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Inflation Bending Extension and Azimuthal Shearing of a Fiber-Reinforced Elastic Sector of a Circular Tube

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Summary — Zusammenfassung

Inflation Bending Extension and Azimuthal Shearing of a Fiber-Reinforced Elastic Sector of a Circular Tube. Universal solutions for the deformation of inflation, bending, extension and azimuthal shearing of a fiber-reinforced sector of a circular tube are obtained by inverse methods. The material is assumed to be isotropic and non-linear elastic, incompressible or compressible. The reinforcement consists of inextensible fibers along the R or Θ or Z directions or in one or two sets of helical path. The surface tractions and the corresponding forces to support the deformation are determined. A discussion is made about the strengthening of the material due to the reinforcement, the sign of the tension in the fibers and the deformed configuration of the fibers.

Aufweitung, Biegung, Längsdehnung und azimutaler Schub eines faserverstärkten elastischen Sektors eines Kreiszylinderrohres. Allgemeine Lösungen dieses Problems werden durch Umkehrmethoden erhalten. Der Werkstoff wird isotrop und nichtlinear-elastisch, kompressibel oder inkompressibel vorausgesetzt, mit Faserverstärkung in R- oder Θ- oder Z-Richtung oder in einer Schar oder in zwei Scharen schraubenförmig. Oberflächenspannungen und die der Deformation entsprechenden Kräfte werden bestimmt. Diskutiert wird die Verstärkung des Körpers durch die Fasern, das Vorzeichen der Normalkraft in den Fasern und die verformte Gestalt der Fasern.

1. Introduction

Universal solutions, that is solutions of controllable deformations independent of the material behavior for fiber-reinforced isotropic elastic materials, incompressible or compressible under large elastic deformations, were obtained by Beskos [1], [2]. In those papers only the first four families of the non-homogeneous deformations of Ericksen [3] yielding universal solutions for incompressible unreinforced elastic materials were investigated. However, it was found later by Singh and Pipkin [4] that there exists one more such family, namely the inflation, bending extension and azimuthal shearing of a sector of a circular tube. For this deformation the possibility of existence of universal solutions, for various systems of reinforcement and for the material being isotropic elastic incompressible, is investigated in this paper.

The fibers are assumed to be thin, flexible and inextensible filling the material continuously and completely. The reinforced body is treated as a material subject to internal constraints according to the theory developed by Ericksen and Rivlin [5] and also contained in Truesdell and Noll [6].

We prove, under these assumptions, that for the incompressible case there exist universal solutions corresponding to reinforcement consisting of fibers along the R, Θ, Z directions and in one or two sets of helical path, while no universal solution exists for the compressible case.

The strengthening of the material due to the fiber reinforcement is of engineering importance. The reinforcement is considered to be significant if the fiber tension is of the same order of magnitude with the maximum extra stress with respect to the constants of the deformation.

For reasons of stability the fiber tension must be positive and this can be the case under certain conditions among the various constants of the deformation and the response coefficients of the constitutive equation of the material. Of course, satisfaction of these conditions, in conjunction with existence of solution of the boundary relations determining the constants of the deformation, depends on the nature of the response coefficients of the material.

2. Preliminaries

The static configuration of a body is determined by the invertible transformation

$$x^i = x^i(X^\alpha)^1. \tag{2.1}$$

The deformation gradient $F_\alpha{}^i$ is defined as

$$F_\alpha{}^i = x^i_{,\alpha} \tag{2.2}$$

and the right and left Cauchy-Green deformation tensors are given by

$$C_{\alpha\beta} = g_{ij} F_\alpha{}^i F_\beta{}^j, \tag{2.3}$$

$$B^{ij} = G^{\alpha\beta} F_\alpha{}^i F_\beta{}^j. \tag{2.4}$$

Here $G^{\alpha\beta}$ and g_{ij} are the metric tensors in the reference and present configuration, respectively.

The stress tensor $\boldsymbol{T}$ for a homogeneous fiber-reinforced elastic material takes the form [6]

$$\boldsymbol{T} = \boldsymbol{S} - \boldsymbol{N}, \tag{2.5}$$

where $\boldsymbol{S}$ is the extra stress given by the constitutive equation of the material and $\boldsymbol{N}$ is the stress due to the constraints. For a compressible material $\boldsymbol{N}$ is a reaction stress due to the reinforcement given by

$$N^{ij} = q F_\alpha{}^i F_\beta{}^j e^\alpha e^\beta \tag{2.6}$$

and for an incompressible material $\boldsymbol{N}$ is a reaction stress plus a hydrostatic pressure due to the incompressibility given by

$$N^{ij} = q F_\alpha{}^i F_\beta{}^j e^\alpha e^\beta + p\delta^{ij}. \tag{2.7}$$

[1] Here and throughout, Greek indices refer to material coordinates X^α, Latin to the spatial coordinates x^i, summation convention is used and a comma indicates differentiation.

In Eqs. (2.6) and (2.7) p and q are scalar functions of the coordinates and $\boldsymbol{e}$ is a unit vector in the reference configuration which is tangent at each point to the fiber passing from that point.

For a compressible isotropic elastic material [6]

$$S_{<ij>} = \beta_0 \delta_{ij} + \beta_1 B_{<ij>} + \beta_{-1} (B^{-1})_{<ij>}, \tag{2.8}$$

while for an incompressible isotropic elastic material

$$S_{<ij>} = \beta_1 B_{<ij>} + \beta_{-1} (B^{-1})_{<ij>}, \tag{2.9}$$

in physical components[2].

In the Eqs. (2.8) and (2.9), the response coefficients β_0, β_1 and β_{-1} are functions of the three principal invariants of the tensor $\boldsymbol{B}$ satisfying the inequalities [6]

$$\beta_0 \leqq 0, \quad \beta_1 > 0, \quad \beta_{-1} \leqq 0. \tag{2.10}$$

The conditions of the constraints of incompressibility and inextensibility imposed on the deformation are

$$\det [B_{<ij>}] = 1, \tag{2.11}$$

$$C_{\alpha\beta} e^\alpha e^\beta = 1, \tag{2.12}$$

respectively.

For our cylindrical deformation the equations of equilibrium in a cylindrical system, r, θ, z, for zero body forces, take the form

$$\left.\begin{aligned}
&\frac{\partial T_{<rr>}}{\partial r} + \frac{1}{r}\frac{\partial T_{<r\theta>}}{\partial \theta} + \frac{\partial T_{<rz>}}{\partial z} + \frac{T_{<rr>} - T_{<\theta\theta>}}{r} = 0 \\
&\frac{\partial T_{<r\theta>}}{\partial r} + \frac{1}{r}\frac{\partial T_{<\theta\theta>}}{\partial \theta} + \frac{\partial T_{<\theta z>}}{\partial z} + \frac{2}{r} T_{<r\theta>} = 0 \\
&\frac{\partial T_{<rz>}}{\partial r} + \frac{1}{r}\frac{\partial T_{<\theta z>}}{\partial \theta} + \frac{\partial T_{<zz>}}{\partial z} + \frac{1}{r} T_{<rz>} = 0
\end{aligned}\right\} \tag{2.13}$$

3. Incompressible Materials

By taking both the reference and present configuration of the body to refer to cylindrical coordinate systems R, Θ, Z and r, θ, z respectively and the constants A, B, C and D to correspond to inflation, azimuthal shearing, bending and extension, respectively, the following deformation of an incompressible elastic body is considered [4]:

$$r = AR, \quad \theta = B \log R + C\Theta, \quad z = DZ. \tag{3.1)^3$$

[2] The physical components of a tensor $\boldsymbol{T}$, symbolized by $< >$, are given in terms of its contravariant components by $T_{<ij>} = \sqrt{g_{ii} g_{jj}}\, T^{ij}$ (unsummed).

[3] It is assumed that $B \neq 0$ because otherwise (3.1) becomes a deformation already studied in [1].

The body in its undeformed state is bounded by the curved surfaces $R = R_1$, $R = R_2$ ($R_2 > R_1$) and the planes $\Theta = \pm\Theta_0$ and $Z = \pm L$, while in the present configuration the strained body is bounded by the curved surfaces $r = r_1 = AR_1$, $r = r_2 = AR_2$ the planes $z = \pm DL$ and the surfaces $\theta = B \log R \pm C\Theta_0$.

A simple computation based on (2.2)—(2.4) yields the deformation gradient and the Cauchy-Green tensors:

$$F_\alpha^{\ i} = \begin{bmatrix} A & 0 & 0 \\ B/R & C & 0 \\ 0 & 0 & D \end{bmatrix}, \tag{3.2}$$

$$[C_{\alpha\beta}] = \begin{bmatrix} A^2(1 + B^2) & A^2B^2C^2 & 0 \\ A^2B^2C^2 & C^2r^2 & 0 \\ 0 & 0 & D^2 \end{bmatrix}, \tag{3.3}$$

$$[B_{<ij>}] = \begin{bmatrix} A^2 & A^2B & 0 \\ A^2B & A^2(B^2 + C^2) & 0 \\ 0 & 0 & D^2 \end{bmatrix}, \tag{3.4}$$

$$[(B^{-1})_{<ij>}] = \frac{1}{(A^2CD)^2} \begin{bmatrix} D^2A^2(B^2 + C^2) & -A^2D^2B & 0 \\ -A^2D^2B & A^2D^2 & 0 \\ 0 & 0 & A^4C^2 \end{bmatrix}. \tag{3.5}$$

The incompressibility condition (2.11) takes the form

$$A^2CD = 1. \tag{3.6}$$

The following types of reinforcement are considered:

a) Fibers Along the R Direction

The inextensibility condition (2.12) takes the form

$$B^2 = (1 - A^2)/A^2 \tag{3.7}$$

and the reaction stress (2.6) the form

$$[N_{<ij>}] = q \begin{bmatrix} A^2 & A^2B & 0 \\ A^2B & A^2B^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.8}$$

indicating that the fiber tension $T_f = qA^2B^2 \left(\frac{r^2 - 1}{B^2 + r^2}\right)$ directed along the tangents to the deformed fibers.

From (2.9), (3.4) and (3.5) we conclude that $S_{<rz>} = S_{<\theta z>} = 0$ and the rest components of the extra stress are constants. Thus the equilibrium Eqs. (2.13) on account of (2.5), (2.7) and (2.9) take the form

$$\left.\begin{aligned}
&-\frac{\partial p}{\partial r} - A^2 \frac{\partial q}{\partial r} - \frac{1}{r} A^2 B \frac{\partial q}{\partial \theta} + \frac{1}{r} A^2(B^2 - 1)\, q \\
&\qquad\qquad + \frac{1}{r} (S_{<rr>} - S_{<\theta\theta>}) = 0 \\
&-\frac{1}{r}\frac{\partial p}{\partial \theta} - \frac{1}{r} A^2 B^2 \frac{\partial q}{\partial \theta} - A^2 B \frac{\partial q}{\partial r} - \frac{2}{r} A^2 B q + \frac{2}{r} S_{<r\theta>} = 0 \\
&-\frac{\partial p}{\partial z} = 0.
\end{aligned}\right\} \tag{3.9}$$

By assuming that p is a function of r only, we can integrate (3.9) and get

$$q = \frac{S_{<r\theta>}}{A^2 B} + \frac{K_1}{r^2}, \tag{3.10}$$

$$p = \left[S_{<rr>} - S_{<\theta\theta>} + \frac{(B^2 - 1)}{B} S_{<r\theta>}\right] \log r - K_1 A^2 (B^2 + 1) \frac{1}{2r^2} + K_2, \tag{3.11}$$

where K_1 and K_2 are constants of integration. Thus deformation (3.1) case a) has a universal solution. The constants that have to be determined from the boundary conditions are A, C, K_1 and K_2, since B and D can be determined in terms of A and C from (3.6) and (3.7). We impose the following conditions: $T_{<rr>}|_{r=r_1} = T_{<rr>}|_{r=r_2} = 0$, $T_{<zz>}|_{r=r_1} = 0$ and $T_{<\theta\theta>}|_{r=r_1} = 0$.

The nonzero stresses of this deformation are

$$\begin{aligned}
T_{<rr>} &= -p - qA^2 + \beta_1 A^2 + \beta_{-1} D^2 A^2 (B^2 + C^2), \\
T_{<\theta\theta>} &= -p - qA^2 B^2 + \beta_1 A^2 (B^2 + C^2) + \beta_{-1} A^2 D^2, \\
T_{<zz>} &= -p + \beta_1 D^2 + \beta_{-1} A^4 C^2, \\
T_{<r\theta>} &= -qA^2 B + \beta_1 A^2 B - \beta_{-1} A^2 D^2 B,
\end{aligned}$$

with q and p given by (3.10) and (3.11), respectively. The deformation is supported by the traction $P_z = T_{<zz>}$ on the planes $z = \pm DL$ and the tractions $P_r = \frac{1}{\sqrt{B^2 + r^2}} (T_{<rr>} B + T_{<r\theta>} r)$ and $P_\theta = \frac{1}{\sqrt{B^2 + r^2}} (T_{<r\theta>} B + T_{<\theta\theta>} r)$ on the surfaces $\theta = B \log (r/A) \pm C\Theta_0$. The scalar $q = \beta_1 - \beta_{-1} D^2 + K_1 r^{-2}$, in view of (2.10), is positive provided K_1 is positive. The reinforcement is significant here and the originally radially placed fibers change horizontal level and become curved in their horizontal plane.

The above analysis includes the special cases of $C = 1$ (no bending) and $D = 1$ (no extension).

b) Fibers Along the Θ Direction

The inextensibility condition (2.12) takes the form $AC = 1$, which combined with (3.6) gives

$$C = D = 1/A. \tag{3.12}$$

The only nonzero component of the reaction stress is

$$N_{<\theta\theta>} = q, \tag{3.13}$$

indicating that the fiber tension $T_f = q$ along the θ direction. The equilibrium Eqs. (2.13) take the form

$$\left.\begin{aligned} &-\frac{\partial p}{\partial r} + \frac{1}{r}\,(q + S_{<rr>} - S_{<\theta\theta>}) = 0, \\ &-\frac{\partial p}{\partial \theta} - \frac{\partial q}{\partial \theta} + 2S_{<r\theta>} = 0, \\ &-\frac{\partial p}{\partial z} = 0. \end{aligned}\right\} \tag{3.14}$$

From $(3.14)_3$ and $(3.14)_1$ it is evident that p and q are independent of z. Thus we are able to integrate $(3.14)_1$ and $(3.14)_2$ and get

$$p = 2S_{<r\theta>}\,\theta - \frac{1}{r}\,\Psi(\theta) + (S_{<rr>} - S_{<\theta\theta>}) \log r + \int \frac{\Phi(r)}{r^2}\,dr + K_1, \tag{3.15}$$

$$q = \frac{1}{r}\,[\Phi(r) + \Psi(\theta)], \tag{3.16}$$

where $\Phi(r)$, $\Psi(\theta)$ are arbitrary functions of r and θ, respectively, and K_1 is a constant. Thus deformation (3.11) case b has a universal solution.

It would be very convenient to impose the boundary conditions $T_{<rr>}|_{r=r_2} = 0$, but this would lead to $S_{<r\theta>} = \beta_1 A^2 B - \beta_{-1} B = 0$, which is impossible in view of (2.10), unless $B = 0$, a case studied in [1]. This observation is also true for all the subsequent cases. The nonzero stresses of this deformation are:

$$\begin{aligned} T_{<rr>} &= -p + \beta_1 A^2 + \beta_{-1}(B^2 + 1/A^2), \\ T_{<\theta\theta>} &= -p + q + \beta_1 A^2(B^2 + 1/A^2) + \beta_{-1}, \\ T_{<zz>} &= -p + \beta_1/A^2 + \beta_{-1} A^2, \\ T_{<r\theta>} &= \beta_1 A^2 B - \beta_{-1} B. \end{aligned}$$

The deformation is supported by assigning pressures $P_1(\theta) = -T_{<rr>}|_{r=r_1}$, $P_2(\theta) = -T_{<rr>}|_{r=r_2}$ and tractions P_r, P_θ on the surfaces $\theta = B \log (r/A) \pm \Theta_0/A$ and P_z on the planes $z = \pm L/A$. These conditions serve to determine the unknown functions $\Psi(\theta)$, $\Phi(r)$ and the constants K_1, A and B.

The contribution of the reinforcement and the sign of q depend on the solution of the above conditions. The fibers change level and experience a rigid body rotation remaining along the θ direction. The above analysis includes the special case of azimuthal shearing ($A = 1$).

c) *Fibers Along the Z Direction*

The inextensibility condition (2.12) takes the form $D = 1$, which combined with (3.6) gives

$$D = 1, \quad C = 1/A^2. \tag{3.17}$$

The only nonzero component of the reaction stress is

$$N_{<zz>} = q, \tag{3.18}$$

indicating that the fiber tension $T_f = q$ along the z-direction. The equilibrium Eqs. (2.13) take the form

$$\left.\begin{aligned} &-\frac{\partial p}{\partial r} + \frac{1}{r}\,(S_{<rr>} - S_{<\theta\theta>}) = 0, \\ &-\frac{\partial p}{\partial \theta} + 2S_{<r\theta>} = 0, \\ &-\frac{\partial p}{\partial z} - \frac{\partial q}{\partial z} = 0. \end{aligned}\right\} \tag{3.19}$$

The general solution of (3.19) is

$$p = 2S_{<r\theta>}\theta + (S_{<rr>} - S_{<\theta\theta>}) \log r + \phi(z), \tag{3.20}$$

$$q = -\phi(z) + f(r, \theta), \tag{3.21}$$

where $\phi(z)$ and $f(r, \theta)$ are arbitrary functions of z and r, θ, respectively. Thus (3.1) for this case has, in general, a universal solution.

The nonzero stresses of this deformation are:

$$\begin{aligned} T_{<rr>} &= -p + \beta_1 A^2 + \beta_{-1} A^2 (B^2 + 1/A^2), \\ T_{<\theta\theta>} &= -p + \beta_1 A^2 (B^2 + 1/A^2) + \beta_{-1} A^2, \\ T_{<zz>} &= -p - q + \beta_1 + \beta_{-1} A^2, \\ T_{<r\theta>} &= A^2 B (\beta_1 - \beta_{-1}). \end{aligned}$$

As in case b) we assign surface tractions to support the deformation and their relations with the stresses serve to determine the unknown functions. The contribution of the reinforcement and the sign of q depend on the values of the above functions. The fibers experience a rigid body rotation remaining straight along the z direction. The above analysis includes the special case of azimuthal shearing ($A = C = D = 1$).

d) One Set of Fibers of Helical Path Inclined at an Angle a to the z Direction (Complete Tube)

The inextensibility condition (2.12) takes the form $C^2 A^2 \sin^2 a + D^2 \cos^2 a = 1$, which combined with (3.6) gives

$$C = \frac{D(1 - D^2 \cos^2 a)}{\sin^2 a}, \qquad A = \frac{\sin a}{D\sqrt{1 - D^2 \cos^2 a}}. \tag{3.22}$$

The reaction stress (2.6) takes the form

$$[N_{<ij>}] = q \begin{bmatrix} 0 & 0 & 0 \\ 0 & A^2C^2 \sin^2 a & ACD \sin a \cos a, \\ 0 & ACD \sin a \cos a & D^2 \cos^2 a \end{bmatrix}, \tag{3.23}$$

indicating that the fiber tension $T_f = q(A^2C^2 \sin^2 a + D^2 \cos^2 a)$ directed along the tangents to the deformed fibers. The equilibrium Eqs. (2.13) take the form

$$\left.\begin{aligned} &-\frac{\partial p}{\partial r} + \frac{1}{r}\left(qA^2C^2 \sin^2 a + S_{<rr>} - S_{<\theta\theta>}\right) = 0, \\ &-\frac{1}{r}\left(\frac{\partial p}{\partial \theta} + A^2C^2 \sin^2 a \frac{\partial q}{\partial \theta}\right) - ACD \sin a \cos a \frac{\partial q}{\partial z} + \frac{2}{r} S_{<r\theta>} = 0, \\ &-\frac{1}{r} ACD \sin a \cos a \frac{\partial q}{\partial \theta} - \frac{\partial p}{\partial z} - D^2 \cos^2 a \frac{\partial q}{\partial z} = 0. \end{aligned}\right\} \tag{3.24}$$

By assuming that q is a constant we can integrate (3.24) and obtain

$$q = K_1, \tag{3.25}$$

$$p = 2S_{<r\theta>}\theta + (K_1A^2C^2 \sin^2 a + S_{<rr>} - S_{<\theta\theta>}) \log r + K_2, \tag{3.26}$$

where K_1 and K_2 and are constants, indicating that the problem has a universal solution.

The nonzero stresses of the deformation are:

$$\begin{aligned} T_{<rr>} &= -p + \beta_1 A^2 + \beta_{-1} D^2 A^2 (B^2 + C^2), \\ T_{<\theta\theta>} &= -p - qA^2C^2 \sin^2 a + \beta_1 A^2 (B^2 + C^2) + \beta_{-1} A^2 D^2, \\ T_{<zz>} &= -p - qD^2 \cos^2 a + \beta_1 D^2 + \beta_{-1} A^4 C^2, \\ T_{<\theta z>} &= -qACD \sin a \cos a. \end{aligned}$$

We assign the pressures $P_1(\theta)$ and $P_2(\theta)$ on the surfaces $r = r_1$ and $r = r_2$, respectively and the force F_z and moment M_z on the planes $z = \pm DL$, given by

$$F_z = \int_0^{2\pi} \int_{r_1}^{r_2} T_{<zz>} r \, dr \, d\theta,$$

$$M_z = \int_0^{2\pi} \int_{r_1}^{r_2} T_{<\theta z>} r^2 \, dr \, d\theta.$$

The relations of these assigned pressures and forces with the stresses serve to compute the constants D, B, K_1 and K_2.

The contribution of the reinforcement is significant and the sign of q depends on the solution of the above relations. The deformed fibers remain helicoidal but have different pitch and radius.

e) Two Sets of Fibers of Helical Path and Symmetrically Inclined at an Angle a to the Z Direction (Complete Tube)

The inextensibility condition is that of case d) and thus C and A are given by (3.22). The only nonzero components of $\boldsymbol{N}$ are now $N_{<\theta\theta>} = 2qA^2C^2 \sin^2 \alpha$ and $N_{<zz>} = 2qD^2 \cos^2 \alpha$. Thus the equilibrium Eqs. (3.24) yield

$$\left.\begin{aligned} & -\frac{\partial p}{\partial r} + \frac{1}{r}\,(2qA^2C^2 \sin^2 a + S_{<rr>} - S_{<\theta\theta>}) = 0\,, \\ & -\frac{1}{r}\left(\frac{\partial p}{\partial \theta} + 2A^2C^2 \sin^2 a\,\frac{\partial q}{\partial \theta}\right) + \frac{2}{r}\,S_{<r\theta>} = 0\,, \\ & -\frac{\partial p}{\partial z} - 2D^2 \cos^2 a\,\frac{\partial q}{\partial z} = 0\,. \end{aligned}\right\} \tag{3.27}$$

By assuming that q is a constant, p and q are given again by (3.25) and (3.26) with $2A^2C^2$ instead of A^2C^2. The rest of the analysis follows that of the previous case with the only difference that $T_{<\theta z>} = 0$.

4. Compressible Materials

We investigate the possibility of existence of universal solutions of the deformation

$$r = f(R)\,, \qquad \theta = B \log R + C\Theta\,, \qquad z = DZ \tag{4.1}$$

of a compressible isotropic elastic material reinforced with various systems of inextensible fibers. The analysis follows very closely that of the incompressible material, so that only the results will be presented here.

a) Fibers Along the R Direction

The inextensibility condition (2.12) takes the form

$$\left(\frac{\partial r}{\partial R}\right)^2 + B^2\left(\frac{r}{R}\right)^2 = 1\,. \tag{4.2}$$

The solution of the above nonlinear equation is found as

$$r = \frac{\sin\phi}{B}\left[\frac{e^{B\phi}}{B\cos\phi - \sin\phi}\right]^{\frac{1}{B^2+1}}, \qquad \phi = \arcsin\left(B\,\frac{r}{R}\right). \tag{4.3}$$

For this case the extra stress components are functions of r and the equilibrium Eqs. (2.13) take the form

$$\left.\begin{aligned} & -\frac{\partial}{\partial r}\,(qf'^2) + \frac{\partial S_{<rr>}}{\partial r} - \frac{B}{R}\,f'\,\frac{\partial q}{\partial \theta} + \frac{1}{r}\,(S_{<rr>} - S_{<\theta\theta>}) \\ & \qquad\qquad - \frac{qf'^2}{f} + q\,\frac{B}{R}\,f' = 0\,, \\ & -\frac{\partial}{\partial r}\left(q\,\frac{B}{R}\,ff'\right) + \frac{\partial S_{<r\theta>}}{\partial r} - f\,\frac{B^2}{R^2}\,\frac{\partial q}{\partial \theta} - 2\,\frac{B}{R}\,f'q + \frac{2}{r}\,S_{<r\theta>} = 0 \end{aligned}\right\} \tag{4.4}$$[4]

where $r = f(R)$ and $f' = \partial r/\partial R$.

[4] The third equilibrium equation is identically satisfied.

Introducing the variable ϕ defined in (4.3), into Eqs. (4.4.), the later equations[5] take the form

$$\left.\begin{aligned}&[G(\cot\phi - 1) - F\sin 2\phi]\, q + F\cos^2\phi\,\frac{\partial q}{\partial\phi} + G\,\frac{\partial q}{\partial\theta}\\ &\qquad = (S_{<rr>} - S_{<\theta\theta>})\,(\cot\phi - 1) + F\,\frac{\partial S_{<rr>}}{\partial\phi},\\ &[2G + F\cos 2\phi]\, q + \frac{1}{2}\,F\sin 2\phi\,\frac{\partial q}{\partial\phi} + \left(\frac{1}{2}\sin 2\phi - \sin^2\phi\right)\frac{\partial q}{\partial\theta}\\ &\qquad = 2S_{<r\theta>}\,(\cot\phi - 1) - \frac{\partial S_{<r\theta>}}{\partial\phi}\,F,\end{aligned}\right\} \tag{4.5}$$

where $G = \cos\phi(\cos\phi - \sin\phi)$ and $F = e^{\varphi/2}(1 - \tan\phi)^2$.

Eqs. (4.5) can be brought into the general form

$$\begin{aligned}&\frac{\partial q}{\partial\phi} + Aq = B,\\ &\frac{\partial q}{\partial\theta} + \bar{A}q = \bar{B},\end{aligned} \tag{4.6}$$

where A and $\bar{A}$ are functions of ϕ and B and $\bar{B}$ are functions of ϕ and the extra stress components $S_{<rr>}$, $S_{<\theta\theta>}$ and $S_{<r\theta>}$.

From Eqs. $(4.6)_1$ and $(4.6)_2$ we can obtain, in general, two different solutions for q, which have to coincide. Thus, in general, we cannot have a solution for q unless some conditions among the extra stress components are valid. This means that there is no universal solution. As an example, the $r = R/\sqrt{1 + B^2}$ is a solution of (4.2) and leads to a solution for $q = \frac{2}{B}\,S_{<r\theta>} + S_{<\theta\theta>} - S_{<rr>}$ under the condition $S_{<r\theta>} = (B/1 - B^2)\,(S_{<rr>} - S_{<\theta\theta>})$.

b) *Fibers Along the Θ Direction*

The inextensibility condition takes the form

$$r = R/C \tag{4.7}$$

and thus the equilibrium take the form

$$\left.\begin{aligned}&S_{<rr>} - S_{<\theta\theta>} + q = 0,\\ &\frac{\partial q}{\partial\theta} + 2S_{<r\theta>} = 0.\end{aligned}\right\} \tag{4.8}$$

Because the extra stress components are constants, (4.8) are satisfied, only if $S_{<rr>} = 0$, indicating that there is no universal solution for this case.

[5] B = 1 has been assumed for simplicity.

c) Fibers Along the Z Direction

The inextensibility condition simply gives $D = 1$ and thus the function f, defined by $r = f(R)$, is not known. This means that the extra stress components are functions of r in this case and the equilibrium equations take the form

$$\left.\begin{aligned} &\frac{\partial S_{<rr>}}{\partial r} + \frac{1}{r}\left(S_{<rr>} - S_{<\theta\theta>}\right) = 0, \\ &\frac{\partial S_{<r\theta>}}{\partial r} + \frac{2}{r} S_{<r\theta>} = 0, \\ &-\frac{\partial q}{\partial z} = 0, \end{aligned}\right\} \tag{4.9}$$

indicating again that there is no universal solution, since the $S_{<ij>}$ have to satisfy certain conditions.

d) One Set of Fibers of Helical Path Inclined at an Angle a to the Z Direction

The inextensibility condition gives

$$r = \frac{\sqrt{1 - D^2 \cos^2 a}}{C \sin a} R \tag{4.10}$$

and the equilibrium equations take the form

$$\left.\begin{aligned} &S_{<rr>} - S_{<\theta\theta>} + qA^2C^2 \sin^2 a = 0, \\ &-\frac{1}{r} A^2C^2 \sin^2 a \frac{\partial q}{\partial \theta} - ACD \sin a \cos a \frac{\partial q}{\partial z} + \frac{2}{r} S_{<r\theta>} = 0, \\ &-\frac{1}{r} ACD \sin a \cos a \frac{\partial q}{\partial \theta} - D^2 \cos^2 a \frac{\partial q}{\partial z} = 0. \end{aligned}\right\} \tag{4.11}$$

Because $S_{<rr>}$ and $S_{<\theta\theta>}$ are constants in $(4.11)_1$, q will be also a constant and thus $(4.11)_2$ yields $S_{<r\theta>} = 0$, indicating that there is no universal solution.

e) Two Sets of Fibers of Helical Path Symmetrically Inclined at an Angle a to the Z Direction

Following the same procedure as in case d) we also conclude that there is no universal solution for this case.

5. Conclusions

For a fiber-reinforced isotropic elastic sector of a circular tube undergoing inflation, bending, extension and azimuthal shearing, universal solutions were determined for the incompressible case. The addition of the reinforcement, even though it makes the equilibrium equations more complicated than those of the unreinforced case, permits the satisfaction of a greater number of boundary conditions. However, the surface tractions supporting the deformation are, in general, functions of both r and θ and thus very complicated, as they were for the unreinforced case. For the compressible case it was shown that there are no universal solutions, even though it was found in [2] that the constraint of inextensibility creates universal solutions for some deformations without such solutions in the unconstrained case.

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