Acta Mechanica 51, 139—149 (1984)

ACTA MECHANICA

A Numerical Study of Similarity Solutions for Combined Forced and Free Convection

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With 7 Figures

(*Received October 5, 1983*)

Summary

The similarity equations for combined forced and free convection flow over a horizontal plate when the wall temperature is inversely proportional to the square root of the distance from the leading edge are solved by introducing a scaling similar to that for the Blasius equation. The technique is also applied to the local similarity equations for the case of a constant wall temperature.

1. Introduction

Different authors have investigated the effect of buoyancy forces on the steady, laminar plane flow over a horizontal plate in the framework of first order boundary layer theory taking into account the pressure variation normal to the plate.

In [6] similarity equations to the boundary layer equations are derived for a wall temperature that is inverse proportional to the square root of the distance from the leading edge. The similarity equations depend on two parameters: The Prandtl number Pr and the buoyancy parameter $K = \mathrm{Ar}/\sqrt{Re}$, where Ar und Re are the Archimedes number and the Reynolds number, respectively. In the process of obtaining a numerical solution to the similarity equations it was observed in [6] that, given a fixed value of Pr, there appear to be more than one solutions for a specific range of K; but only a part of the branch of solutions was actually computed.

In this paper we propose a method to compute the complete branch of solutions of the similarity equations. This is achieved by a scaling of the similarity equations which essentially reduces the boundary value problem on the half line to an initial value problem.

Numerical approximations for the boundary layer equations in the case of a constant temperature of the plate were obtained in [1] by applying a shooting technique to the local similarity equations, and the local nonsimilarity equations corresponding to the second level of truncation. We will demonstrate the use of our scaling procedure in this situation by computing the complete solution to the local similarity equations.

2. The Similarity Equations

In [6] the following form of the boundary layer equations was derived:

$$\psi_Y \psi_{XY} - \psi_X \psi_{XY} - K \int_Y^\infty \theta_X \, dY = \psi_{YYY} \tag{2.1}$$

$$\psi_Y \theta_X - \psi_X \theta_Y = \frac{1}{\mathrm{Pr}} \theta_{YY} \tag{2.2}$$

where ψ is a stream function, θ is a (scaled) temperature, $K = \mathrm{Ar}/\sqrt{\mathrm{Re}}$ is the buoyancy parameter and Pr is the Prandtl number. Eq. (2.1) is valid for the flow above the plate. For the flow below the pate K must be replaced by $-K$.

Eqs. (2.1), (2.2) must be satisfied for $0 < X,\ Y < \infty$, with the boundary conditions

$$\psi = \psi_Y = 0, \qquad \theta = \theta_W(X) \qquad \text{on} \qquad Y = 0, \quad X > 0 \tag{2.3}$$

$$\psi_Y = 1, \qquad \theta = 0 \qquad \text{as} \qquad Y \to \infty, \tag{2.4}$$

where $\theta_W(X)$ is the (scaled) temperature of the plate.

When

$$\theta_W(X) = X^{-1/2},$$

the similarity transformation

$$\eta = YX^{-1/2}, \qquad \psi = X^{1/2} f(\eta), \qquad \theta = \theta_W \delta(\eta)$$

yields the system of ordinary differential equations

$$2f''' + ff'' + K\eta\delta = 0 \tag{2.5}$$

$$0 \leqq \eta < \infty$$

$$\frac{2}{\mathrm{Pr}} \delta'' + f\delta' + f'\delta = 0 \tag{2.6}$$

From (2.3), (2.4) the boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1$$

$$\delta(0) = 1, \qquad \delta(\infty) = 0$$

are obtained.

Using the exponential decay of δ, Eq. (2.6) can be integrated once, leading to the final form of the boundary value problem

$$2f''' + ff'' + K\eta\delta = 0 \tag{2.7}$$

$$\frac{2}{\mathrm{Pr}}\,\delta' + f\delta = 0 \qquad 0 \leqq \eta < \infty \tag{2.8}$$

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1, \qquad \delta(0) = 1. \tag{2.9}$$

To analyze the case when θ_W is constant, in [1] the transformation

$$\xi = K\,|\theta_W|\,X^{1/2}, \qquad \eta = YX^{-1/2}$$

$$F(\xi, \eta) = \psi(X, Y)\,X^{-1/2}, \qquad \Theta(\xi, \eta) = \theta(X, Y)/\theta_W$$

is applied to (2.1), (2.2) yielding

$$2F''' + FF'' \pm \xi\left[\eta\Theta + \int_\eta^\infty \Theta\,d\eta + \xi\int_\eta^\infty \frac{\partial\Theta}{\partial\xi}\,d\eta\right] = \xi\left[F'\,\frac{\partial F'}{\partial\xi} - F''\right] \tag{2.10}$$

$$\frac{2}{\mathrm{Pr}}\,\Theta'' + F\Theta' = \xi\left[F'\,\frac{\partial\Theta}{\partial\xi} - \Theta'\,\frac{\partial F}{\partial\xi}\right], \tag{2.11}$$

for $0 < \xi, \eta < \infty$. The boundary conditions (2.3), (2.4) become

$$F(\xi, 0) = F'(\xi, 0) = 0, \qquad \Theta(\xi, 0) = 1, \tag{2.12}$$

$$F'(\xi, \eta) = 1, \qquad \Theta(\xi, \eta) = 0 \quad \text{as} \quad \eta \to \infty. \tag{2.13}$$

Here "$'$" denotes differentiation with respect to η. Regarding the sign in (2.10) we have: + for the flow above the plate when θ_W is positive and for the flow below the plate when θ_W is negative, and − otherwise.

To facilitate the solution of the problem, the integral term in (2.10) is removed by differentiation with repect to η, which leads to the following boundary value problem:

$$2F'''' + (FF''' + F'F'') \pm \xi\eta\Theta' \mp \xi^2\Phi = \xi[F'G'' - F'''G] \tag{2.14}$$

$$\frac{2}{\mathrm{Pr}}\,\Theta'' + F\Theta' = \xi[F'\Phi - \Theta'G], \tag{2.15}$$

$$2F'''(\xi, 0) = \mp\xi\int_0^\infty \Theta\,d\eta \mp \xi^2\int_0^\infty \Phi\,d\eta \tag{2.16}$$

$$F'(\xi, \infty) = 1, \qquad \Theta(\xi, \infty) = 0, \tag{2.17}$$

where $G = \partial F/\partial\xi$, $\Phi = \partial\Theta/\partial\xi$. The integral boundary condition (2.16) was obtained by evaluating (2.10) at $\eta = 0$.

When terms involving G, G'' and Φ are neglected in the above equations, the following boundary value problem in the variable η, with ξ as a parameter, is obtained:

$$2F'''' + FF''' + F'F'' \pm \xi\eta\Theta' = 0 \tag{2.18}$$

$$0 \leqq \eta < \infty$$

$$\frac{2}{\mathrm{Pr}}\,\Theta'' + F\Theta' = 0 \tag{2.19}$$

$$F(0) = F'(0) = 0, \qquad \Theta(0) = 1 \tag{2.20}$$

$$2F'''(0) = \mp\xi \int\limits_{\infty}^{0} \Theta\, d\eta \tag{2.21}$$

$$F'(\infty) = 1, \qquad \Theta(\infty) = 0. \tag{2.22}$$

Note that by (2.18),

$$2\big(F'''(0) - F'''(\infty)\big) + (FF'')\,(0) - (FF'')\,(\infty)$$

$$= \pm\xi\eta\Theta\,\Big|_{\eta=0}^{\eta=\infty} \mp \xi \int\limits_{0}^{\infty} \Theta(\eta)\, d\eta,$$

whence (2.21) is equivalent to requiring that $F''(\eta)$ tends to zero exponentially as $\eta \to \infty$. Hence, we will replace (2.21) by

$$F''(\infty) = 0. \tag{2.23}$$

A discussion of the solutions of the boundary value problems (2.7), (2.8), (2.9) and (2.18), (2.19), (2.20), (2.22), (2.23) comprises the remainder of the paper.

Results on the asymptotic behaviour of solutions to (2.7), (2.8) satisfying $f'(\infty) = 1$, valid for large η, are given in [4], essentially generalizing the well known results on the asymptotic behaviour of the Blasius equation, [2]. The techniques used in [4] can also be applied to resolve the asymptotic behaviour of solutions to (2.18), (2.19) with $F'(\infty) = 1$, $F''(\infty) = 0$, $\Theta(\infty) = 0$.

3. The Scaling Procedure

With (2.7)—(2.9) we associate the initial value problem

$$2f''' + ff'' + K\eta\delta = 0, \qquad 0 \leqq \eta < \infty \tag{3.1}$$

$$\frac{2}{\mathrm{Pr}}\,\delta' + f\delta = 0 \tag{3.2}$$

$$f(0) = f'(0) = 0, \qquad f''(0) = s, \qquad \delta(0) = 1, \tag{3.3}$$

where s is a real parameter. Setting

$$f(\eta) = \varepsilon\Phi(\varepsilon\eta), \qquad \delta(\eta) = \Psi(\varepsilon\eta), \qquad \varepsilon > 0 \tag{3.4}$$

Eqs. (3.1), (3.2) yield

$$2\varepsilon^4\Phi'''(\varepsilon\eta) + \varepsilon^4\Phi(\varepsilon\eta)\,\Phi''(\varepsilon\eta) + \frac{K\varepsilon\eta}{\varepsilon}\,\Psi(\varepsilon\eta) = 0$$

$$\frac{2\varepsilon}{\Pr}\,\Psi'(\varepsilon\eta) + \varepsilon\Phi(\varepsilon\eta)\,\Psi(\varepsilon\eta) = 0.$$

With $\tau = \varepsilon\eta$ we obtain

$$2\Phi'''(\tau) + \Phi(\tau)\,\Phi''(\tau) + \frac{K}{\varepsilon^5}\,\tau\Psi(\tau) = 0 \tag{3.5}$$

$$0 \leqq \tau < \infty$$

$$\frac{2}{\Pr}\,\Psi'(\tau) + \Phi(\tau)\,\Psi(\tau) = 0 \tag{3.6}$$

and the initial conditions (3.3) become

$$\Phi(0) = \Phi'(0) = 0, \qquad \Phi''(0) = \frac{f''(0)}{\varepsilon^3} = \frac{s}{\varepsilon^3}, \qquad \Psi(0) = 1. \tag{3.7}$$

Given some values $\bar{K}$, $\bar{s}$ consider the curve in the $K - s$ plane,

$$K = \bar{K}\varepsilon^5, \qquad s = \bar{s}\varepsilon^3, \qquad \varepsilon > 0. \tag{3.8}$$

All values K, s on this curve lead to one and the same solution $\Phi(\tau)$, $\Psi(\tau)$. By (3.4) we have for K, s satisfying (3.8)

$$f(\eta) = \varepsilon\Phi(\varepsilon\eta), \qquad \delta(\eta) = \Psi(\varepsilon\eta),$$

whence

$$f'(\infty) = \varepsilon^2\Phi'(\infty).$$

So, provided $\Phi'(\infty)$ defined via the initial value problem

$$2\Phi'''(\tau) + \Phi(\tau)\,\Phi''(\tau) + \bar{K}\tau\Psi(\tau) = 0 \tag{3.9}$$

$$0 \leqq \tau < \infty,$$

$$\frac{2}{\Pr}\,\Psi'(\tau) + \Phi(\tau)\,\Psi(\tau) = 0 \tag{3.10}$$

$$\Phi(0) = \Phi'(0) = 0, \qquad \Phi''(0) = \bar{s}, \qquad \Psi(0) = 1 \tag{3.11}$$

exists and is positive, then there is a unique $\varepsilon > 0$ satisfying

$$1 = \varepsilon^2\Phi'(\infty), \tag{3.12}$$

implying that on the curve (3.8) there is a unique pair of parameters

$$K = \bar{K}\Phi'(\infty)^{-5/2}, \qquad s = \bar{s}\Phi'(\infty)^{-3/2} \tag{3.13}$$

such that the solution of the initial value problem (3.1)—(3.3) satisfies the boundary value problem (2.7)—(2.9).

It is easy to show that the boundary value problem has no solution in the fourth quadrant of the $K-s$ plane, i.e. for $K \geqq 0$, $s \leqq 0$: From (3.2) and the initial condition for δ we have

$$\delta(\eta) = \exp\left(-\frac{\Pr}{2}\int_0^\eta f(t)\,dt\right),$$

and hence by (3.1),

$$2f''' + ff'' = -K\eta\exp\left(-\frac{\Pr}{2}\int_0^\eta f(t)\,dt\right).$$

This implies that $f''(\eta) \leqq 0$ when $f''(0) = s \leqq 0$ and $K \geqq 0$. Consequently $f'(\eta) \leqq 0$ for $\eta \geqq 0$, and the condition $f'(\infty) = 1$ cannot be satisfied.

To use the scaling introduced above as the basis of a computational procedure for the remaining three quadrants of the $K-s$ plane, we must choose in each quadrant a curve of $\bar{K}$, $\bar{s}$ values such that when $\bar{K}$, $\bar{s}$ vary, the family of curves (3.8) covers the whole quadrant. For the computations reported in the next section we have chosen straight lines

$$\bar{s} = a\bar{K} + b, \qquad a, b \in R, \tag{3.14}$$

with $\bar{K} \in [-b/a, 0]$ if $b/a > 0$ and $\bar{K} \in [0, -b/a]$ otherwise. This is illustrated in Fig. 3.1 for the third quadrant ($a = -1$, $b = -0.1$).

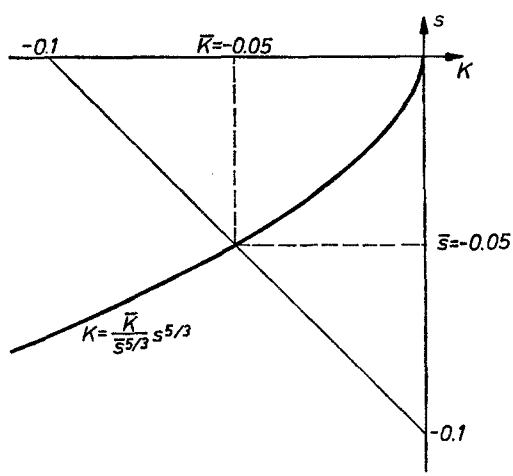

Fig. 3.1

We now briefly discuss the application of the scaling technique to the problem (2.18), (2.19), (2.20), (2.22), (2.23). Replacing only the condition $F'(\infty) = 1$ by the initial condition $F''(0) = s$, and not changing the second condition in (2.22), we obtain

$$F(0) = F'(0) = 0, \qquad F''(0) = s, \qquad \Theta(0) = 1, \qquad F''(\infty) = 0, \qquad \Theta(\infty) = 0.$$

Setting

$$F(\eta) = \varepsilon\Phi(\tau), \qquad \Theta(\eta) = \Psi(\tau), \qquad \tau = \varepsilon\eta,$$

we get (on taking the positive sign in (2.18)),

$$2\Phi''''(\tau) + \Phi(\tau)\,\Phi'''(\tau) + \Phi'(\tau)\,\Phi''(\tau) + \bar{\xi}\tau\Psi'(\tau) = 0, \qquad 0 \leqq \tau < \infty, \quad (3.15)$$

$$\frac{2}{\mathrm{Pr}}\,\Psi''(\tau) + \Phi\Psi'(\tau) = 0 \quad (3.16)$$

$$\Phi(0) = \Phi'(0) = 0, \qquad \Phi''(0) = \bar{s}, \qquad \Psi(0) = 1 \quad (3.17)$$

$$\Phi''(\infty) = 0, \qquad \Psi(\infty) = 0, \quad (3.18)$$

where $\bar{\xi} = \xi/\varepsilon^5$, $\bar{s} = s/\varepsilon^3$. Note that this is a boundary value problem, so it is not clear if the scaling has brought any gain. However as discussed in the next section, it is advantageous to use this problem rather than the original one for the purpose of numerical computation.

4. Numerical Results

We restrict our attention to the second and the third quadrant of the $K-s$ plane, since the first quadrant is well understood, [6], [1]. We first discuss the problem (2.7), (2.8), (2.9). For a discrete set of points $(\bar{K}, \bar{s})$ on the line segments $\bar{s} = \bar{K} + 0.1$, $\bar{K} \in [-0.1, 0]$ and $\bar{s} = -\bar{K} - 0.1$, $\bar{K} \in [-0.1, 0)$ the initial value problem (3.9), (3.10), (3.11) was solved using the initial value solver DVERK of IMSL. The computation of the initial value problem was terminated when the magnitude of $\Phi''(\tau)$ reached 10^{-6}. By computing with different error tolerances we made sure that the numerical solution was correct to four digits after the comma. The initial value problem becomes very sensitive when $\bar{K}$ approaches zero in the third quadrant, as is illustrated in Fig. 4.1, where $\Phi'(\infty)$ is plotted versus $\bar{K}$. The initial value solver fails for $\bar{K}$ very close to zero.

Then K and s were determined according to (3.13), leading to the curves depicted in Fig. 4.2. For K and s on such a curve the solution of the initial value problem (3.1), (3.2), (3.3) is also a solution of the original boundary value problem (2.7), (2.8), (2.9).

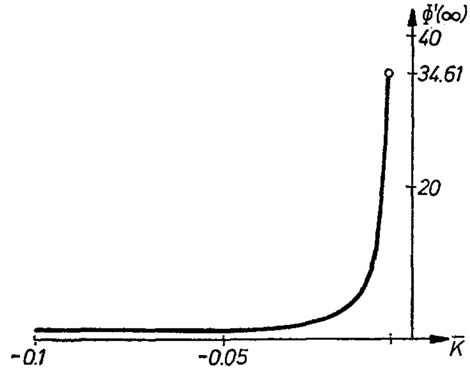

Fig. 4.1

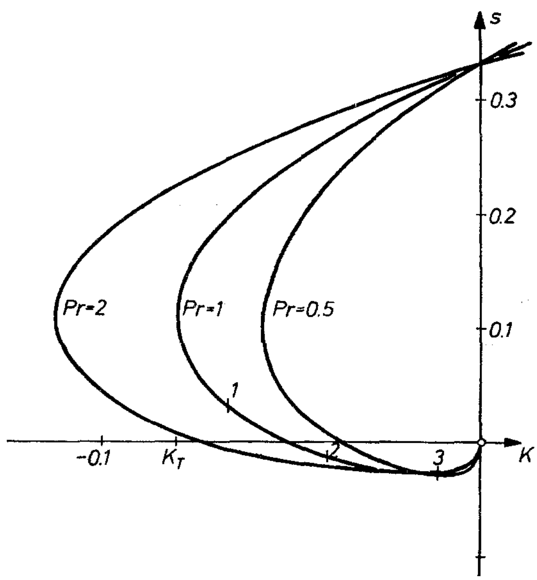

Fig. 4.2

The intersection of the solution branch with the line $s = 0$ is also of interest (separation point). Our value for K in the case Pr $= 1$ agrees well with that given in [6].

In Figs. 4.3 and 4.4 we give for Pr $= 1$ the velocity and temperature profiles corresponding to the points in Fig. 4.2 labeled 1, 2, 3 with the following coordinates. 1: $K = -0.0676$, $s = 0.0294$; 2: $K = -0.0411$, $s = -0.0134$; 3: $K = -0.0110$, $s = -0.0277$.

Each of the curves depicted in Fig. 4.2 has two "critical points":

Firstly, there is a turning point (for Pr $= 1$ its abscissa is denoted by K_T), such that there is no solution to the left of K_T and there are two solutions to the

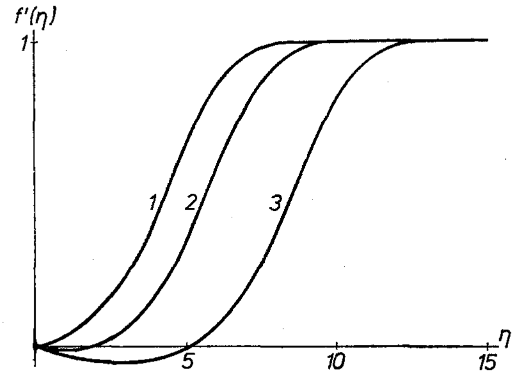

Fig. 4.3. Pr = 1

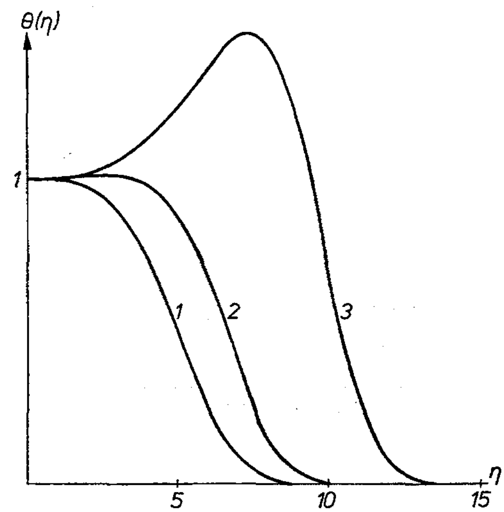

Fig. 4.4. Pr = 1

right of K_T. The method of parametric differentiation as used in [6] fails when a turning point is approached. However, this method can be rescued when techniques along the line of [3, 5, 7] are incorporated.

The second critical point is $K = s = 0$, which, of course, does not lie on the curve. When this point is approached, ε defined by (3.8) tends to zero very rapidly, and according to (3.4) the structure of f and δ is shifted further and further towards infinity. Here a shooting method, or any other boundary value technique, is bound to run into trouble for two reasons: The boundary value problem is extremely sensitive to perturbations, and the finite interval to replace $[0, \infty)$ must be a very long one. It is here that our approach is of particular advantage.

We now briefly discuss our computations for the problem (2.18), (2.19), (2.20), (2.22), (2.23). The *boundary value problem* (3.15)—(3.18) was solved by the shooting method for a set of discrete values $\bar{\xi}$, $\bar{s}$ on the line segments $\bar{s} = \bar{\xi} + 0.15$, $\bar{\xi} \in [-0.15, 0]$ and $\bar{s} = -\bar{\xi} - 0.15$, $\bar{\xi} \in [-0.15, 0)$. This boundary value problem becomes very sensitive when $\bar{\xi} = 0$ is approached in the third quadrant, as is apparent from Fig. 4.5, where $\Phi'(\infty)$ is plotted versus $\bar{\xi}$. For different Prandtl numbers the curves

$$\xi = \bar{\xi}\Phi'(\infty)^{-5/2}, \qquad s = \bar{s}\Phi'(\infty)^{-3/2}$$

representing solutions of the *original* boundary value problem are plotted in Fig. 4.6. Note that $\varepsilon = \Phi'(\infty)^{-1/2}$ tends to zero rapidly as the point $\xi = s = 0$ is approached. Hence, a very long interval would be needed for the solution of the

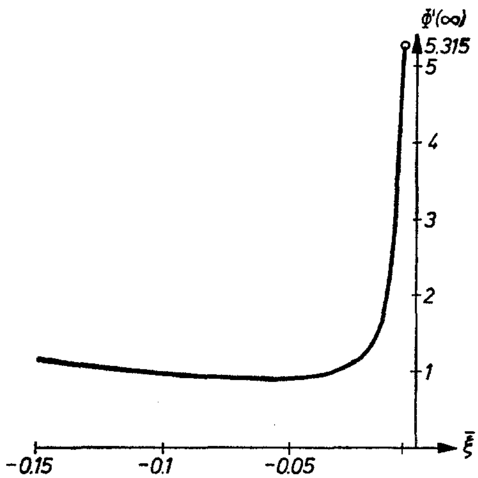

Fig. 4.5

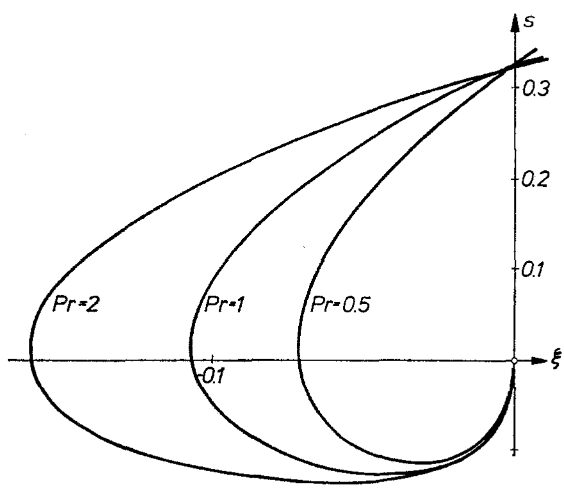

Fig. 4.6

original boundary value problem close to this point. On the other hand, the computations have shown that the structure of the solution of (3.15)—(3.18) is not shifted towards ∞ when $\bar{\xi} = 0$ is approached on the line segment in the third quadrant.

Acknowledgement

We are grateful to Prof. W. Schneider for suggesting this problem to us and for his advice in the course of the work. Thanks are also due to Dr. P. Markowich for valuable discussions.

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