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# On plastic potentials for anisotropic metals and their derivation from the texture function

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**Summary.** Plastic yielding of anisotropic metals can be either described by a macroscopic constitutive relation or assessed by means of a model which correlates single and polycrystal behaviors. The mathematical identification of the plastic work rate derived from the two approaches, for all strain rate tensors, leads to a fit of the polycrystal yield surface by an analytical function. When a quadratic form is assumed, the macroscopic anisotropy parameters become explicit functions of the texture coefficients. This identification method is applied to calculate yield surfaces and $R$-values of rolled and annealed steel sheets: the $R$-values and in general the flow rule, are more significantly modified by the fitting than the yield surface. Thus, it is worth extending the method to more general constitutive relations which may be given by the form of their work function: alternative forms of the work function for plastic materials are explored, especially in the bearing of convexity and homogeneity where quadratic forms have a distinct advantage. Finally, it is shown that the identification of the work function allows to express the phenomenological coefficients as analytical functions of the texture parameters for many forms of the work function; in the other cases, these coefficients may be obtained by linear or non-linear regression.

## 1 Introduction

Two approaches are generally used to describe the plastic anisotropy of polycrystals. The crystallographic approach takes explicitly into account the texture of the material by considering the polycrystal as a collection of grains, each of one having a specific orientation. Plastic properties of the aggregate are then calculated from the response of each of its constituents to a given loading. This approach lies on several levels of assumptions: the microscopic deformation mechanisms, the microscopic hardening law, the link between the imposed boundary conditions and stress and strain rate in each grain. The Taylor [1]/Bishop and Hill [2] model belongs to this category and is interesting for several reasons: i) first, the calculated yield surface is an upper bound of the real response of the material, ii) the agreement between calculated and experimental properties is quite satisfactory and iii) among the polycrystal plasticity models, it is one of the simplest to use. For all these reasons, this model has been widely employed but it seems still impossible to introduce it in FEM calculations because of prohibitive computer times. The situation is even worse for more complex and more exact models.

Continuum mechanics, on the other hand, considers the material as a whole and describes the anisotropy of the aggregate by phenomenological expressions such as Hill's quadratic [3] or non quadratic [4] criteria. Because of the relative simplicity of such expressions, they have been widely used in metal forming applications. However, the Hill quadratic criterion has been restricted to orthotropic textures, i.e. to the sheet symmetry, which disappears during the usual forming processes. The homogeneous (non necessarily

quadratic) Hill criterion, in its most convenient form, is subject to additional restrictions. Furthermore, the phenomenological approach does not take explicitly into account the first source of anisotropy of the material, i.e. the crystallographic texture, which evolves with strain.

It seems then interesting to develop a method which combines the advantages of both approaches: the simplicity of continuum mechanics together with the account of the texture of material. Montheillet et al. [5] (see also [6] and [7]) and Arminjon [8]—[10] have presented such methods. The first work proposes an approximation of the single crystal yield surface by a Hill type criterion (quadratic or not) but reduces the texture of the material to its principal components. The second one takes into account the whole texture (described by the orientation distribution function) but the approximation of the polycrystal behaviour is limited to a quadratic yield criterion, although a possible way of extension was briefly outlined in [10].

The aims of the present paper are:

i) to recall in a more geometrical way the principles of the Arminjon method [8]—[10], underlining some differences in the analytical presentation and some recent improvements in the numerical treatment,

ii) to discuss its application to yield surfaces and $R(\alpha)$ values calculations in the case of steel rolling textures,

iii) and last but not least to propose several extensions of this method to calculations of yield surfaces for different types of criteria. In doing so, some general considerations are delivered, regarding the possible analytical forms of the work function and yield criterion in plasticity.

## 2 Principle of the method

The method used here provides a fitting of the polycrystal yield surface calculated through a crystallographic approach by an analytical expression. The originality of the method consists in fitting, not the conventional yield surface, but the associated work function. This leads to analytical calculations in important cases. The fitting process will be described for the case of a Hill quadratic criterion and the Taylor-Bishop and Hill model: the extension to other types of functions and to other polycrystalline models will be discussed in some detail in the discussion part of this paper.

*The crystallographic approach*

All the grains in the polycrystal are supposed to be plastically deformed, hence the formulation is not valid during unloading or at the first stage of loading. Only the permanent part of the strain-rate is considered, which derives from the velocity of the permanent displacement: this latter is the displacement which would be measured after a total unloading from the current macroscopic stress. Thus the "plastic" (or permanent) strain-rate of a grain, $\boldsymbol{D}^g$, contains the rate of the residual elastic strain in this grain and we admit that the maximum work principle[1] holds with this phenomenological definition of $\boldsymbol{D}^g$. This is consistent with the nature of the Schmid law [11].

[1] $\boldsymbol{\sigma}_*:\boldsymbol{D}^g \leqq \boldsymbol{\sigma}^g:\boldsymbol{D}^g$ for any $\boldsymbol{\sigma}_*$ not exterior to the yield surface of the grain

If we designate by $\boldsymbol{D}$ the macroscopic plastic strain rate, the assumption of strain rate homogeneity [1], [2] can be written:

$$\boldsymbol{D}^g = \boldsymbol{D} \tag{1}$$

where $\boldsymbol{D}^g$ designates the strain rate tensor in grain $g$. We assume furthermore that deformation is accommodated by slip (of the pencil glide type, for steels) and that this slip mode obeys the Schmid law. The critical resolved shear stress $\tau_C$ will be taken the same for all slip systems in all the grains. Having thus characterized the microscopic behavior, and knowing the strain rate in each grain, it is then possible to calculate the plastic work rate $\dot{W}^g(\boldsymbol{D}^g)$ in each grain (for simplicity, the letter $g$ is used to designate the grain as well as its orientation) which is defined as[2]:

$$\dot{W}^g(\boldsymbol{D}^g) = S^g_{ij} \cdot D^g_{ij} = \boldsymbol{S}^g : \boldsymbol{D}^g \tag{2}$$

where $\boldsymbol{S}^g$, the deviatoric stress tensor in grain $g$, is determined through the maximum work rate principle associated with the Schmid law. This stress tensor can be normalized by $\tau_C$ and $\boldsymbol{D}^g$ can be also normalized by a certain norm $D_n$, which will be defined later. Having done that, $\dot{W}^g(\boldsymbol{D}^g)$ can be rewritten as:

$$\dot{W}^g(\boldsymbol{D}^g) = D_n \tau_c \frac{S^g_{ij}}{\tau_c} \frac{D^g_{ij}}{D_n} = D_n \tau_c M^g \left(\frac{\boldsymbol{D}^g}{D_n}\right) \tag{3}$$

where $M^g(\boldsymbol{D}^g/D_n)$ is the well known Taylor factor. To define $D_n$, we use here the 5 dimensional notation introduced by Lequeu et al. [6] to transform stress and strain rate tensors into vectors:

$$\begin{aligned} \boldsymbol{S} &= \left(\frac{S_{11} - S_{22}}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, S_{33}, \sqrt{2}\, S_{23}, \sqrt{2}\, S_{13}, \sqrt{2}\, S_{12}\right) \\ \boldsymbol{D} &= \left(\frac{D_{11} - D_{22}}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, D_{33}, \sqrt{2}\, D_{23}, \sqrt{2}\, D_{13}, \sqrt{2}\, D_{12}\right) \end{aligned} \tag{4}$$

which are work-conjugate, i.e.:

$$\dot{W}^g = S^g_{ij} \cdot D^g_{ij} = S_i{}^g \cdot D_i{}^g . \tag{5}$$

To avoid heavy notations, the same symbol is (abusively) employed for the deviatoric tensor and the corresponding vector in $\mathbb{R}^5$. Thus, the components will depend on one or two indices. The norm $D_n$ of the vector $\boldsymbol{D}$ in this $5D$ space is simply:

$$D_n = \|\boldsymbol{D}\| = \sqrt{(D_i D_i)} . \tag{6}$$

In this way, $M^g$ corresponds to the plastic work rate associated with a unit vector $\boldsymbol{D}^g$ and with a stress vector normalized by $\tau_C$. (In [8]—[10], the direct representation of general stress and strain-rate tensors by six-dimensional vectors is used. Incompressibility and the associated pressure-independency of the yield criterion leave only five independent components. A different norm is also used.) The average plastic work rate for the aggregate, $\dot{W}(\boldsymbol{D})$, can then be calculated from:

$$\dot{W}(\boldsymbol{D}) = \int_g \dot{W}^g(\boldsymbol{D}^g)\, f(\boldsymbol{g})\, d\boldsymbol{g} . \tag{7}$$

[2] The Einstein convention of summation on repeated indices $i$ and $j$ (but not on $g$) is used troughout the paper

Under the general conditions of macro-homogeneity [12] for the stress field and the permanent strain-rate field (the latter deriving from a continuous velocity field), the average (7) of the product $\boldsymbol{S}^g : \boldsymbol{D}^g$ equals the product of the averages $\boldsymbol{S} : \boldsymbol{D}$. Thus (7) is also the macroscopic rate of plastic work, independently of the Taylor assumption. In this equation $f(\boldsymbol{g})$ designates the orientation distribution function (ODF) which describes the texture of the material [13], [14] and the integration is performed over the whole orientation space. Using the series expansion method, Bunge [14] has shown that this function can be written as:

$$f(\boldsymbol{g}) = \sum_{l=0}^{l_{\max}} \sum_{\mu=1}^{M(l)} \sum_{\nu=1}^{N(l)} C_l^{\mu\nu} \ddot{T}_l^{\mu\nu}(\boldsymbol{g}). \tag{8}$$

The $\ddot{T}_l^{\mu\nu}$ functions form a base of harmonic functions on which the texture of the material can be expressed and described by the $C_l^{\mu\nu}$ coefficients. The chosen formalism assumes cubic symmetry of the crystal and orthotropic symmetry of the deformation process. This choice has been made because the method is applied in the following pages to rolling textures of steels but is by no means compulsory [10]. For more details concerning this series expansion method, the reader is referred to the original work of Bunge [13], [14].

Since the $\dot{W}$ function is homogeneous of degree one with respect to positive multipliers (i.e. $\dot{W}(\lambda \boldsymbol{D}) = \lambda \dot{W}(\boldsymbol{D})$ for $\lambda \geqq 0$), it is sufficient to work only with unit vectors $\boldsymbol{D}$ and/or $\boldsymbol{D}^g$ and normalized stress vectors $\boldsymbol{S}/\tau_C$ and replace the average plastic work rate $\dot{W}(\boldsymbol{D})$ by the average Taylor factor $M(\boldsymbol{D})$ which is calculated in the same manner as $\dot{W}(\boldsymbol{D})$ by:

$$M(\boldsymbol{D}) = \int_g M^g(\boldsymbol{D}^g)\, f(\boldsymbol{g})\, d\boldsymbol{g}. \tag{9}$$

Combining Eq. (8) and (9) leads to the following expression for $M(\boldsymbol{D})$:

$$M(\boldsymbol{D}) = \sum_{l\mu\nu} C_l^{\mu\nu} \int_g M^g(\boldsymbol{D}^g)\, \ddot{T}_l^{\mu\nu}(\boldsymbol{g})\, d\boldsymbol{g} \tag{10}$$

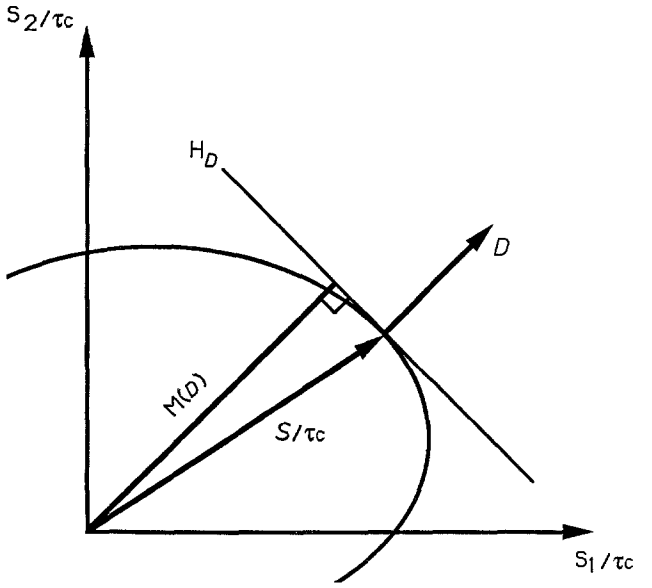


**Fig. 1.** The mean Taylor factor $\overline{M}(\boldsymbol{D})$ averaged over the whole orientation distribution function is represented by the distance to the tangent hyperplane associated with a given applied strain rate (unit vector). $2D$ schematic representation. The stresses are normalized by $\tau_c$

or more simply:

$$M(\boldsymbol{D}) = \sum_{l\mu\nu} C_l{}^{\mu\nu} M_l{}^{\mu\nu}(\boldsymbol{D}) = \sum_{i=1}^{i_{\max}} C_i M_i(\boldsymbol{D}) \tag{11}$$

using a single index notation. When $l_{\max}$ is taken equal to 22, which is usually the case in texture calculations, $i_{\max}$ is equal to 125. Here the coefficients $C_i$ describe the texture of the material and the $M_i(\boldsymbol{D})$ terms depend only on the macroscopic strain rate. Under Taylor's model and isotropic strain hardening assumptions, these strain rate functions can be easily calculated a priori and saved into a file for several values of $\boldsymbol{D}$ [8], [15]. It is worth noting that $M(\boldsymbol{D})$ can also be written as:

$$M(\boldsymbol{D}) = S_i \cdot D_i/\tau_c \tag{12}$$

where $\boldsymbol{S}$ is the macroscopis stress vector and $M(\boldsymbol{D})$ is equal to the distance from the origin to the tangent plane of normal $\boldsymbol{D}$, since $\boldsymbol{D}$ is a unit vector (see Fig. 1). In this way, the yield surface is defined as the inner envelope of all the hyperplanes $H_{\boldsymbol{D}}$ obtained by varying the direction of $\boldsymbol{D}$ in the $5D$ space.

*The phenomenological approach*

Here, the yield surface of the polycrystal is described by a Hill quadratic criterion which, in the case of orthotropic symmetry, is written as:

$$f(\boldsymbol{S}) = F(S_{33} - S_{22})^2 + G(S_{11} - S_{33})^2 + H(S_{22} - S_{11})^2 + 2\{LS_{23}^2 + MS_{13}^2 + NS_{12}^2\} = \tau_c{}^2. \tag{13}$$

Thus, the amount of isotropic hardening is given by the value of $\tau_C$, the CRSS defined previously and not by the flow stress in uniaxial tension as it is usually the case. The method may also be applied to a general (non-orthotropic) quadratic criterion, which in the case of incompressibility contains 15 independent coefficients [10].

The associated flow rule is[3]:

$$D_{ij} = \lambda \frac{\partial f}{\partial S_{ij}}. \tag{14}$$

$\lambda$ can be calculated from.

$$\dot{W}_h = S_{ij} \cdot D_{ij} = S_{ij}\lambda \frac{\partial f}{\partial S_{ij}} = 2\lambda\tau_c{}^2 \tag{15}$$

where the index $h$ stands for Hill. Since $f(S_{ij})$ is a quadratic function of the $S_{ij}$ components, it is possible to calculate these components as functions of 5 independent components of $\boldsymbol{D}$ and using (15), to write:

$$\frac{\dot{W}_h{}^2(\boldsymbol{D})}{\tau_c{}^2} = \frac{(F+H)D_{11}^2 + 2HD_{11}D_{22} + (G+H)D_{22}^2}{FH + GH + GF} + \frac{2D_{23}^2}{L} + \frac{2D_{13}^2}{M} + \frac{2D_{12}^2}{N} \tag{16}$$

where $F$, $G$, $H$, $L$, $M$ and $N$ characterize the anisotropy of the material and thus depend strongly on its texture and of course the $D_{ij}D_{kl}$ terms depend only on the imposed strain rate. Here again, it is sufficient to consider unit vectors $\boldsymbol{D}$ and $\dot{W}_h(\boldsymbol{D})/\tau_c$ can then be replaced

[3] To calculate the partial derivative, $S_{ij}$ has to be distinguished from $S_{ji}$

by $M_h(\boldsymbol{D})$. Equation (16) takes a simpler form if the following notation is introduced:

$$\begin{aligned} X_1 &= D_{11}^2, & \alpha_1 &= (F + H)/(FG + FH + GH) \\ X_2 &= D_{11}D_{22}, & \alpha_2 &= 2H/(FG + FH + GH) \\ X_3 &= D_{22}^2, & \alpha_3 &= (G + H)/(FG + FH + GH) \\ X_4 &= D_{23}^2, & \alpha_4 &= 2/L \\ X_5 &= D_{13}^2, & \alpha_5 &= 2/M \\ X_6 &= D_{12}^2, & \alpha_6 &= 2/N \end{aligned} \tag{17}$$

thus:

$$M_h^2(\boldsymbol{D}) = \alpha_k X_k \tag{18}$$

where the $\alpha_k$'s depend only on the anisotropy and the $X_k$ depend only on the strain rate. $M_h(\boldsymbol{D})$ can be used to represent the yield surface by the inner envelope of tangent planes, even if this procedure is rarely used in this case since we have an analytical expression of this surface (Eq. (13)).

*Principle of the identification procedure*

It has been seen that in both crystallographic and analytical approaches, it is possible to express $M(\boldsymbol{D})$ or $M^2(\boldsymbol{D})$ as functions of the same variable $\boldsymbol{D}$ and to separate texture (or anisotropy) and imposed strain rate. Both expressions contain some coefficients (on one hand, the $C_i$ coefficients and on the other, the $\alpha_k$) which depend only on the anisotropy (here the texture) of the material.

It has been seen also that the quantity $M(\boldsymbol{D})$ is equal to the distance from the origin to the yield surface. If we denote by $M_c(\boldsymbol{D})$ and $M_h(\boldsymbol{D})$ the normalized macroscopic plastic work rates as calculated in the two previous paragraphs ($c$ stands for crystallographic and $h$ for Hill), a good approximation of the crystallographic yield surface by a quadratic criterion can be obtained by minimizing the function (Fig. 2):

$$f_1 = \sum_{\boldsymbol{D}} \{M_c(\boldsymbol{D}) - M_h(\boldsymbol{D})\}^2 = \text{Min.} \tag{19}$$

with respect to the coefficients $\alpha_k$ introduced in Eq. (18).

However, due to the square root sign present in the expression of $M_h(\boldsymbol{D})$ (Eq. (18) defines the square of this quantity), this minimization does not correspond to a linear regression problem and should then be performed for each texture. Having noticed that, it was decided to replace (19) by the following minimum:

$$f_2 = \sum_{\boldsymbol{D}} \{M_c^2(\boldsymbol{D}) - M_h^2(\boldsymbol{D})\}^2 = \text{Min.} \tag{20}$$

which can be rewritten as:

$$f_2 = \sum_{\boldsymbol{D}} \{C_i C_j M_i(\boldsymbol{D})\, M_j(\boldsymbol{D}) - \alpha_k X_k\}^2 = \text{Min.} \tag{21}$$

In Eqs. (19) to (21), the sum is performed on unit vectors $\boldsymbol{D}$, uniformly distributed in $5D$ space. Differentiating $f_2$ with respect to $\alpha_k$ leads to a set of 6 linear equations with 6 unknowns from which we can easily get the coefficients $F$, $G$, $H$, $L$, $M$ and $N$ with the aid of Eq. (17). Arminjon [8] has also shown that minimizing the function $f_2$ as defined by

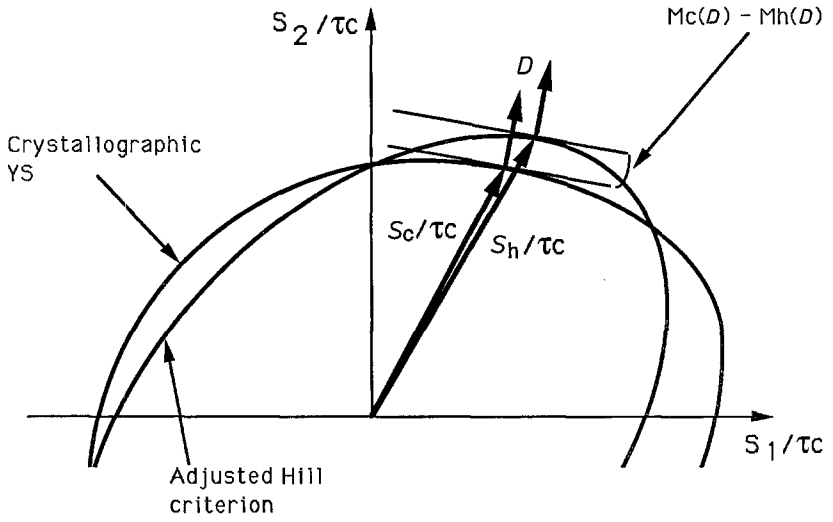


**Fig. 2.** Fitting of the crystallographic yield surface by a quadratic criterion. The minimization is performed with respect to the distance between tangent planes to the two surfaces for varying $\boldsymbol{D}$ vectors. The stresses are normalized by $\tau_c$

Eq. (21) was equivalent to minimizing the functions $G_{ij}$:

$$G_{ij} = \sum_{\boldsymbol{D}} \{M_i(\boldsymbol{D})\, M_j(\boldsymbol{D}) - \beta_k^{ij} X_k\}^2 \tag{22}$$

for each pair $(i, j)$. These functions depend only on the strain rate $\boldsymbol{D}$ and this minimization can be done *independently of any texture*. Again Eq. (22) leads to a set of 6 equations with 6 unknowns $\beta_k^{ij}$, which can be easily solved and the $\beta_k^{ij}$ stored into a file. For a given texture the $\alpha_k$ can then be obtained from:

$$\alpha_k = \sum_{i,j} \beta_k^{ij} \cdot C_i C_j. \tag{23}$$

It has been shown by several authors [8], [13]—[15] that a development of the texture function $f(\boldsymbol{g})$ up to $l_{\max} = 8$ is sufficient to calculate average plastic properties with satisfactory accuracy even though this value is too low to describe the texture. When $l_{\max} = 8$, $i_{\max} = 13$ and only $91 \times 6\beta_k^{ij}$ coefficients need to be calculated once ($i$ and $j$ vary from 1 to 13 and $k$ varies from 1 to 6).

## 3 Numerical procedure

The first step is to choose the distribution of $\boldsymbol{D}$ vectors necessary to perform the minimization of Eq. (22). It was decided here to sweep the $5D$ space uniformly. A unit vector $\boldsymbol{D}$ can be described in a $5D$ space by 4 angles $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ such that [16]:

$$\begin{aligned} D_1 &= \cos\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \\ D_2 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \\ D_3 &= \cos\theta_2 \sin\theta_3 \sin\theta_4 \\ D_4 &= \cos\theta_3 \sin\theta_4 \\ D_5 &= \cos\theta_4. \end{aligned} \tag{24}$$

$\theta_1$ ranges from 0 to $2\pi$ and the other angles from 0 to $\pi$. The element of area on the unit hypersphere is consequently equal to:

$$dS = d\theta_1 d\cos\theta_2 d\left(\frac{1}{2}\theta_3 - \frac{1}{4}\sin 2\theta_3\right) d\left\{\frac{1}{3}(2 + \sin^2\theta_4)\cos\theta_4\right\} \tag{25}$$

which leads to $8\pi^2/3$ for the surface area of the unit sphere. Because of the orthotropic symmetry of the texture, it is possible to reduce the domain of variation of the different angles as follows: $\theta_1$ varies between 0 and $2\pi$, $\cos\theta_2$ between $-1$ and 1, $(1/2\theta_3 - 1/4\sin 2\theta_3)$ between 0 and $\pi/4$ and $\{1/3(2 + \sin^2\theta_4)\cos\theta_4\}$ between 0 and 2/3 in regular intervals. In this way, only a quarter of the sphere is described. In what follows, we have considered in our calculations $40 \times 20 \times 10 \times 10$ unit vectors $\boldsymbol{D}$.

The second step is to calculate the functions $M_i(\boldsymbol{D})$ for all the vectors $\boldsymbol{D}$ considered. Let us recall first that $M_i(\boldsymbol{D})$ is equal to:

$$M_i(\boldsymbol{D}) = \int_g M^g(\boldsymbol{D})\, T_i(\boldsymbol{g})\, d\boldsymbol{g} \tag{26}$$

which can be regarded as the series expansion coefficients of the Taylor factor on the base of harmonic functions [14]. Theses coefficients can be calculated for every direction $\boldsymbol{D}$ but can also be calculated for specific forms of the strain rate tensor only and then approximated by polynomial forms. This procedure has been explained in details by Bunge et al. [15] and is recalled in Appendix 1, since the coefficients published by these authors were used in this work.

Once the functions $M_i(\boldsymbol{D}) \cdot M_j(\boldsymbol{D})$ are calculated for all vectors $\boldsymbol{D}$ uniformly distributed in $5D$ space, it is then possible to calculate the coefficients $\beta_k{}^{ij}$. It is worth mentioning at this point that this procedure varies from the one originally proposed by Arminjon [8] who selected a different distribution of strain rate vectors which were unit vectors with respect to the other norm he introduced. The procedure described above has the advantage of better reproducing the possible symmetries of the textures which are considered.

It can be useful as this point to compare several minimization conditions:

— $f_1$ versus $f_2$, i.e. the minimization performed on distances rather than on the squares of these distances.

— $f_2$ versus $G_{ij}$, i.e. the direct minimization performed on the yield surface of the polycrystal rather than the one performed on the mathematical functions $M_i(\boldsymbol{D}) \cdot M_j(\boldsymbol{D})$.

These different procedures were compared for 3 different textures: i) a texture composed of only one ideal orientation $\{100\}\langle 010\rangle$ with a Gaussian spread of 7° around it, ii) an isotropic polycrystal and iii) the experimental texture of a rolled mild steel. For each of these 3 textures, the minimizations performed on $f_1$ and $f_2$ (Eqs. (19) and (20)), with or without the aid of Eqs. (22) and (23) (direct or non-direct minimization). The results are presented in Table 1 and speak for themselves: the minimization carried out on the $M_iM_j$ functions leads to completely equivalent results as the so-called "direct" minimization (this has already been shown by Arminjon [8]). On the other hand, a slight difference can be observed when comparing the results obtained with $M$ or $M^2$. This difference is too small to be relevant here, and only the minimization on $M^2$ achieved with the aid of the $M_iM_j$ functions will be performed in what follows, since it has the advantage of considerably reducing the computing time.

The average quadratic difference $\delta$ between the Taylor factor $M_c(\boldsymbol{D})$ and the obtained fitting $M_h(\boldsymbol{D})$ was also calculated, for ideal and for experimental textures. For the cube

**Table 1.** Calculated values of the Hill criterion coefficients for 3 textures and 3 minimization procedures

| Texture | Minimization of | | |
|---|---|---|---|
| | $\sum \{M_c(D) - M_h(D)\}^2$ | $\sum \{M_c^2(D) - M_h^2(D)\}^2$ | $\sum \{M_i(D)\, M_j(D) - \beta_{ij} X_k\}^2$ |
| Single crystal {100} ⟨010⟩ Gaussian spread 7° | $F = 0.1528$ $L = 0.1449$<br>$G = 0.1625$ $M = 0.1466$<br>$H = 0.1623$ $N = 0.1496$ | $F = 0.1744$ $L = 0.1420$<br>$G = 0.1741$ $M = 0.1445$<br>$H = 0.1740$ $N = 0.1484$ | $F = 0.1744$ $L = 0.1420$<br>$G = 0.1741$ $M = 0.1445$<br>$H = 0.1740$ $N = 0.1484$ |
| Isotropic polycrystal | $F = 0.0684$ $L = 0.2078$<br>$G = 0.0684$ $M = 0.2078$<br>$H = 0.0684$ $N = 0.2078$ | $F = 0.0693$ $L = 0.2077$<br>$G = 0.0693$ $M = 0.2072$<br>$H = 0.0693$ $N = 0.2072$ | $F = 0.0693$ $L = 0.2077$<br>$G = 0.0693$ $M = 0.2075$<br>$H = 0.0693$ $N = 0.2075$ |
| Rolled and annealed mild steel | $F = 0.0591$ $L = 0.2319$<br>$G = 0.0597$ $M = 0.2306$<br>$H = 0.0769$ $N = 0.1909$ | $F = 0.0592$ $L = 0.2319$<br>$G = 0.0597$ $M = 0.2302$<br>$H = 0.0768$ $N = 0.1909$ | $F = 0.0592$ $L = 0.2319$<br>$G = 0.0597$ $M = 0.2302$<br>$H = 0.0768$ $N = 0.1909$ |

orientation, $\delta$ reached its maximum value, namely 0.17, whereas $\delta = 0.09$ for the {111} fibre. For experimental textures, $\delta$ increases from 0.02 to 0.06 when going from a weakly to a strongly textured steel. These values are worth to be compared with the average value of $M_c$ which is approximately 3. Thus, the fitting of the crystallographic work function by the phenomenological one (derived from the Hill criterion) is quite satisfactory, even for ideal orientations.

## 4 Application to yield surfaces and $R(\alpha)$ calculations

The identification method described above has been applied to several "ideal" and experimental textures of rolled and annealed steels. Yield surface sections as well as $R(\alpha)$ values have been calculated using both the Taylor model and the adjusted Hill criterion. The "ideal" textures are composed of one of the principal components of textures usually found in steels with a Gaussian spread taken equal to 7° around the principal component.

A measure of the anisotropy of a rolled sheet can be gained by carrying out a tensile test on a flat specimen cut in the rolled sheet at an angle $\alpha$ of the rolling direction and by calculating the so-called Lankford coefficient (or strain rate ratio) which is by definition equal to:

$$R(\alpha) = D_{22}(\alpha)/D_{33}(\alpha) \tag{27}$$

where $D_{22}(\alpha)$ and $D_{33}(\alpha)$ are components of the strain rate tensor expressed in the reference system linked to the tensile test. The boundary conditions generally used to characterize such a test in a Taylor approach can be described by the following tensors (expressed in the tensile test axes):

$$\boldsymbol{D}_T = \begin{vmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{vmatrix} \Rightarrow \boldsymbol{\sigma}_T = \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{28}$$

which can be considered to be rigorously valid only for short samples (because of the condition $D_{12} = 0$, which would not be true in the middle of a long sample). The derivation of $R(\alpha)$ from these boundary conditions is explained in Appendix 2.

*Ideal textures*

Most of the ideal orientations belonging to the $\alpha = \{hkl\}\ \langle 110\rangle$ and $\gamma = \{111\}\ \langle uvw\rangle$ fibres have been examined here. Sections of yield surfaces have been calculated for $\tau_C = 1$ in the closed sub-space $(S_{11}, S_{22}, S_{12})$ [17]. The expression "closed sub-space" refers to the fact that when a strain rate state with $D_{13} = D_{23} = 0$ is imposed to the material, the resulting deviatoric stress state has the special form $S_{13} = S_{23} = 0$ due to the orthotropic symmetry of the sheet. The section of yield surface obtained in this way is called the "tricomponent plane stress yield surface" [18] and can be represented as these authors did (see also [19], [20]) as sections (in the stress space) with planes parallel to the $(\sigma_{11}, \sigma_{22})$ plane for various values of $\sigma_{12}$ (with the additional condition $\sigma_{33} = 0$).

Such sections are shown in Fig. 3 for 6 ideal orientations as well as a fibre texture and the isotropic case. The yield surfaces calculated with the Taylor model have been drawn as the inner envelope of 800 tangent planes (which is in fact a too small number to obtain a precise description and sometimes gives to the sections presented in Fig. 3 an exaggeratly "angular" shape). The corresponding $R(\alpha)$ variations are presented in Fig. 4. Looking at Figs. 3 and 4, the ideal textures can be classified into several groups, depending on the quality of the adjustment of both yield surfaces ($YS$) and $R(\alpha)$ variations:

i) For "single crystals" such as $\{110\}\ \langle 001\rangle$ and $\{110\}\ \langle 110\rangle$, the 2 predicted $R(\alpha)$ curves are very close to each other whereas the $YS$ sections are quite different.

ii) On the contrary, for the $\{111\}\ \langle uvw\rangle$ orientations, the $YS$ sections are reproduced quite satisfactorily, whereas the $R(\alpha)$ variations calculated with the aid of the Taylor model can by no means be predicted by a Hill criterion: it is well known that a quadratic Hill criterion cannot predict more than 4 ears (associated with the maxima of the $R(\alpha)$ curve) whereas the variations shown in Fig. 4 for these components correspond to 6 ears. Similarly, the component $\{11\ 11\ 8\}\ \langle 4\ 4\ 11\rangle$ (located at 7° from the $\{111\}\ \langle 112\rangle$ component in orientation space) has to be placed also in this category, since it produces 6 ears of different heights.

iii) The isotropic texture illustrates the case of a perfect agreement in terms of $R(\alpha)$ ($R = 1 = C$st in both cases) and of a good adjustment of the $YS$. In this case, only the first $C_i$ coefficient is non-zero and thus only the $G_{11}(\boldsymbol{D})$ function (see Eq. (22)) plays a role in the minimization: this function is in fact almost quadratic and thus the adjustment is quite acceptable.

iv) finally the $\{112\}\ \langle 110\rangle$ component represents the more general case: the sharp "corners" of the crystallographic $YS$ are rounded whereas the variations of the strain rate coefficient are somewhat attenuated by the quadratic criterion.

It is worth mentioning that since these textures are synthetic, nothing is known about the real shape of the $YS$ or of the $R(\alpha)$ curve. In particular, it is difficult to establish experimentally whether a single crystal of orientation $\{111\}\ \langle 110\rangle$ produces 6 ears or not.

*Experimental textures*

22 experimental textures of rolled and annealed mild and stainless steels were then studied. They have been selected for the wide variety of experimental $R(\alpha)$ they represent: the

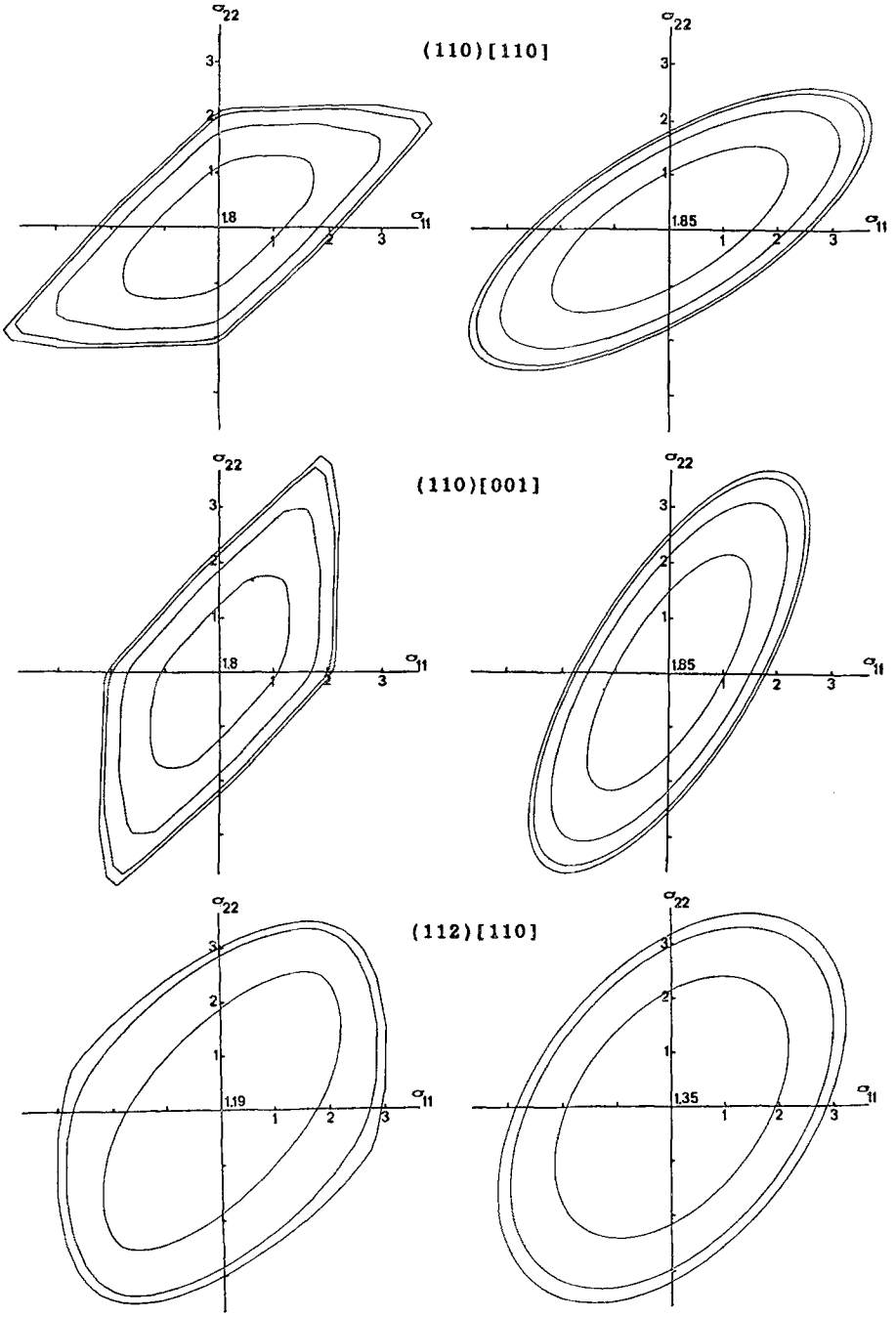


**Fig. 3.** Tricomponent plane stress surfaces obtained for ideal textures with the Taylor model (left) and adjusted Hill criterion (right). The different sections correspond to $\sigma_{12} = 0.$, 0.5, 1.0 and 1.5 ($\times \tau_c$) and the number written in the center is the value of $\sigma_{12}$ on the axis (i.e., for $\sigma_{11} = \sigma_{22} = 0$)

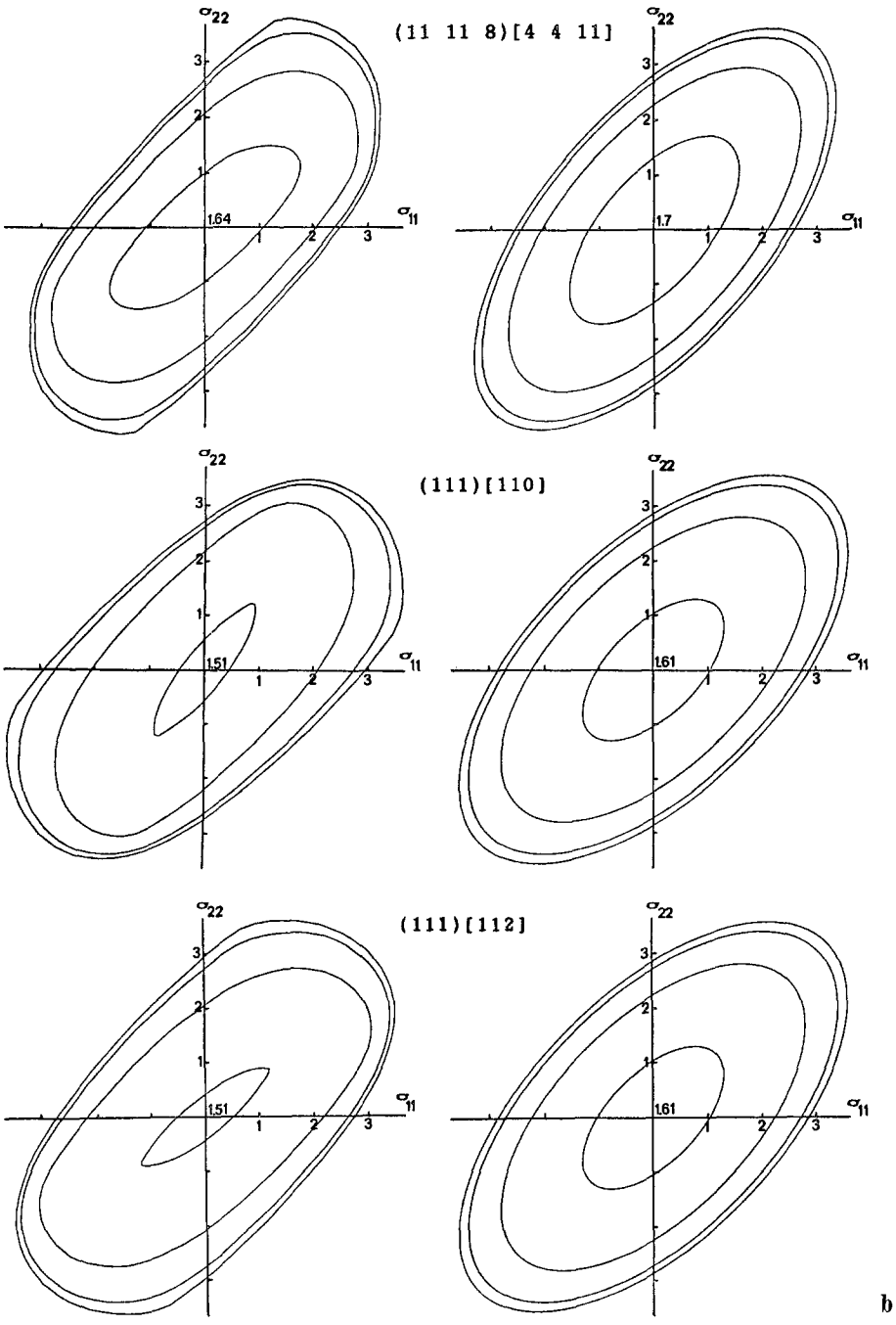


Fig. 3.

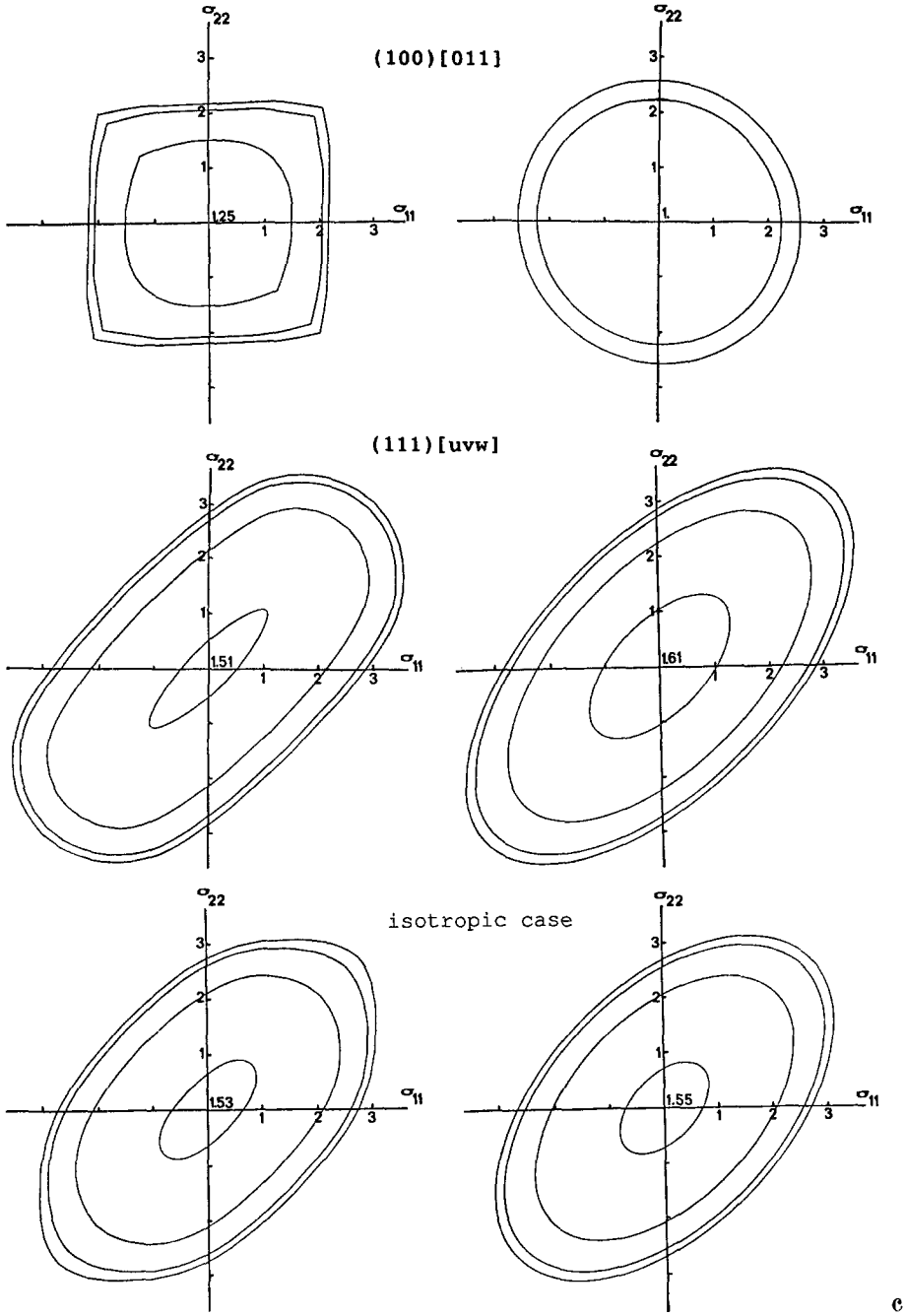


**Fig. 3.**

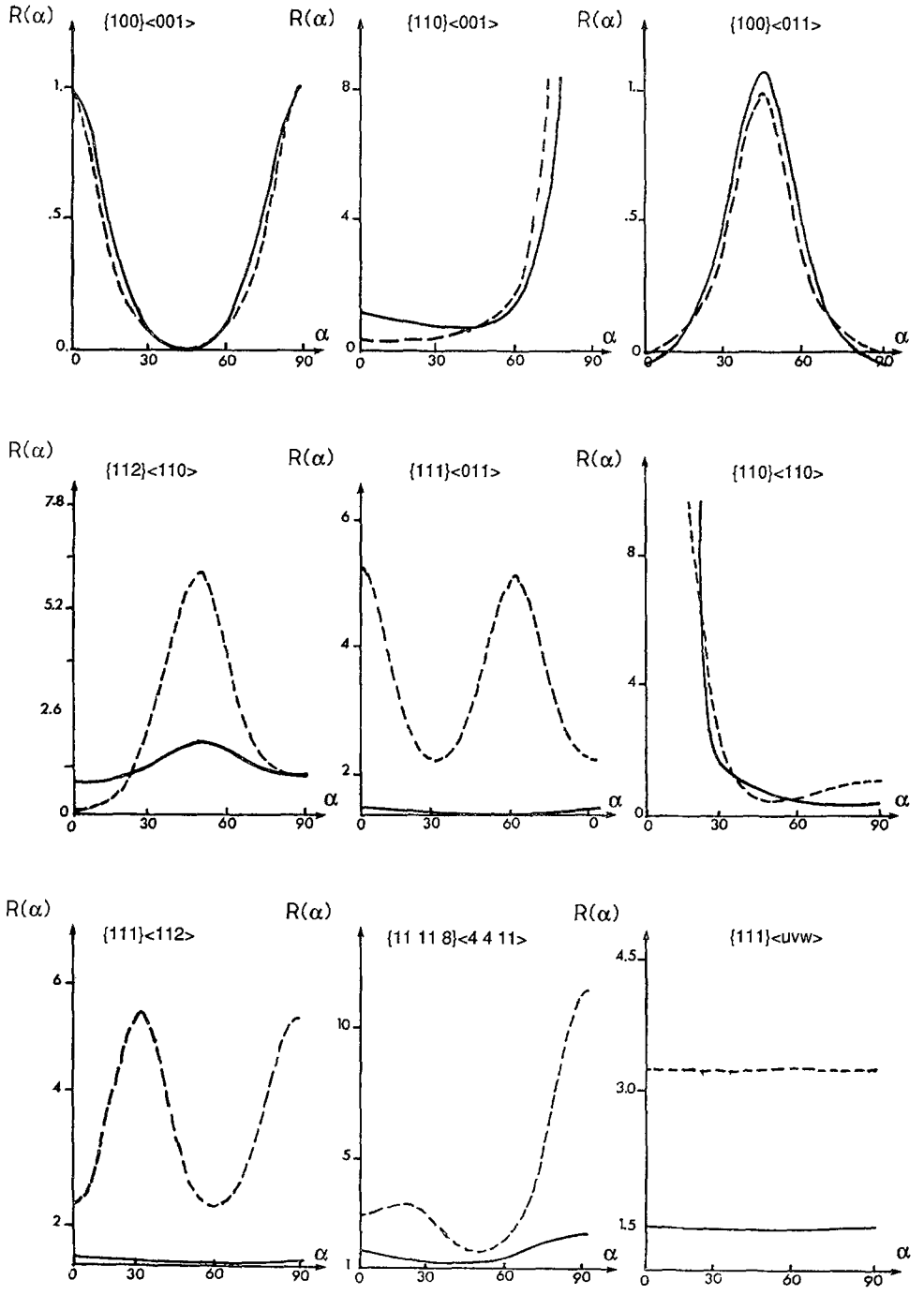


**Fig. 4.** $R(\alpha)$ variations for the 9 ideal textures investigated (Taylor model and adjusted Hill criterion)

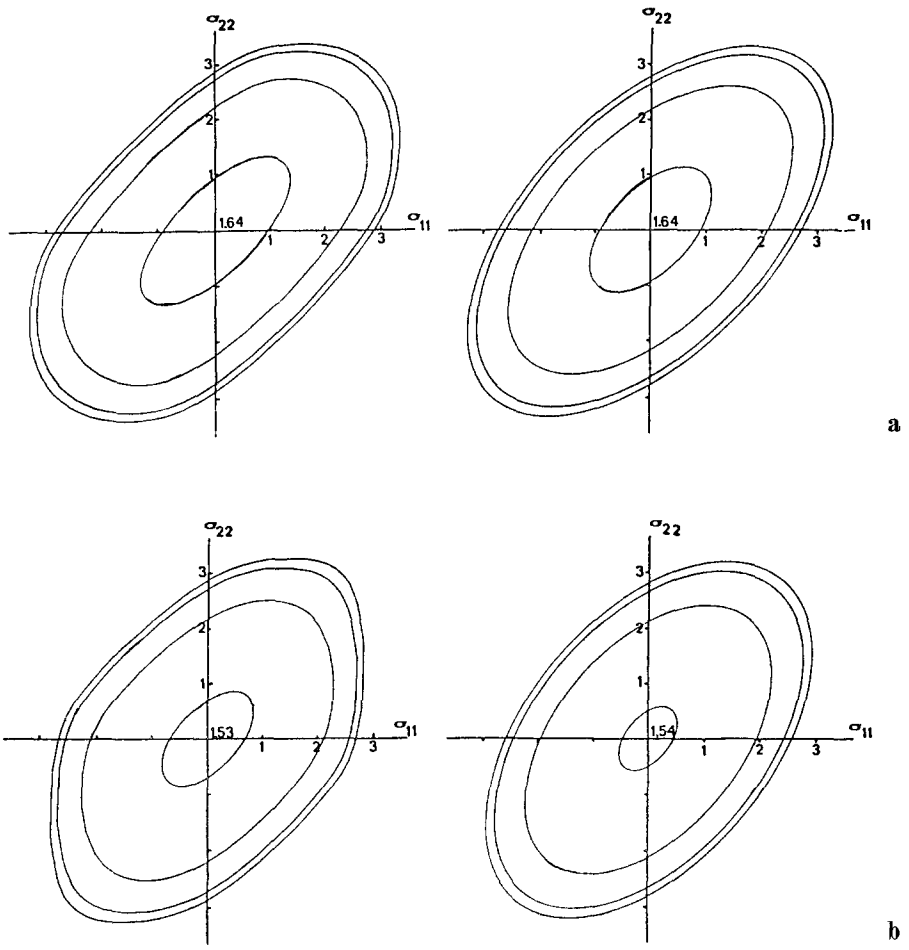


**Fig. 5.** Tricomponent plane stress surfaces for **a** a rolled and annealed steel and **b** a partially recrystallized stainless steel

high $R$ values found in mild steels indicate a good drawability of these steels whereas the low $R$ values found sometimes in annealed stainless steels correspond to a partially recrystallized structure and are thus an indication of a bad selection of annealing parameters. This last point was not relevant for our study but has to be kept in mind, since the inhomogeneity of the microstructure could be linked to the inhomogeneity of the texture through the thickness of the sample. In order to better compare experimental $R$ values to predictions, this texture variation should have been taken into account, which was not the case in our investigation.

Two examples are presented in Figs. 5 and 6: one rolled and annealed mild steel and one partially recrystallized stainless steel. In the first case, the texture presents a high proportion of $\{111\}\langle uvw\rangle$ fibre, resulting in rather high $R$ values whereas in the second case, the $R$ values are very low and the $\Delta R$ (equal to $1/2\big(R(0°) - 2R(45°) + R(90°)\big)$) parameter is negative. In both cases the adjustment of the $YS$ is acceptable and the variation of $R$ is also well reproduced even if the values found with the adjusted Hill criterion are lower than the experimental ones.

The $\bar{R}$ and $\Delta R$ values calculated for the 22 steels with the 2 models are compared with experimental values in Figs. 7 and 8 ($\bar{R} = \big(R(0°) + 2R(45°) + R(90°)\big)/4$ is a "mean" value

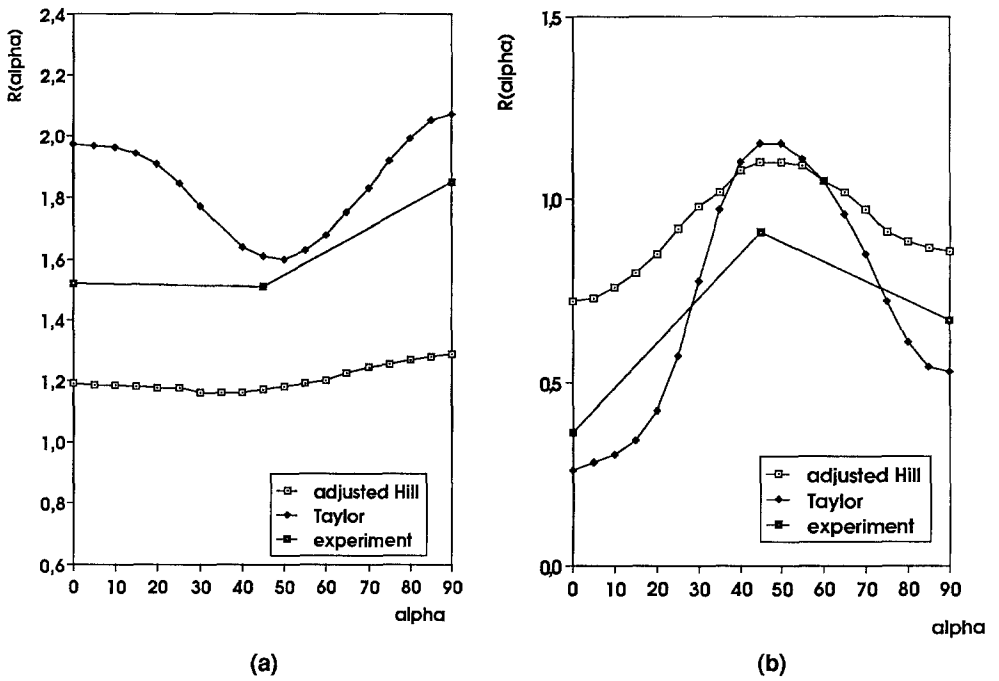


**Fig. 6.** $R(\alpha)$ variations for **a** a rolled and annealed steel and **b** a partially recrystallized stainless steel

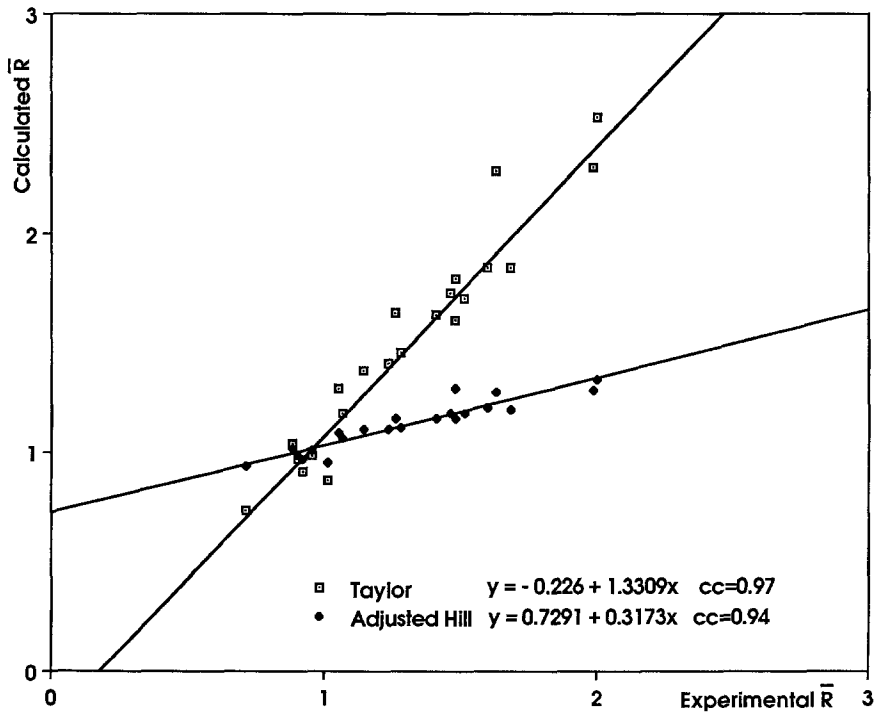


**Fig. 7.** Correlation between experimental and predicted $\overline{R}$ values

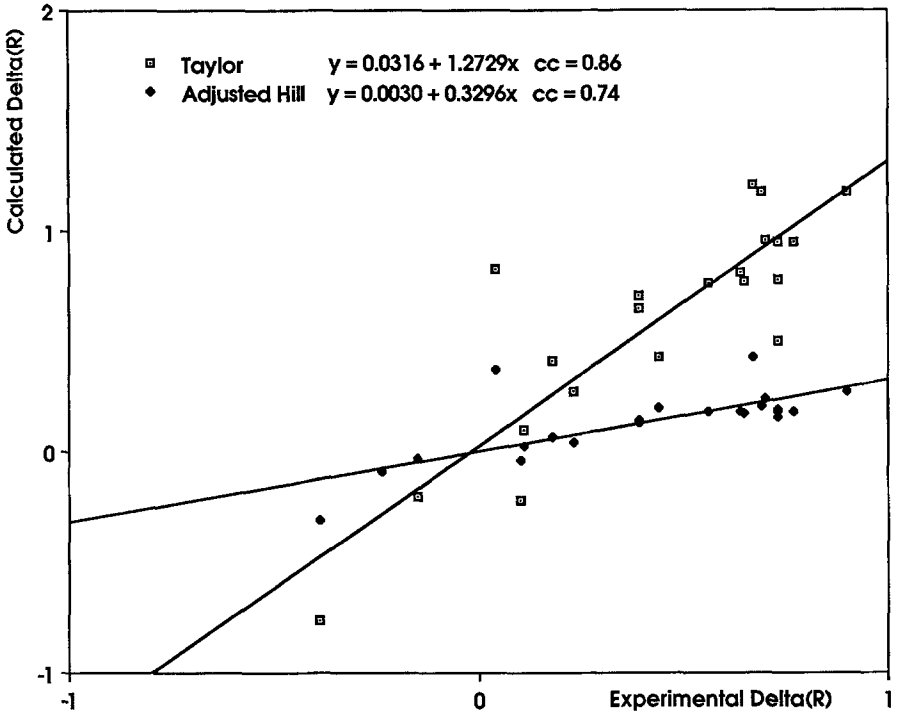


**Fig. 8.** Correlation between experimental and predicted $\Delta R$ values

of $R$ in the sheet plane). These figures call for 2 general comments:

i) the anisotropy, as measured by the $(R - 1)$ values, is systematically overestimated by the Taylor model and underestimated by the adjusted Hill criterion.

ii) in both cases, a linear correlation between calculated and experimental values can be established: the correlation coefficient is close to one for $\bar{R}$, which is an indication of a reliable correlation.

*Discussion*

It could be argued however that a reliable correlation is not sufficient and that the equations of the linear correlations obtained here do not speak in favor of the adjusted Hill criterion. Some remarks are worth mentioning:

i) The worst case was in fact investigated here, i.e. most of the experimental textures contain a proportion of the {111} fibre which is badly described by a quadratic criterion (see Figs. 3 and 4). Even for steel sheets, the experimental $R(\alpha)$ values are, generally speaking, close to the arithmetic average of the two calculated $R$-values [11]. Hence, the proposed method generally gives as accurate predictions as does the classical Taylor model, with the considerable advantage of being derived from an easy implemented yield criterion. Furthermore, when a calculation of the $R$-values from the texture is the unique aim, the joined use of the classical method and this method provides thus more accurate predictions than does the classical one alone. On the other hand, this method should be also tested in the case of recrystallization textures of Al sheets for example which contain a high proportion of the cube texture (which is nothing else than the {110} ⟨100⟩ single crystal, rotated 45° around the normal direction, described in Figs. 3 and 4).

ii) More realistic polycrystal models than the simple Taylor model may also be used in the identification procedure (21), (22), e.g. the "inhomogeneous extremum-based model" [9], [21]. However, the obtained Hill coefficients and the corresponding $R(\alpha)$ values are very close to those obtained using the Taylor model in the identification procedure [9], [11], although the predicted deformation textures, and also the $R$-values obtained by the classical minimization described in Appendix 2, are quite different [11], [21].

iii) Though the yield surface predicted by the classical Taylor model is more angular than our fitted Hill yield surface (whence the observed differences in the $R(\alpha)$-values), these two surfaces are close to each other, except for some ideal textures. Thus, if one looks at the predicted *stresses*, the two models should exhibit smaller differences. This is confirmed for industrial steel sheets [11]; moreover, the theoretical stresses obtained with our texture-adjusted Hill criterion have been found to be in a better experimental agreement than the stresses corresponding to the Taylor yield surface.

iv) The two preceding remarks seem to imply that the very stable adjustment which is performed by the present method, is not far from being the best fitting of the *experimental* yield surface by a quadratic criterion. Now, the less precise description of the $R$-values (deduced from the yield surface by the normality rule) simply means that the ellipsoidal shape of the quadratic criterion is too poor to describe simultaneously the yield stresses and the flow rule within the experimental accuracy. Moreover, as noted by Stout et al. [22], the normality flow-rule is unstable with respect to small perturbations of the yield surface (see also [9] in the case of the $R(\alpha)$ calculations). In the classical adjustment of the Hill coefficients from uniaxial tension tests, this shortcoming of the quadratic criterion appears differently, since the $R(\alpha)$ values ($R(0°)$, $R(45°)$ and $R(90°)$) are usually the input data (e.g. together with one yield stress, as far as plane stress states are considered); then, the observed discrepancies concern the values of the yield stresses [22], [23] but also the strain ratios for other loading paths [22].

In view of these remarks, the most important limitation of the proposed method is confirmed to be the use of the Hill quadratic criterion. The extension to a non-orthotropic quadratic criterion [10], theoretically necessary, could be a first practical improvement [24]. The possibility of extending this method to non-quadratic yield criteria has now to be examined separately.

## 5 Extension to non-quadratic criteria

In classical plasticity, the yield condition is usually expressed as function of the stress tensor:

$$\phi(\sigma) = \tau. \tag{29}$$

It is then rarely possible to express $\dot{W}(\boldsymbol{D})$ as an explicit function of the $\boldsymbol{D}$ tensor, due to the difficulty of inverting the normality rule. This is why the previous work has been limited to a quadratic criterion which is the only case where this inversion can be easily performed.

It is worth mentioning that the above mentioned difficulty can be overcome by remembering that the $\dot{W}(\boldsymbol{D})$ function completely describes the yield surface (Fig. 1). If the function $\phi$ is taken as homogeneous of degree one with respect to positive multipliers and is symmetrized in $\sigma_{ij}$ and $\sigma_{ji}$, the normality rule uniquely determines:

$$\dot{W}(\boldsymbol{D}) = \sigma_{ij} D_{ij} = \tau\psi(\boldsymbol{D}) \tag{30}$$

where $\psi$ is convex, symmetrized and similarly homogeneous [11], [25]—[27]. The rule itself can now be written as:

$$D_{ij} = \psi(\boldsymbol{D}) \cdot \frac{\partial \phi}{\partial \sigma_{ij}} \tag{31}$$

from where it follows that [11], [25]—[27]:

$$\sigma_{ij} = \tau \frac{\partial \psi}{\partial D_{ij}} \tag{32}$$

showing that the function $\psi$ acts as a "work function" or potential for the stress tensor. Equation (32) can be more simply rewritten as:

$$\boldsymbol{\sigma} = \frac{\partial \dot{W}}{\partial \boldsymbol{D}}. \tag{33}$$

The function $\psi(\boldsymbol{D})$ is a "dual" function of $\phi(\boldsymbol{\sigma})$ [26], [27]. The concept of a dual surface allows us to describe the plastic behaviour of a material with the aid of $\dot{W}(\boldsymbol{D})$, and to deduce the stress from $\dot{W}(\boldsymbol{D})$, using Eqs. (30) and (33). There is now a complete correspondence between the analytical and crystallographic approaches: in both cases, the strain-rate $\boldsymbol{D}$ is imposed and the stress tensor is deduced from $\dot{W}(\boldsymbol{D})$.

*On analytical forms of the work function and yield criterion*

Let us examine the restrictions to be imposed to the $\dot{W}(\boldsymbol{D})$ function. First, it must be convex. From the flow rule and the volume invariance during plastic deformation, it follows that $\dot{W}$ depends only on the deviatoric part of $\boldsymbol{D}$. It must also obey the material symmetries. Finally, the $\dot{W}$ function must be positively homogeneous. Conversely, any function $\dot{W}(\boldsymbol{D})$ satisfying these four requirements defines a unique yield surface $(\Sigma_{W'})$ of a standard, incompressible plastic material obeying the imposed material symmetries: $(\Sigma_{W'})$ is the inner envelope of the hyperplanes $H_D$ (Eq. (30)). Here we notice that any function $\phi$ satisfying the four requirements provides also the *direct* expression of a different yield surface $(\Sigma_\phi)$ (Eq. (29)), when it is taken as the yield function, depending on the deviatoric *stress* tensor. The homogeneity of $\phi$ is not necessary in the definition of the yield surface (29); it means that the parameter $\tau$ is a scalar measure of the stress magnitude i.e. an equivalent stress naturally associated with the shape of the yield surface $(\Sigma_\phi)$ [27]. Moreover, any closed convex surface $(\Sigma)$ defines naturally a homogeneous and convex function $\phi$: for any deviatoric tensor $\boldsymbol{\sigma}$, $\phi(\boldsymbol{\sigma}) = \tau$ is the unique positive number such that $\boldsymbol{\sigma}/\tau$ belongs to $(\Sigma)$ [28]. Thus, the search of a *general* analytical expression for the work function or the search for the yield function are *completely* equivalent. Any possible form of the work function may also be used as an analytical form of the yield function $\phi(\boldsymbol{\sigma})$ and conversely (but not for the same material, since the duality correspondence between the work function and the yield function of a given material [27] generally alters the analytical form).

The most expedient way to satisfy the material symmetries is to assume that the material has *no* particular symmetry (at the macroscopic scale), i.e. to develop the model for the most general anisotropy. Once the coefficients of the $\dot{W}$ function are expressed as functions of the texture coefficients, the material symmetry affects these coefficients [14] and thus transmits automatically to the $\dot{W}$ function. In applications however, it is generally better to take into account the largest possible symmetry group since it considerably reduces the number of coefficients in the formulae. Then, the incompressibility constraint

is directly accounted for if one defines the $\dot{W}$ function in the 5dimensional strain-rate space. *The essential point here is thus to find an analytical form of* $\dot{W}(\boldsymbol{D})$ *ensuring that the requirements of convexity and homogeneity are simultaneously satisfied.*

When the analytical expression of $\dot{W}$ is directly given as a homogeneous function of five variables, the convexity of $\dot{W}$ is easier to recognize. A general method is to express $\dot{W}(\boldsymbol{D})$ as follows:

$$\dot{W}(\boldsymbol{D}) = L(\boldsymbol{D}) + [P(\boldsymbol{D})]^{1/n}, \tag{34}$$

where $L(\boldsymbol{D}) = \boldsymbol{\sigma}_0 : \boldsymbol{D}$ is a linear form which allows to take into account the Bauschinger effect and $P$ is a positive function, homogeneous of degree $n$ (i.e. $P(\lambda \boldsymbol{D}) = |\lambda|^n \cdot P(\boldsymbol{D})$ for any $\lambda$) [10]. The problem is now to explore the possible forms of $P$. If $P$ is a quadratic form (which is a particular case of the case $n = 2$):

$$P(\boldsymbol{D}) = \alpha_{kl} D_k D_l \quad \text{with} \quad 1 \leqq k, \qquad l \leqq 5 \tag{35}$$

then the associated yield criterion $\phi$ is also a (shifted) quadratic form, whose coefficients are proportional to the inverse matrix $B = \alpha^{-1}$ [11] (the "shift" is $\boldsymbol{\sigma}_0$). In contrast, if the work function of a given material is assumed to have the form (34) with $n \neq 2$ (or with $n = 2$ but $P$ not quadratic), then the yield criterion has no simple analytical expression; obviously, the converse is also true: if one assumes a $n$-homogeneous yield criterion (shifted, if desired), the work function can only be obtained numerically. The quadratic case has one still more essential particularity: a bounded quadratic surface (in the deviatoric plane) is necessarily an ellipsoid, and thus is convex. Hence, any reasonable fitting of a yield surface by a quadratic one saves automatically the desired convexity. Dually, this applies to work functions as well; these are fitted on the unit sphere, as we have seen. It is easy to verify that a quadratic form is convex if and only if it has a positive lower bound on the unit sphere. Obviously, the work function of a real material (e.g. the average Taylor factor) must have this latter property: thus, when fitted by a quadratic form, the work function remains (or even becomes) convex. This is not true in general for non-quadratic expressions, e.g. it is wrong for quartics, as these have been proposed by Gotoh [29] for the yield criterion.

*Identification procedure and analytical formulae*

An "analytical expression" of the $\dot{W}(\boldsymbol{D})$ function involves a finite number of coefficients $\alpha_k$ and writes most generally:

$$\phi(\boldsymbol{D}, \alpha_k) = \dot{W}_a(\boldsymbol{D}), \tag{36}$$

where $\phi$ might depend non-linearly on the $\alpha_k$ coefficients. In this case, an iterative identification procedure might be used, providing the best fitting values of the $\alpha_k$ for a given texture, i.e. for a given "crystallographic work function" (Eq. (11)). We can now restrict $\phi$ to the following expression (since all known expressions are particular cases of this expression):

$$\dot{W}_a(\boldsymbol{D}) = L(\boldsymbol{D}) + \left\{ \sum_{k=1}^{N} \alpha_k \psi_k(\boldsymbol{D}) \right\}^{1/n}, \tag{37}$$

where $L$ is a linear form and the $\psi_k$ linearly independent, positive and $n$-homogeneous functions of the (deviatoric) tensor $\boldsymbol{D}$. The linear form would be of interest if the "crystallographic" work function $\dot{W}_c(\boldsymbol{D})$ is unsymmetrical as a result of different critical shear stresses in the two different slip directions of some glide systems [8]. The fitting procedure has then

simply to be applied separately to $\dot{W}_{c,1}(\boldsymbol{D}) = 1/2[\dot{W}_c(\boldsymbol{D}) - \dot{W}_c(-\boldsymbol{D})]$, giving $L$, and to $\dot{W}_{c,2}(\boldsymbol{D}) = 1/2[\dot{W}_c(\boldsymbol{D}) + \dot{W}_c(-\boldsymbol{D})]$, giving the symmetrical sum in (37). Hence, we can restrict the discussion to "unshifted" expressions ($L = 0$). Van Houtte et al. [30], [31] propose the following form for $\dot{W}$:

$$\dot{W}(\boldsymbol{D}) = a_{pq}D_pD_q + b_{pqrs}D_pD_qD_rD_s + c_{pqrstu}D_pD_qD_rD_sD_tD_u + \cdots \tag{38}$$

when $\|\boldsymbol{D}\| = 1$. Expression (38) falls into the category represented by Eq. (37) if $n = 1$ and $\psi_k$ are the functions obtained by the homogeneous extension of the independent terms in (38), thus:

$$\psi_{pq} = \frac{D_pD_q}{\|\boldsymbol{D}\|}, \qquad \psi_{pqrs} = \frac{D_pD_qD_rD_s}{\|\boldsymbol{D}\|^3} \cdots. \tag{39}$$

If $n$ is some positive number, any expression of the form (37) may be used as a fitting function for the crystallographic work function and the $\alpha_k$ coefficients are then obtained by a *linear* regression, i.e. a least square procedure applied to the $n$th power, $\dot{W}^n_{c,2}$. Thus, we have extended our identification procedure to a very wide class of anisotropic constitutive relations, given by their work function. Moreover, if $n$ is an *integer* (which does not imply that the $n$-homogeneous term in (37) is a polynomial), the identification of $\dot{W}^n_{c,2}$ becomes *analytical*, as well as in the case of the quadratic criterion. Indeed, we have for $\|\boldsymbol{D}\| = 1$:

$$\dot{W}^n_{c,2}/\tau_c{}^n = M^n_{c,2} = \sum C_{i_1} \ldots C_{i_n} M_{i_1}(\boldsymbol{D}) \ldots M_{i_n}(\boldsymbol{D}) \tag{40}$$

so that the $\alpha_k$ coefficients in (37), as obtained by the direct least square procedure applied to fit $\dot{W}^n_{c,2}$ by (37), are exactly:

$$\alpha_k = \tau_c{}^n \sum_{i_1,\ldots,i_n} \beta_k^{i_1 \ldots i_n} C_{i_1} \ldots C_{i_n}, \tag{41}$$

where the $\beta$ are obtained by the same least square procedure, applied to fit the function $M_{i_1}(\boldsymbol{D}) \ldots M_{i_n}(\boldsymbol{D})$, instead of (40), by expression (37). In particular, analytical formulae of the form (41) are obtained for the expression (38) used by van Houtte et al. [30], [31] (although this has not been noticed by these authors). This is also true for Hill's new expression [4] (when $m$, our $n$, is an integer) and Gotoh's quartics [29], provided that these are used as a *work function* depending on $\boldsymbol{D}$.

## 6 Conclusions

1) In this paper, the identification method of the work function, initially proposed by Arminjon [8]—[10], has been presented afresh. The obtained *analytical* adjustment of a quadratic yield criterion from the texture coefficients has been reassessed and experimentally checked: whereas the yield surface and above all the work function itself are satisfactorily fitted, the associated flow rule of the fitted yield surface differs much more from the flow rule which is associated to the "crystallographic" yield surface (i.e. the yield surface derived from the texture function by using the classical Taylor model). Thus the theoretical values of the Lankford coefficient, as deduced from the "crystallographic" or "fitted" yield surface, differ significantly. The experimental values turn out to be close to the average of these two calculations. The quadratic yield function results in much simpler calculations and may be implemented in FEM codes. Moreover, the simple analytical relationship between the coefficients of the quadratic criterion and the texture coefficients enables to take into account the texture evolution when simulating forming processes.

2) Since the constitutive relationship derives from the work function, the phenomenological expression of this latter may be a priori set (instead of postulating the form of the yield function). The general restrictions which must be imposed to the work function have been discussed. It turns out that these are exactly the same for the work function and the yield function. The way to build analytical expressions satisfying the requirements has been examined: this is hence valid for yield functions as well. Here we focused on the requirements of convexity and homogeneity. As regards the convexity, the quadratic forms have a robust behaviour.

3) The identification method of the work function might be used for any analytical form of the latter: the corresponding "phenomenological" parameters would then be deduced from the texture coefficients by an iterative regression. However, all the expressions examined here lead to a *linear* regression. Furthermore, the *analytical* relationships between the phenomenological parameters and the texture coefficients, previously obtained in the quadratic case, have been extended to the case where the phenomenological expression of the work function is the linear combination of linearly independent, a priori given functions, when all of these are homogeneous of degree $n$ with the same integer $n$.

## Appendix 1

*Calculation of the $M_i(\boldsymbol{D})$ functions for an arbitrary tensor $\boldsymbol{D}$*

For any tensor $\boldsymbol{D}$, it is possible to find the orientation $\boldsymbol{g}_0$ and the quantities $K$ and $q$ such that in the reference system:

$$\boldsymbol{D} = K\boldsymbol{g}_0 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & q-1 \end{vmatrix} \cdot {}^t\boldsymbol{g}_0 \tag{A1.1}$$

$\boldsymbol{g}_0$ is nothing else than the orientation of the principal axes of the strain rate tensor with respect to the external reference system and $K$ and $q$ are calculated from the eigenvalues of the tensor (it has to be remembered that the tensor $\boldsymbol{D}$ is defined from a unit vector $\boldsymbol{D}$ and consequently the largest eigenvalue is not necessarily equal to 1). The diagonal tensor is called $\boldsymbol{D}_0$. Bunge et al. [15] have shown that the plastic work rate associated with this strain rate tensor $\boldsymbol{D}$ for a grain of orientation $\boldsymbol{g}$ could be calculated from:

$$\frac{1}{\tau_c}\dot{W}^g(\boldsymbol{D}) = \frac{K}{\tau_c} \cdot \dot{W}(q, \boldsymbol{g})_{\boldsymbol{g}_0} = K \cdot \sum_{l\mu\nu} m_l{}^{\mu\nu}(q)_{\boldsymbol{g}_0} \cdot \dot{\ddot{T}}_l{}^{\mu\nu}(\boldsymbol{g}) \tag{A1.2}$$

with:

$$m_l{}^{\mu\nu}(q)_{\boldsymbol{g}_0} = \sum_{s=0(2)}^{l} m_l{}^{\mu s}(q) \cdot \ddot{T}_l{}^{sn}(\boldsymbol{g}_0), \tag{A1.3}$$

where the coefficients $m_l{}^{\mu\nu}(q)$ are the series expansion coefficients of the plastic work rate associated with the strain rate $\boldsymbol{D}_0 = \boldsymbol{D}_0(q)$:

$$\frac{1}{\tau_c}\dot{W}^g(\boldsymbol{D}_0) = \sum_{l\mu\nu} m_l{}^{\mu\nu}(q) \cdot \dot{\ddot{T}}_l{}^{\mu\nu}(\boldsymbol{g}). \tag{A1.4}$$

The coefficients $m_l{}^{\mu\nu}(q)$ can be calculated for any value of $q$. Bunge et al. [15] have calculated these coefficients for $q$ equal to 0, 0.1, 0.2, 0.3, ..., 1 and various slip modes, and pro-

posed interpolation formulas for the intermediate values:

$$m_l^{\mu\nu}(q) = \sum_{j=0}^{5} m_{l,j}^{\mu s} \cdot q^j. \tag{A1.5}$$

Combining Eqs. (A1.2) to (A1.5) leads to the following expression for $M_l^{\mu\nu}(\boldsymbol{D})$:

$$M_l^{\mu\nu}(\boldsymbol{D}) = K \cdot \sum_{s=0(2)}^{l} \left( \sum_{j=0}^{5} m_{l,j}^{\mu s} \cdot q^j \right) \cdot \ddot{T}_l^{sn}(\boldsymbol{g}_0). \tag{A1.6}$$

In this expression, $K$, $\boldsymbol{g}_0$ and $q$ characterize the imposed strain rate, the $\ddot{T}_l^{sn}$ terms are mathematical functions which are easily calculated and the coefficients $m_{l,j}^{\mu s}$ are known from Bunge et al. [15]. Similar formulae have been obtained independently by Arminjon [8].

## Appendix 2

*Calculation of* $\boldsymbol{R}(\alpha)$

In its principal axes, the strain rate tensor has the following shape:

$$\boldsymbol{D} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & q-1 \end{vmatrix} \tag{A2.1}$$

and we want to determine $q$ such that the corresponding stress tensor is a tensile stress state along the 1 axis. In other words, the stress tensor has the special form:

$$\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{A2.2}$$

Geometrically, we have to find the normal to the yield surface at the point characterized by $D_{12} = 0$ and $\sigma_{22} = 0$ in the sub-space $\sigma_{13} = \sigma_{23} = 0$ [6]. The plastic work rate associated with such a strain rate, expressed in the tensile reference system, is equal to:

$$\dot{W}(q, \alpha) = \sigma_{11} = \boldsymbol{\sigma} : \boldsymbol{D}. \tag{A2.3}$$

Let us consider now another value of $q$, named $q^*$ which does not correspond to the tensile test and thus is associated with another stress tensor $\boldsymbol{\sigma}^*$. The plastic work rate $\dot{W}(q^*, \alpha)$ is then equal to:

$$\dot{W}(q^*, \alpha) = (\sigma_{11}^* - \sigma_{33}^*) - q^*(\sigma_{22}^* - \sigma_{33}^*) = \boldsymbol{\sigma}^* : \boldsymbol{D}^*. \tag{A2.4}$$

The principle of maximum work rate allows us to write now:

$$\boldsymbol{\sigma} : \boldsymbol{D}^* \leqq \boldsymbol{\sigma}^* : \boldsymbol{D}^* \tag{A2.5}$$

using the fact that $\dot{W}(q, \alpha) = \boldsymbol{\sigma} : \boldsymbol{D} = \boldsymbol{\sigma} : \boldsymbol{D}^*$ due to the special shape of $\boldsymbol{\sigma}$, we can write:

$$\dot{W}(q, \alpha) \leqq \dot{W}(q^*, \alpha). \tag{A2.6}$$

It has thus been demonstrated that the value of $q$ associated with a tensile test is found by minimizing $\dot{W}(q^*, \alpha)$ with respect to $q^*$. Then, $R(\alpha)$ is calculated from:

$$R(\alpha) = q/(1 - q). \tag{A2.7}$$

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## References

[1] Taylor, G. I.: Plastic strain in metals. J. Inst. Metals **62**, 307—324 (1938).
[2] Bishop, J. F. W., Hill, R.: A theory of the plastic distorsion of a polycrystalline aggregate under combined stress. Phil. Mag. **42**, 414—427 (1951).
[3] Hill, R.: A theory of the yielding and plastic flow of anisotropic metals. Proc. Roy. Soc. **A 193**, 281—297 (1948).
[4] Hill, R.: Theoretical plasticity of textured aggregates. Math. Proc. Cambr. Phil. Soc. **85**, 179—191 (1979).
[5] Montheillet, F., Gilormini, P., Jonas, J. J.: Relation between axial stresses and texture development during torsion testing: a simplified theory. Acta Metall. **33**, 705—717 (1985).
[6] Lequeu, Ph., Gilormini, P., Montheillet, F., Bacroix, B., Jonas, J. J.: Yield surfaces for textured polycrystals. I. Crystallographic approach. Acta Metall. **35**, 439—451 (1987).
[7] Lequeu, Ph., Gilormini, P., Montheillet, F., Bacroix, B., Jonas, J. J.: Yield surfaces for textured polycrystals. II. Analytical approach. Acta Metall. **35**, 1159—1174 (1987).
[8] Arminjon, M.: Contribution à l'étude des relations entre les paramètres d'anisotropic et les fonctions de répartition des orientations cristallographiques des polycristaux métalliques. Thèse de Doctorat de 3ème cycle, Université de Grenoble, France (1981).
[9] Arminjon, M.: Explicit relationships between texture coefficients and three-dimensional yield criteria of metals. In: Textures of materials (Brakman, C. M., et al., eds.) pp. 31—37. Zwijndrecht: Netherlands Soc. Mater. Sci. 1984.
[10] Arminjon, M., Boehler, J. P.: Comportement plastique et texture cristallographique des métaux anisotropes. In: Comportement plastique des solides anisotropes (Boehler, J.P., ed.) pp. 133—155. Paris: Editions du CNRS 1985.
[11] Arminjon, M.: Lois de comportement homogénéisées pour la plasticité des polycristaux. Mémoire d'Habilitation, Université Paris-Nord, France (1988).
[12] Hill, R.: The essential structure of constitutive laws for metal composites and polycrystals. J. Mech. Phys. Solids **15**, 79—95 (1967).
[13] Bunge, H. J.: Mathematische Methoden der Texturanalyse. Berlin: Akademie Verlag 1969.
[14] Bunge, H. J.: Texture analysis in materials science. Mathematical methods. London: Butterworths 1982.
[15] Bunge, H. J., Schulze, M., Grzesik, D.: Calculation of the yield locus of polycrystalline materials according to the Taylor theory. Peine-Salzgitter Berichte, Sonderheft (1980).
[16] Gilormini, P., Bacroix, B., Jonas, J. J.: Theoretical analyses of ⟨111⟩ pencil glide in BCC crystals. Acta Metall. **36**, 231—256 (1988).
[17] Canova, G. R., Kocks, U. F., Tome, C. N., Jonas, J. J.: The yield surface of textured polycrystals. J. Mech. Phys. Solids **33**, 371—397 (1985).
[18] Barlat, F., Lian, J.: Plastic behavior and stretchability of sheet metals. Part I. A yield function for orthotropic sheets under plane stress conditions. Int. J. Plast. **5**, 51—66 (1989).
[19] Barlat, F.: Crystallographic texture, anisotropic yield surfaces and forming limits of sheet metals. Mat. Sci. Eng. **91**, 55—72 (1987).
[20] Barlat, F., Richmond, O.: Prediction of tricomponent plane stress yield surfaces and associated flow and failure behavior of strongly textured Fcc polycrystalline sheets. Mat. Sci. **95**, 15—29 (1987).
[21] Arminjon, M.: Théorie d'une classe de modèles de Taylor « hétérogènes ». Application aux textures de déformation des aciers. Acta Metall. **35**, 615—630 (1987).

[22] Stout, M. G., Hecker, S. S., Bourcier, R.: An evaluation of anisotropic effective stress-strain criteria for the biaxial yield and flow of 2024 Aluminum tubes. J. Eng. Mater. Techn. **105**, 242—249 (1983).
[23] Vial, C., Hosford, W. F., Caddell, R. M.: Yield loci of anisotropic sheet metals. Int. J. Mech. Sci. **25**, 899—915 (1983).
[24] Legait, J.: Prise en compte de l'anisotropie en vue du calcul par éléments finis de l'emboutissage. Mém. Et. Sci. Rev. Mét. **86**, 171—175 (1989).
[25] Hill, R.: New horizons in the mechanics of solids. J. Mech. Phys. Solids **5**, 66—74 (1956).
[26] Hill, R.: Extremal paths of plastic work and deformation. J. Mech. Phys. Solids **34**, 511—523 (1986).
[27] Hill, R.: Constitutive dual potentials in classical plasticity. J. Mech. Phys. Solids **35**, 23—33 (1987).
[28] Arminjon, M.: Textures de déformation et écrouissage anisotrope de tôles d'acier. IRSID external report RE1186, pp. 97—98 (1985).
[29] Gotoh, M.: A theory of plastic anisotropy based on a yield function of fourth order (plane stress). Int. J. Mech. Sci. **19**, 505—520 (1977).
[30] van Houtte, P., van Bael, A., Aernoudt, E., Pillinger, I., Hartley, P., Sturgess, C. E. N.: The introduction of anisotropic yield loci derived from texture measurements in elasto plastic finite elements calculations. In: Advances in plasticity 1989. Proc. 2nd Int. Symp. on Plasticity and its Current Applications. London: Pergamon Press 1989.
[31] van Houtte, P., Mols, K., van Bael, A., Aernoudt, E.: Application of yield loci calculated from texture data. Textures and Microstructures **11**, 23—39 (1989).

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