bar{t})$ is the unit step function and $\bar{q}$ is a constant. The stress boundary conditions are

$$\bar{P}(1, \bar{t}) = -\left(\frac{a}{A}\right)^2 \bar{q}H(\bar{t}) \tag{9}$$

and

$$\bar{P}(n, \bar{t}) = 0, \tag{10}$$

where $a = a(t)$ is the inside radius in the deformed state and a/A is the circumferential stretch at $\bar{R} = 1$. It is assumed that the strain energy function satisfies the conditions

$$(\bar{W}_{\delta\delta})_{\lambda=1} > 0 \quad \text{and} \quad (\bar{W}_{\delta\delta\delta})_{\lambda=1} < 0,$$

and as indicated in [1] a shock is initiated at $\bar{t} = 0$, $\bar{R} = 1$ for a spatially uniform application of internal pressure given by Eq. (8).

IV. Consideration of Reflected Shock Waves

In [1] a numerical procedure is given for the determination of the shock path and field of characteristics in the $(\bar{R}, \bar{t})$ plane for the expansion of a spherical cavity in an unbounded medium. An extension of this procedure to consider a thick-walled spherical shell is given in this section and reference is made to Fig. 1 which gives a diagrammatic representation of the $(\bar{R}, \bar{t})$ plane.

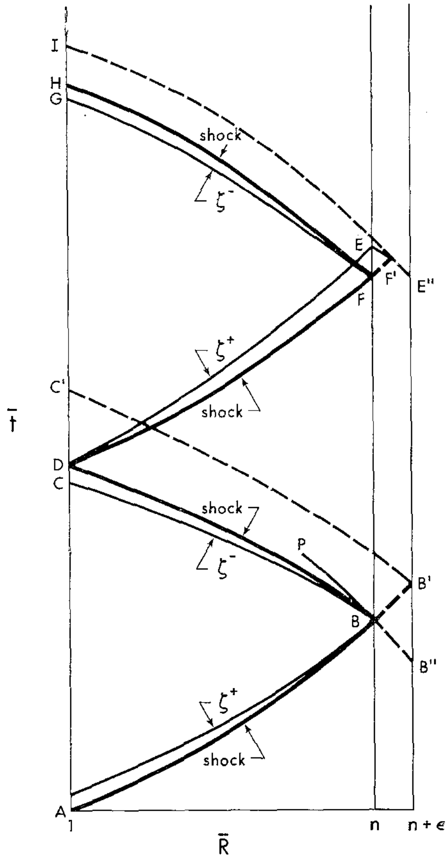

Fig. 1. Schematic representation of field of characteristics and the extensions required to determine the path of reflected shock at $\bar{R} = n$

The shock which is initiated at $\bar{R} = 1$ and $\bar{t} = 0$ propagates radially outwards into undeformed material at rest until it undergoes the first reflection at $\bar{R} = n$ and $\bar{t}_B$, where subscripts refer to points in Fig. 1. Until the first reflection occurs the response is identical to that for an unbounded medium and the shock path AB and the field of characteristics in region ABC are obtained as described in [1]. It may be deduced from the boundary condition (10), the jump relations (6) and (7) and the continuity of λ that the value of $\bar{V}$ for the reflected shock at B is the same as for the incident shock and that $\bar{V} < \bar{C}$ at B where $\bar{C}$ is evaluated just behind the incident shock or just ahead of the reflected shock at B. It follows that the reflected shock path BD lies above the ζ^- characteristic BC. These conclusions are also valid for subsequent reflections of the shock at $\bar{R} = n$. The shock path BD and the field of characteristics in the region BDE can be determined by the numerical procedure described in [1] if the particle velocity and stretches (or stresses) ahead of she shock are known since the slope of BD at B and the stress boundary condition (10) are known. This means the field of characteristics in BCD must be determined. This is done by extending the field of characteristics in ABC into $BB'C'C$ and for the first reflection the field in $ABB'C'$ is the same as for the unbounded medium. When this field in $BB'C'C$ has been obtained the shock path BD and field of characteristics in BDE bounded by BD, the ζ^0 line BE and the ζ^+ line through D is found from the numerical procedure. The extension ε in the $\bar{R}$ direction must be sufficiently large that $t_{C'} > t_D$. After the field in BDE has been determined the part $B'C'DB$ of the original field obtained in $BB'CC'$ is discarded. An extension $BB''E''E$ of the field in BDE is required and this is easily obtained by using the data along BE and noting that BB'' is an extension of the ζ^- PB through B with $\bar{P}(n + \varepsilon, \bar{t}_{B''}) = 0$. This extension into region $B''BEE''$ represents the response of a sphere with the ratio of undeformed outer radius to undeformed inner radius equal to $(n + \varepsilon)/n$ with zero prescribed stress at the inner radius and particle velocity at inner radius given by data along BE. At time t_D the shock front is reflected from the inner surface. It may be deduced from the boundary condition (9), the jump relations (6) and (7) and the continuity of $\dot{\lambda}$ that $\bar{V}$ for the reflected shock is the same as for the incident shock and that $\bar{V} > \bar{C}$ at D where $\bar{C}$ is evaluated just behind the incident shock at D or just ahead of the reflected shock. It follows that the reflected shock path DF lies below the ζ^+ line DE. These conclusions are also valid for subsequent reflections of the shock at $\bar{R} = 1$. Consequently the shock path DF and the field of characteristics in DFG can be obtained by the numerical method since initial slope of DF is known and the stretches and particle velocity ahead of the shock are known. When the field in DFG is obtained the original field in DFE is discarded.

The procedure for further reflections is the same as for the first two reflections except for a minor difference for reflections at $\bar{R} = \bar{n}$. For example for the second reflection at $\bar{R} = \bar{n}$ the extension of field in DFG into $FF'IG$ requires the data in the extension into $B''BEE''$ of the field in BDE whereas for the first reflection at B the extended field in $BB'CD$ is the same as for the unbounded medium. It is necessary that the extension ε be sufficiently large that $t_I > t_H$ and this can only be insured by trial and error.

V. Numerical Results

Numerical results were obtained from a Fortran IV program in double precision mode for the strain energy function

$$\bar{W} = \frac{1}{2}\left\{(I_1 - 3) + \frac{(1 - 2\nu)}{\nu}\left(I_3^{\frac{\nu}{2\nu - 1}} - 1\right)\right\}, \tag{11}$$

where ν is Poisson's Ratio for infinitesimal deformation from the undeformed state and

$$I_1 = 2\lambda^2 + \delta^2 \quad \text{and} \quad I_3 = \lambda^4\delta^2$$

for spherically symmetric deformation. This is the strain energy function for which results were presented in [1] for the expansion of a spherical cavity in an unbounded medium. The neo-Hookean strain energy function is obtained as a limiting case of Eq. (11) as ν approaches 0.5.

It is well known that an equilibrium state for a neo-Hookean spherical shell subjected to non-dimensional internal pressure $\bar{q}$ is possible only if $\bar{q}$ does not exceed a critical value which depends on B/A. For example the critical value of $\bar{q}$ for $B/A = 2$ is $\bar{q}_{\text{crit}} = 0.8164$. A compressible spherical shell with strain energy function (11) also has a critical internal pressure above which there is no equilibrium state and this pressure, obtained from a numerical method due to Haddow and Faulkner [8], is $\bar{q}_{\text{crit}} = 0.6456$ for $B/A = 2$ and $\nu = 0.3$. It is clear that a necessary but not sufficient condition for an oscillatory response when a spatially uniform step function application of pressure is applied at the inner surface is that this pressure be less then $\bar{q}_{\text{crit}}$. Necessary and sufficient conditions for the periodic oscillation of an incompressible walled shells have been given Guo and Solecki [4]. The results presented in this paper are for pressures which are sufficiently low that an oscillation occurs about the equilibrium state.

Results for neo-Hookean spherical shells are presented for comparison purposes and these results were obtained from numerical integration of the governing non-linear differential equation obtained by Guo and Solecki [4].

Results obtained for $n = 2$, $\bar{q} = 0.001$ and $\nu = 0.3$ indicate that this non-dimensional pressure produces small deformation. These results are almost

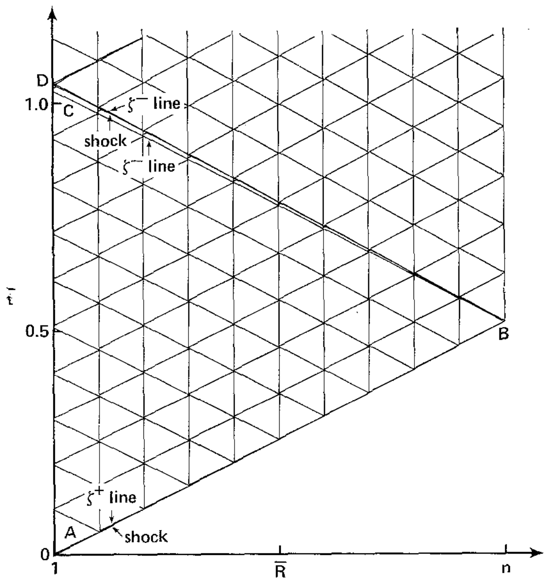

Fig. 2. Field of characteristics for $n = 2$, $\nu = 0.3$, $\bar{q} = 0.25$

identical to those for linear elasticity theory and this provides a check on the procedure since the response of an isotropic hyperelastic solid approaches that for an isotropic linearly elastic solid as the deformation approaches zero.

In Fig. 2 the field of characteristics and shock front path, obtained by the procedure described in section IV is shown for $n = 2$, $\bar{q} = 0.25$ and $\nu = 0.3$. The points lettered in Fig. 2 correspond to those in Fig. 1.

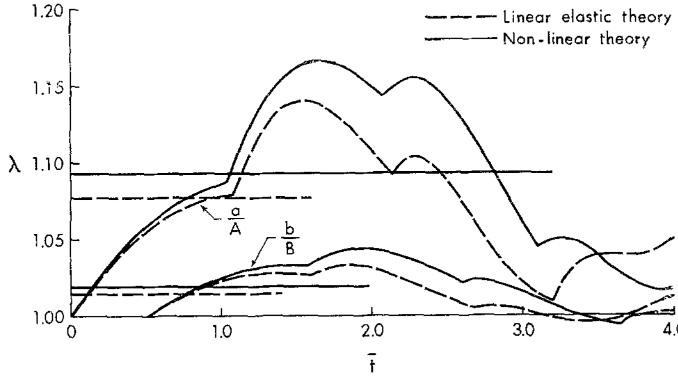

Fig. 3. Response of inner and outer surfaces for $n = 2$, $\nu = 0.3$, $\bar{q} = 0.25$

The stretches at the inner and outer radii along with the quasi-static or equilibrium stretches obtained from an extension of the numerical method described in [8] are shown graphically in Fig. 3 for $n = 2$, $\bar{q} = 0.25$ and $\nu = 0.3$. Cusps in the curves correspond to reflections of the shock. Also shown in Fig. 3 are results, for linear elasticity, obtained by the method of characteristics described by Chou and Koenig [7], adapted to consider a spherical shell. In Fig. 3 b denotes the outer radius in the deformed state.

The response of a neo-Hookean spherical shell with $n = 2$, $\bar{q} = 0.25$ is shown for comparison in Fig. 4.

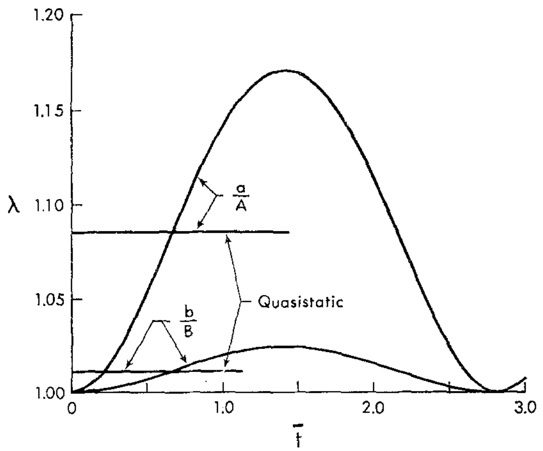

Fig. 4. Response of inner and outer surfaces for neo-Hookean shell with $n = 2$, $\bar{q} = 0.25$

In Fig. 5, stretches at the inner radius are shown graphically for $n = 2$, $\bar{q} = 0.1$, $\nu = 0.45$, $\nu = 0.48$ and the neo-Hookean solid. These results indicate that as ν approaches 0.5 in strain energy function (11) the response of the shell approaches that of a neo-Hookean shell. However, as ν approaches 0.5 difficulties arise in the numerical method since $\bar{V}$ and $\bar{C}$ approach infinity and it was not possible to obtain results for $\nu > 0.48$ because of numerical difficulties.

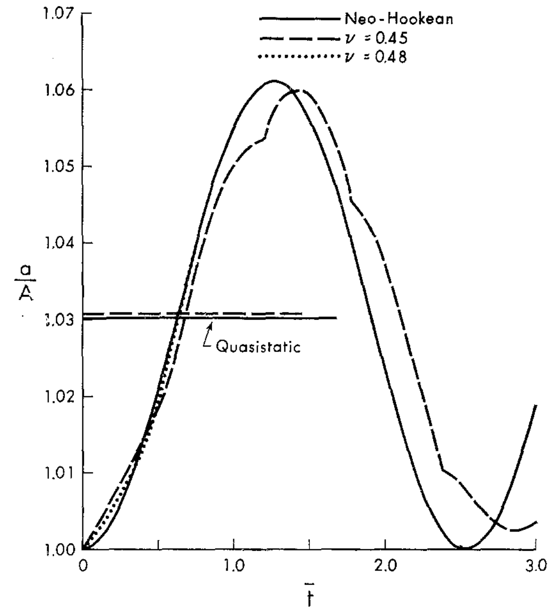

Fig. 5. Response of inner surface for $n = 2$, $\bar{q} = 0.1$, $\nu = 0.45$, $\nu = 0.48$ and neo-Hookean

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