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The Poynting Effect

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Summary

The predictions regarding the Poynting effect, obtained by way of the constitutive equation of a simple elastic solid expressed in terms of the spatial description using the Cauchy stress are compared with those obtained by way of the constitutive equation of a simple elastic solid expressed in terms of the referential description using the second referential stress tensor. It is shown that the quantitative predictions of the constitutive equation of a simple elastic solid expressed in terms of the referential description are in good accord with the observations of experiment.

1. Introduction

For many solids, in particular rubber-like solids, polymeric solids and metals, the mechanical response of a rod or tube twisted in simple torsion is characterised by the Poynting effect. This effect relates to the observation [1], [2] that the lengths of various steel, copper and brass wires increased when twisted in the elastic range, and that the elongation was proportional to the square of the twist. The development of a nonlinear theory of finite elasticity which predicts the form of the Poynting effect is one of the outstanding successes of modern nonlinear continuum mechanics. The development of the appropriate class of simple elastic material has been directed almost exclusively to rubber-like solids, thus recognising that this type of material displays the Poynting effect to a marked extent, (see for example [3]). The present paper is concerned with the extent to which the quantitative predictions of theory are in accord with the observations of experiment. The discussion is restricted to isotropic, incompressible solids, no account being taken of thermodynamic restrictions, the proposed constitutive equation being purely mechanical. Furthermore, no attempt is made to take account of the subcontinuum, that is the micromechanics of the material.

2. Constitutive Equation

Referential coordinates, denoted X^α $(\alpha = 1, 2, 3)$, and spatial coordinates, denoted x^i $(i = 1, 2, 3)$, are set up in space by adjoining to the separate origins O and o the bases $G = \{\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3\}$ and $g = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$. Using standard notation and conventions [4], the deformation gradient tensor $\mathbf{F} = \operatorname{grad} \boldsymbol{x}$, has the component form $\mathbf{F} = F^p{}_\mu \boldsymbol{g}_p \otimes \boldsymbol{G}^\mu$ where $F^i{}_\alpha = x^i{}_{,\alpha}$.

The constitutive equation for a simple elastic solid in the spatial description will be taken to be,

$$\mathbf{T} = \beta_0 \mathbf{I} + \beta_1 \mathbf{B} + \beta_{-1} \mathbf{B}^{-1}, \qquad (\mathbf{B} = \mathbf{F}\mathbf{F}^\top), \tag{2.1}$$

where $\mathbf{T}$ is the Cauchy stress, $\mathbf{B}$ the left Cauchy-Green deformation tensor, and where the response coefficients β_a $(a = 0, \pm 1)$ are scalar functions of the principal invariants $I_\mathbf{B}$, $II_\mathbf{B}$ and $III_\mathbf{B}$ of $\mathbf{B}$.

It has been shown [5] that the constitutive equation for a simple elastic solid can be expressed in the referential description in the form

$$\hat{\mathbf{T}} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{C} + \alpha_{-1} \mathbf{C}^{-1}, \qquad (\mathbf{C} = \mathbf{F}^\top \mathbf{F}), \tag{2.2}$$

where

$$\hat{\mathbf{T}} = J \mathbf{U}^{1/2} \mathbf{F}^{-1} \mathbf{T} (\mathbf{F}^{-1})^\top \mathbf{U}^{1/2}, \tag{2.3}$$

is the second referential stress, $J = \det \mathbf{F} > 0$, $\mathbf{U}$ the right stretch tensor, $\mathbf{C}$ $(= \mathbf{U}^2)$ the right Cauchy-Green deformation tensor, and where the response coefficients, α_k $(k = 0, \pm 1)$ are scalar functions of the principal invariants $I_\mathbf{C}$, $II_\mathbf{C}$ and $III_\mathbf{C}$ of $\mathbf{C}$.

Using the Cayley-Hamilton theorem, the constitutive equation (2.2) can be rearranged into the alternative form

$$\hat{\mathbf{T}} = \hat{\alpha}_0 \mathbf{I} + \hat{\alpha}_1 \hat{\mathbf{E}} + \hat{\alpha}_2 \hat{\mathbf{E}}^2, \tag{2.4}$$

where,

$$\hat{\mathbf{E}} = (\mathbf{C} - \mathbf{C}^{-1})/4, \tag{2.5}$$

and where the response coefficients,

$$\hat{\alpha}_0 = \alpha_0 + \alpha_1 + \alpha_{-1}, \qquad \hat{\alpha}_1 = 4 \frac{[\alpha_1(1 + II_\mathrm{c}) - \alpha_{-1}(1 + I_\mathrm{c})]}{(2 + I_\mathrm{c} + II_\mathrm{c})},$$
$$\hat{\alpha}_2 = \frac{16(\alpha_1 + \alpha_{-1})}{(2 + I_\mathrm{c} + II_\mathrm{c})}. \tag{2.6}$$

Equation (2.2) can be expressed in the form,

$$\hat{\mathbf{T}}' = 2\hat{G}(\varphi_0 \mathbf{I} + \varphi_1 \hat{\mathbf{E}}' + \varphi_2 \hat{\mathbf{E}}'^2) = 2\hat{G}\hat{\mathbf{M}}'(\hat{\mathbf{E}}'), \tag{2.7}$$

where a prime denotes a deviator, and where $\hat{G} = \hat{G}(K_2', K_3')$ is a non-negative factor of proportionality, it being noted that,

$$K_2' = \frac{1}{2} \operatorname{tr} \hat{\mathbf{E}}'^2, \qquad K_3' = \det \hat{\mathbf{E}}', \tag{2.8}$$

and that the response coefficients,

$$\varphi_0 = -\frac{2}{3} K_2' \varphi_2, \qquad \varphi_1 = \left(\hat{\alpha}_1 + \frac{2}{3} I_{\hat{\mathbf{E}}} \hat{\alpha}_2\right) \Big/ (2\hat{G}), \qquad \varphi_2 = \hat{\alpha}_2/(2\hat{G}), \tag{2.9}$$

use having been made of the Cayley-Hamilton theorem.

It has been shown [6] that the biaxial stretching of a thin rubber sheet can be described by the ground state form of the constitutive equation (2.7) which is characterised by having $\varphi_2 = \hat{\alpha}_2 = 0$, which limiting conditions give $\varphi = 1$. Setting $\varphi_1 = 1$, $\varphi_2 = 0$ in Eq. (2.7) gives the ground state form of the constitutive equation:

$$\hat{\mathbf{T}}' = 2\hat{G}\hat{\mathbf{E}}'. \tag{2.10}$$

Alternatively, noting from Eq. (2.6) that the condition $\varphi_2 = \hat{\alpha}_2 = 0$ gives $\alpha_1 = -\alpha_{-1}$, the ground state form of the constitutive equation can be expressed in the reduced form of Eq. (2.2):

$$\hat{\mathbf{T}} = \alpha_0 \mathbf{I} + \alpha_1(\mathbf{C} - \mathbf{C}^{-1}), \qquad (\alpha_1 = 1/2\hat{G}). \tag{2.11}$$

3. Extension and Torsion of a Solid Rod

Consider a right circular solid rod of incompressible material. Let (R, Θ, Z) be the cylindrical referential coordinates in the initial state of a particle that is located in the deformed configuration by the cylindrical spatial coordinates (r, θ, z).

For a twisted rod with its principal axis aligned with the Z axis, the simple deformations to be considered are:

$$r = R/\sqrt{F}, \qquad \theta = \Theta + DZ, \qquad z = FZ, \tag{3.1}$$

where $F = l/L$ is the ratio of the current length l to the undeformed length L of a solid rod. Using Eq. (3.1),

$$[F^i{}_\alpha] = \begin{bmatrix} \frac{1}{\sqrt{F}} & 0 & 0 \\ 0 & 1 & D \\ 0 & 0 & F \end{bmatrix}, \tag{3.2}$$

and hence the non-vanishing physical components $B\langle ij\rangle$, $(B^{-1})\langle ij\rangle$ of $\mathbf{B}$ and its inverse $\mathbf{B}^{-1}$ can be obtained in the form,

$$[B\langle ij\rangle]=\begin{bmatrix}\frac{1}{F} & 0 & 0\\ 0 & \frac{1}{F}(1+R^2D^2) & rDF\\ 0 & rDF & F^2\end{bmatrix},\quad [(B^{-1})\langle ij\rangle]=\begin{bmatrix}F & 0 & 0\\ 0 & F & -rD\\ 0 & -rD & \frac{(1+R^2D^2)}{F^2}\end{bmatrix}. \tag{3.3.1, 2}$$

Similarly, the physical components of $\mathbf{C}$ and its inverse $\mathbf{C}^{-1}$ can be obtained in the form:

$$[C\langle\alpha\beta\rangle]=\begin{bmatrix}\frac{1}{F} & 0 & 0\\ 0 & \frac{1}{F} & \frac{RD}{F}\\ 0 & \frac{RD}{F} & \frac{(F^3+R^2D^2)}{F}\end{bmatrix},\quad [(C^{-1})\langle\alpha\beta\rangle]=\begin{bmatrix}F & 0 & 0\\ 0 & \left(F+\frac{R^2D^2}{F^2}\right) & -\frac{RD}{F^2}\\ 0 & -\frac{RD}{F^2} & \frac{1}{F^2}\end{bmatrix}. \tag{3.4.1, 2}$$

The non-vanishing physical components $T\langle ij\rangle$ of the Cauchy stress $\mathbf{T}$ and the non-vanishing physical components $\hat{T}\langle\alpha\beta\rangle$ of the second referential stress tensor are:

$$[T\langle ij\rangle]=\begin{bmatrix}\sigma_{rr} & 0 & 0\\ 0 & \sigma_{\theta\theta} & \tau_{\theta z}\\ 0 & \tau_{z\theta} & \sigma_{zz}\end{bmatrix},\qquad [\hat{T}\langle\alpha\beta\rangle]=\begin{bmatrix}\hat{\sigma}_{RR} & 0 & 0\\ 0 & \hat{\sigma}_{\Theta\Theta} & \hat{\tau}_{\Theta Z}\\ 0 & \hat{\tau}_{Z\Theta} & \hat{\sigma}_{ZZ}\end{bmatrix}. \tag{3.5.1, 2}$$

For zero applied traction on the curved surface at $r=r_1$, such that $[\sigma_{rr}]_{r=r_1}=0$, the stress relations in the spatial description follow from Eqs. (2.1), (3.3), (3.5.1):

$$\sigma_{rr}=\int_r^{r_1}[(\sigma_{\theta\theta}-\sigma_{rr})/r]\,dr=-D^2\int_r^{r_1}\beta_1 r\,dr, \tag{3.6}$$

$$\sigma_{\theta\theta}=\sigma_{rr}+\beta_1 r^2D^2, \tag{3.7}$$

$$\sigma_{zz}=\sigma_{rr}+\frac{(F^3-1)}{F}\left(\beta_1-\frac{1}{F}\beta_{-1}\right)+\beta_{-1}\frac{R^2}{F^2}D^2, \tag{3.8}$$

$$\tau_{\theta z}=\left(\beta_1-\frac{1}{F}\beta_{-1}\right)FrD=\tau_{z\theta}. \tag{3.9}$$

The stress relations in the referential description follow from Eqs. (2.11), (3.4), (3.5.2):

$$\hat{\sigma}_{RR} = \alpha_0 - \alpha_1 \frac{(F^2 - 1)}{F}, \tag{3.10}$$

$$\hat{\sigma}_{\Theta\Theta} = \hat{\sigma}_{RR} - \alpha_1 \frac{R^2D^2}{F^2}, \tag{3.11}$$

$$\hat{\sigma}_{ZZ} = \hat{\sigma}_{RR} + \alpha_1 \frac{(F+1)\,(F^3 - 1)}{F^2} + \alpha_1 \frac{R^2D^2}{F}, \tag{3.12}$$

$$\hat{\tau}_{\Theta Z} = \alpha_1 \frac{(F+1)}{F^2} RD = \hat{\tau}_{Z\Theta}, \tag{3.13}$$

it being noted that $\alpha_1 = \hat{G}/2$.

Equation (2.3) can be rearranged, using the right polar decomposition $\mathbf{F} = \mathbf{RU}$, into the form,

$$\mathbf{T} = \mathbf{F}\hat{\mathbf{T}}\mathbf{R}^{\mathsf{T}}, \tag{3.14}$$

where $\mathbf{R}$ is the rotation tensor. Noting that $\mathbf{B} = \mathbf{RCR}^{\mathsf{T}}$, it follows from Eqs. (3.3.1) and (3.4.1), that $\mathbf{R}$ must be of the form,

$$[\mathbf{R}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}, \tag{3.15}$$

where,

$$\tan\psi = \frac{RD}{(F^{3/2} + 1)}. \tag{3.16}$$

Entering the form for $\mathbf{R}$ given by Eq. (3.15) into Eq. (3.14) gives:

$$\sigma_{rr} = \frac{1}{\sqrt{F}} \hat{\sigma}_{RR}, \tag{3.17}$$

$$\sigma_{\theta\theta} = \frac{\cos\psi}{\sqrt{F}} (\hat{\sigma}_{\Theta\Theta} + RD\hat{\tau}_{Z\Theta}) + \frac{\sin\psi}{\sqrt{F}} (\hat{\tau}_{\Theta Z} + RD\hat{\sigma}_{ZZ}), \tag{3.18}$$

$$\sigma_{zz} = F\cos\psi\hat{\sigma}_{ZZ} - F\sin\psi\hat{\tau}_{Z\Theta}, \tag{3.19}$$

$$\tau_{\theta z} = \frac{\cos\psi}{\sqrt{F}} (\hat{\tau}_{\Theta Z} + RD\hat{\sigma}_{ZZ}) - \frac{\sin\psi}{\sqrt{F}} (\hat{\sigma}_{\Theta\Theta} + RD\hat{\tau}_{Z\Theta}) = \tau_{z\theta}, \tag{3.20}$$

$$\tau_{z\theta} = F\cos\psi\hat{\tau}_{Z\Theta} + F\sin\psi\hat{\sigma}_{ZZ} = \tau_{\theta z}, \tag{3.21}$$

For simple torsion, ($F \geqq 1$ for $D \geqq 0$), Eq. (3.13) can be expressed in the form

$$\hat{\tau}_{\Theta Z} = \hat{G} \frac{(F+1)}{2F^2} RD, \qquad (\alpha_1 = 1/2\,\hat{G}), \tag{3.22}$$

where, using the form for $\hat{G}$ given in Eq. (6.1) of [6], that is,

$$\hat{G} = G_0 \frac{\ln(1 + \phi\hat{\varepsilon}'')}{\phi\hat{\varepsilon}''}, \tag{3.23}$$

and where, from equation (6.2) of [6].

$$\phi = \frac{\phi_1}{1 - (\phi_2/\hat{\nu})}, \qquad \hat{\varepsilon}'' = \left(\frac{2}{3}\,\mathrm{tr}\,\hat{\mathbf{M}}'^2\right)^{1/2}, \tag{3.24.1, 2}$$

it being noted that for simple torsion,

$$\hat{\nu} = \frac{3\hat{\varepsilon}_1'}{\hat{\varepsilon}_3' - \hat{\varepsilon}_2'} = -\frac{(F-1)\,[(F^2+F+1)\,(F+1) + R^2D^2]}{(F+1)\,\{4R^2D^2 + [(F^3-1) + R^2D^2]^2\}^{1/2}}, \tag{3.25}$$

the $\hat{\varepsilon}_\alpha'$ ($\alpha = 1, 2, 3$) being the proper numbers of $\hat{\mathbf{E}}'$. As $F \to 1$, $\hat{\nu} \to 0$, and hence from Eq. (3.24.1), $\phi \to 1$ which condition, when entered into Eq. (3.23), gives $\hat{G} \to G_0$. Hence for $F \to 1$, $\hat{G} \to G_0$ and Eq. (3.22) reduces to,

$$\hat{t}_{\Theta Z} = G_0 \frac{(F+1)}{2F^2} RD. \tag{3.26}$$

From Eq. (3.26) it follows that,

$$\hat{\Gamma} = 2\pi \int\limits_0^{R_1} R^2 \hat{t}_{\Theta Z}\, dR = \frac{1}{4}\pi G_0 R_1^4 \frac{(F+1)}{F^2} D. \tag{3.27}$$

where R_1 is the outer radius of the undeformed solid rod.

4. Resultant Longitudinal Force on Plane End of Solid Rod

4.1 The Spatial Description

The identity,

$$\int\limits_r^{r_1} r\sigma_{rr}\, dr = \frac{1}{2} r_1^2 [\sigma_{rr}]_{r=r_1} - \frac{1}{2} r^2 \sigma_{rr} - \frac{1}{2}\int\limits_r^{r_1} r^2 \left(\frac{d\sigma_{rr}}{dr}\right) dr, \tag{4.1}$$

can be rearranged using the condition of equilibrium in the radial direction,

$$\frac{\partial \sigma_{rr}}{\partial r} = \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r}, \tag{4.2}$$

to give,

$$\int\limits_r^{r_1} r(\sigma_{rr} + \sigma_{\theta\theta})\, dr = r_1^2 [\sigma_{rr}]_{r=r_1} - r^2 \sigma_{rr}. \tag{4.3}$$

The right-circular solid rod is to be deformed by simple torsion into a right-circular solid rod, the deformation being achieved with zero applied traction on the outer curved surface at $r = r_1$, such that $[\sigma_{rr}]_{r=r_1} = 0$ for all D. This condition reduces Eq. (4.3) to,

$$\int_r^{r_1} r(\sigma_{rr} + \sigma_{\theta\theta})\, dr = -r^2\sigma_{rr}, \tag{4.4}$$

which applies to the annular region bounded by the outer curved surface at $r = r_1$ and the surface at some arbitrary r. The condition of equilibrium in the radial direction must still apply in this annular region, and hence it follows from Eqs. (4.2) and (4.4) that at any arbitrary r, must have,

$$\sigma_{rr} + \sigma_{\theta\theta} = 0, \tag{4.5}$$

for all values of D.

Entering the condition of Eq. (4.5) into Eq. (3.7) gives,

$$\sigma_{rr} = -\frac{1}{2}\, \beta_1 r^2 D^2 = -D^2 \int_r^{r_1} \beta_1 r\, dr, \tag{4.6}$$

which can be expressed in the form,

$$\sigma_{rr} = -\frac{\beta_1}{2\left(\beta_1 - \frac{1}{F}\, \beta_{-1}\right)^2 F^2}\, \tau_{\theta z}^2, \tag{4.7}$$

where use has been made of Eq. (3.9). Eqs. (4.6) and (4.7) give the following conditions for β_1:

$$\beta_1 r^2 = 2 \int_r^{r_1} \beta_1 r\, dr, \qquad [\beta_1]_{r=r_1} = 0, \tag{4.8}$$

subject to the conditions

$$[\beta_{-1}]_{r=r_1} \neq 0, \qquad [\tau_{\theta z}]_{r=r_1} \neq 0. \tag{4.9}$$

The condition of Eq. (4.6) can be entered into Eq. (3.8) to give:

$$\sigma_{zz} = \frac{(F^3 - 1)}{F}\left(\beta_1 - \frac{1}{F}\, \beta_{-1}\right) - \frac{1}{2}\left(\beta_1 - \frac{2}{F}\, \beta_{-1}\right) r^2 D^2. \tag{4.10}$$

The resultant longitudinal force on the plane end of a solid rod,

$$N_z = 2\pi \int_0^{r_1} r\sigma_{zz}\, dr. \tag{4.11}$$

Entering the form for σ_{zz} given by Eq. (4.10) into Eq. (4.11) gives

$$N_z = \pi \int_0^{r_1} r \left[2 \frac{(F^3 - 1)}{F} \left(\beta_1 - \frac{1}{F} \beta_{-1} \right) - \left(\beta_1 - \frac{2}{F} \beta_{-1} \right) r^2 D^2 \right] dr. \quad (4.12)$$

It follows from Eq. (4.8) that,

$$\int_0^{r_1} \beta_1 r \, dr = 0, \quad (4.13)$$

which condition can be entered into Eq. (4.12) to give,

$$N_z = -2\pi e_{zz} \frac{(F^2 + F + 1)}{F^2} \int_0^{r_1} \beta_{-1} r \, dr - \pi D^2 \int_0^{r_1} \left(\beta_1 - \frac{2}{F} \beta_{-1} \right) r^3 \, dr, \quad (4.14)$$

where the axial strain,

$$e_{zz} = F - 1. \quad (4.15)$$

Eq. (4.12) can be expressed in the form,

$$N_z = 2\pi e_{zz} \frac{(F^2 + F + 1)}{DF^2} \int_0^{r_1} \tau_{\theta z} \, dr - \pi \frac{D}{F} \int_0^{r_1} \tau_{\theta z} r^2 \, dr + \frac{\pi D^2}{F} \int_0^{r_1} \beta_{-1} r^3 \, dr, \quad (4.16)$$

where use has been made of Eq. (3.9). In the limit of infinitely small strains, (for which $F \to 1$ as $D \to 0$), use can be made of the theory of infinitesimal, linear elasticity, to give to a first approximation,

$$\int_0^{r_1} \tau_{\theta z} \, dr = FD \int_0^{r_1} \left(\beta_1 - \frac{1}{F} \beta_{-1} \right) r \, dr \simeq G_0 D \int_0^{r_1} r \, dr = \frac{1}{2} G_0 r_1^2 D, \quad (4.17)$$

and the resultant torque,

$$\Gamma = 2\pi \int_0^{r_1} r^2 \tau_{\theta z} \, dr = 2\pi DF \int_0^{r_1} \left(\beta_1 - \frac{1}{F} \beta_{-1} \right) r^3 \, dr \simeq 2\pi G_0 D \int_0^{r_1} r^3 \, dr = \frac{1}{2} \pi G_0 D r_1^4, \quad (4.18)$$

where G_0 is the classical shear modulus. Substituting in Eq. (4.16) from Eqs. (4.17) and (4.18) gives,

$$\frac{N_z}{\pi R_1^2} = 3 G_0 e_{zz} - \frac{1}{4} G_0 R_1^2 D^2 + \frac{D^2}{R_1^2} \int_0^{r_1} \beta_{-1} r^3 \, dr. \quad (4.16)$$

4.2 *The Referential Description*

Eq. (3.19) can be rearranged using Eqs. (3.10), (3.12), (3.13) and (3.16) to give,

$$\sigma_{zz} = F\left\{\hat{\sigma}_{RR} + \alpha_1\left[\frac{(F+1)(F^3-1)}{F^2} + \frac{(F^{5/2}-1)}{(F^{3/2}+1)}\frac{R^2D^2}{F^2}\right]\right\}\cos\psi. \quad (4.20)$$

Using Eq. (4.5), the sum of Eqs. (3.17) and (3.18) can be rearranged to give:

$$\hat{\sigma}_{RR} = -\alpha_1 \frac{(F^3 + F^2 + F^{3/2} + 1 + R^2D^2)\,R^2D^2}{F\{(F^{3/2}+1) + [(F^{3/2}+1) + R^2D^2]\cos\psi\}}\cos\psi. \quad (4.21)$$

The form for $\hat{\sigma}_{RR}$ given by Eq. (4.21) can be entered into Eq. (4.20). The resulting form for σ_{zz} can be entered into Eq. (4.11) to give N_z.

For infinitely small strains, $(D \to 0,\ F \to 1)$, Eqs. (4.20) and (4.21) reduce to,

$$\sigma_{zz} = 3G_0 e_{zz} - \frac{1}{2}\,G_0 r^2 D^2, \qquad (r \simeq R), \quad (4.22)$$

$$\hat{\sigma}_{RR} = -\frac{1}{2}\,G_0 r^2 D^2, \qquad (r \simeq R), \quad (4.23)$$

where use has been made of Eq. (3.23) and the condition that $\alpha_1 = 1/2\,\hat{G}$. Entering the form for σ_{zz} given by Eq. (4.22) into Eq. (4.11) gives,

$$\frac{N_z}{\pi R_1^2} = 3G_0 e_{zz} - \frac{1}{4}\,G_0 R_1^2 D^2, \qquad (r_1 \simeq R_1), \quad (4.24)$$

which is to be compared with Eq. (4.19).

5. The Poynting Effect

The Poynting effect is characterised by an axial elongation,

$$e_{zz} = F - 1\ (\equiv e_{(P)}), \quad (5.1)$$

which is observed for the condition that the resultant longitudinal force on the plane end of a solid rod is zero.

Setting $N_z = 0$ in Eqs. (4.19) and (4.24) gives,

(a) *Spatial description:*

$$e_{(P)} = \frac{1}{12}\,R_1^2 D^2 - \frac{D^2}{3G_0 R_1^2}\int_0^{r_1} \beta_{-1} r^3\,dr, \quad (5.2)$$

(b) *Referential description:*

$$e_{(P)} = \frac{1}{12} R_1^2 D^2. \tag{5.3}$$

For sufficiently small strains, the constitutive Eqs. (2.1) and (2.2), if compatible, should give the same relation for N_z, and hence for the ratio $e_{(P)}/R_1^2D^2$. It is evident from Eqs. (4.19) and (4.24) and Eqs. (5.2) and (5.3) that this condition implies that either $\beta_{-1} = 0$, or

$$\int_0^{r_1} \beta_{-1} r^3\, dr = 0, \qquad (D \to 0,\ F \to 1). \tag{5.4}$$

Eq. (4.17) implies,

$$G_0 = \beta_1 - \beta_{-1} = \text{const.}, \qquad (D \to 0,\ F \to 1), \tag{5.5}$$

for infinitely small strains. The condition $\beta_{-1} = 0$, $\beta_1 = G_0$, reduces Eq. (2.1) to the constitutive equation for a neo-Hookean solid [7]; these conditions, however, are not in accord with Eqs. (4.8) and (4.9).

6. Material Response

The prediction of Eq. (3.27) is in good accord with the experimental studies of Rivlin and Saunders [8].

With N_z adjusted to give $e_{zz} = 0$ for all D, the prediction of Eq. (4.24) in the form $N_z = -\pi G_0 R_1^4 D^2/4$ is in good accord with the experimental studies of Rivlin and Saunders [8] for small strains. However, for large strains, [3], the relation between N_z and D^2, (with $e_{zz} = 0$ for all D) is no longer linear, an observation in accord with Eq. (4.19).

The results of the experimental studies of Poynting [1], [2], Foux [9], and of Lenoe et al. [3] are in accord with the prediction that in simple torsion, the associated axial extension $e_{(P)}$ is proportional to $\hat{\gamma}^2$ in the elastic range of deformation, it being noted that,

$$\hat{\gamma} = R_1 D. \tag{6.1}$$

Although the weight of the apparatus may be very small, it is of interest to approximate its effect by assuming the existence of a small tensile force N_z which is constant for all D. With $N_z = \text{const.}$, for all D, Eq. (4.24) is of the form

$$e_{(P)} = e_0 + b\hat{\gamma}^2, \qquad (b = 1/12), \tag{6.2}$$

where

$$e_0 = \frac{N_z}{3G_0 \pi R_1^2} = \text{const.} \tag{6.3}$$

The presence of the term e_0 necessitates a slightly different interpretation of the measurements given by Foux [9] in his table 2. The positive and negative cycles have been independently analysed to give two values of b; the average of these two values of b is given in the present table 1 for wire numbers 4, 6, 7 and 9, in the as-drawn state and after heating. Also given in table 1 are the corresponding values of $(R_1/L)^2$ where R_1 is the radius of the wire and L its undeformed length. Wire numbers 2, 5 and 8 are not shown because they are from a different source. From an examination of the dependence of the values of b on the quantity E and the ultimate tensile strength given in table 1 of [9], it is concluded that wire numbers 1 and 3 are in a significantly different material state and are therefore also omitted from the present table 1. Wire number 10 is omitted because the effect of reversing the direction of shear of the as-drawn specimen is to reverse the direction of the change in length of the wire. This is taken to imply that for this radius of wire, the material properties are markedly different, and hence there is the possibility that heat treatment will not necessarily produce the required change in material properties.

It is evident from table 1 that the values of b are dependent upon the cross-section of the wires. There are two possible explanations for such a dependence upon the cross-section. This type of measurement is subject to an error arising from an apparent strengthening by the material nearer the longitudinal axis of the specimen. A second contributing factor is the linear dimension of the grains which for some materials may approach that of the diamter of the wire.

The variation of the average values of b given in table 1 with $(R_1/L)^2$ can be represented to a good degree of approximation by the linear relation,

$$b = b_0 + c(R_1/L)^2,$$

Table 1. *Values of the parameters for the Poynting effect*

Wire No.	$(R_1/L)^2 \times 10^6$	b*	b†
4	0.098	0.555	
	0.099		0.253
6	0.122	0.650	
	0.122		0.259
7	0.142	0.756	
	0.145		0.307
9	0.154	0.806	
	0.157		0.355

* As-drawn state
† After heat treatment

where,

$$b_0 = \begin{cases} 0.104 \\ 0.069 \end{cases} \quad \text{for} \quad \begin{cases} \text{as-drawn state} \\ \text{after heat treatment} \end{cases}$$

and c is a constant characteristic of material properties. These values of b_0 are in general accord with the value $b = 0.0833$, their average value $b_0 = 0.0865$ being, (within the limits of experimental accuracy), in good accord with the value of b predicted by Eq. (5.3).

The non-linear mechanical response of polyurethane rubbers with a high volume percentage of inorganic filler has been studied by Lenoe et al. [3]. A value of $b = 0.095$ at a strain rate of $0.0013\ \mathrm{s}^{-1}$, using a solid-rod specimen for which $(R_1/L)^2 = 0.04$, is in good accord with the value $b = 0.0833$ predicted by equation (5.3). For a strain rate of $0.00013\ \mathrm{s}^{-1}$. Lenoe et al [3] give $b = 0.14$. There are two possible explanations for the significant difference between this value of b and the value predicted by equation (5.3). Examination of their results for this strain rate show what appears to be a significant value for e_0 Also, the scatter in the experimental results for this strain rate is much greater than for the measurements at the higher strain rate. At this much lower strain rate, the axial elongation could be affected by the onset of the phenomenon of creep. That this is the most likely explanation can be inferred from the discussion given by Lenoe et al. [3] of their torsion creep tests with a free end.

With regard to the use of the generally non-symmetric second referential stress tensor, it is to be noted that in the context of Eq. (2.2), it is symmetric for isotropic materials. It is also of interest to note that, in the context of Section 3, the Cartesian components of the referential stress tensor have been introduced by Biot [10], (see also [11], [12]). Its symmetric part is called the Jaumann stress tensor, (see Koiter [13]). The conjugate strain measure associated with the Jaumann stress is the right stretch tensor, in terms of which the strain energy density can be expressed directly as a function.

7. Conclusions

The available measurements of the Poynting effect for both metals and rubber-like solids are in good quantitative agreement with the prediction of the proposed constitutive equation for a simple elastic solid, as formulated in the referential description using the second referential stress tensor.

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