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Wave Propagation in a Steady Supersonic Flow of a Radiating Gas Past Plane and Axisymmetric Bodies

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With 1 Figure

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Summary

Using the method of wavefront analysis the paper presents an analysis of shock wave formation in a two-dimensional steady supersonic flow of a radiating gas past plane and axisymmetric bodies such as a beak and sharp edged ring. Transport equations are derived which lead to the determination of the shock formation distance and also to conditions which insure that no shock will ever evolve on the wavefront. It is assessed as to how the shock formation distance is influenced by the presence of thermal radiation, the initial body curvature and upstream flow Mach number $M_0 > 1$.

1. Introduction

The general behaviour and, in particular, the steepening of waves are described by quasi-linear hyperbolic system of equations. During the past two decades there has been considerable theoretical and experimental research in this field and several methods have been used for the analysis of wave-propagation processes. In recent years the formation of shock waves has received considerable attention in the literature with the shock formation time or distance being used as important parameters characterizing the relative importance of convective nonlinear steepening and flattening, and setting a limit for the use of certain approximate theoretical approaches.

The nonlinear breaking of wavefronts and its analysis within the context of unsteady radiating gasdynamics have received considerable attention in the past (see, for example, Schmitt [4], Srinivasan and Ram [6], and Sharma et al. [5]).

However, the corresponding analysis for two-dimensional steady supersonic flow has not been treated until now. The main academic interest of the present paper is to study the analysis of shock wave formation in a two-dimensional steady supersonic flow past a plane beak or a sharp edged ring in radiative gas-dynamics, and to assess as to how the shock formation distance is influenced by the presence of thermal radiation, the initial boundary curvature and the up-stream flow Mach number. The medium is taken to be sufficiently hot for effects of thermal radiation to be significant, which are, of course, treated by the optically thin approximation to the radiative transfer equation.

2. Basic Equations and Characteristics

The basic equations for a two-dimensional steady axisymmetric flow of a radiating gas near the optically thin limit, assuming that the gas is inviscid, optically grey and in thermodynamic equilibrium, can be written down in the familiar form (see, Pai [2], Penner and Olfe [3]).

$$u\varrho_x + v\varrho_r + \varrho(u_x + v_r + mv/r) = 0, \tag{1}$$

$$\varrho u u_x + \varrho v u_r + p_x = 0, \tag{2}$$

$$\varrho u v_x + \varrho v v_r + p_r = 0, \tag{3}$$

$$up_x + vp_r - a^2(u\varrho_x + v\varrho_r) + (\gamma - 1) F = 0. \tag{4}$$

Here, x is the distance along the axis of the symmetry from the body tip in the direction of oncoming flow, and r the radial distance from the x-axis; u and v denote respectively the velocity components along the x and r axis; ϱ the density of the gas, p the pressure, a the speed of sound given by $a^2 = \gamma p/\varrho$ with γ as the adiabatic index; F is the rate of energy loss by the gas per unit volume through radiation, which is given by,

$$F = 4\alpha\sigma(T^4 - T_b{}^4), \tag{5}$$

where α is the Planck mean absorption constant depending on the density and temperature T of the gas, σ is the Stefan-Boltzmann constant and T_b is the uniform body temperature. The letter subscripts in Eqs. (1) to (4) denote partial differentiation, unless stated otherwise. Here m is a constant, which takes values 0 and 1 for plane and axisymmetric flows resprectively.

Using matrix notation, Eqs. (1) to (4), can be written in the following form,

$$U_x + AU_r + B = 0, \tag{6}$$

where U, A and B are as follows

$$U = \begin{bmatrix} \varrho \\ u \\ v \\ p \end{bmatrix}, \quad A = (u^2 - a^2)^{-1} \begin{bmatrix} \dfrac{v(u^2 - a^2)}{u} & -\varrho v & \varrho u & \dfrac{v}{u} \\ 0 & uv & -a^2 & -\dfrac{v}{\varrho} \\ 0 & 0 & \dfrac{v(u^2 - a^2)}{u} & \dfrac{u^2 - a^2}{\varrho u} \\ 0 & -\varrho v a^2 & \varrho u a^2 & uv \end{bmatrix},$$

$$B = (u^2 - a^2)^{-1} \begin{bmatrix} m\varrho u v a^2/r + (\gamma - 1)\, F/u \\ -m v a^2/r - (\gamma - 1)\, F/\varrho \\ 0 \\ m\varrho u v a^2/r + u(\gamma - 1)\, F \end{bmatrix}.$$

Let $\lambda^{(i)}$ be the eigenvalues of A and $L^{(i)}$ the corresponding left eigen vectors. We then have

$$\begin{gathered} \lambda^{(1,2)} = \left(uv \pm a^2(M^2 - 1)^{1/2}\right)/(u^2 - a^2), \qquad \lambda^{(3,4)} = v/u, \\ L^{(1)} = [0, 1, -u/v, -(M^2 - 1)^{1/2}/\varrho v], \\ L^{(2)} = [0, 1, -u/v, (M^2 - 1)^{1/2}/\varrho v], \\ L^{(3)} = [1, 0, 0, -1/a^2], \qquad L^{(4)} = [0, 1, v/u, 1/\varrho u]. \end{gathered} \tag{7}$$

Here $M = q/a$, with $q = (u^2 + v^2)^{1/2}$, is the upstream flow Mach number.

It is evident from (7) that the system (6) possesses, except along stream line on which $\lambda = v/u$, two families of characteristics along which $dr/dx = \lambda^{(1,2)}$; these characteristics represent waves propagating in opposite directions with characteristic speeds $\lambda^{(1,2)}$. It may be noted that these characteristic velocities are real if, and only if, the flow Mach number $M > 1$, i.e., the flow is supersonic.

3. Transport Equations for the Discontinuities

Let us suppose that $\lambda^{(1)}$ describes the initial wavefront $\xi(x, r) = 0$, which passes through (x_0, r_0). The medium ahead of $\xi = 0$, is assumed to have a uniform temperature $T_0 = T_b$ and a uniform velocity u_0 in the x direction with $v_0 = 0$. In the rest of the paper, we shall use the suffix-0 to denote a quantity in the region ahead of $\xi = 0$.

We now derive the transport equations for jump discontinuities in U as they move along the wavefront $\xi = 0$. As in [1] we introduce new curvilinear coordinates ξ, r' defined as

$$\xi_x + \lambda^{(1)}\xi_r = 0, \tag{8.1}$$

$$\xi(x, r_0) = x - x_0, \tag{8.2}$$

and $r = r'$. Then ξ has the required coordinate property that ξ is positive (negative) behind (ahead of) the leading characteristic on which $\xi = 0$.

In terms of these new coordinates, Eq. (6) on premultiplying by $L^{(i)}$ becomes

$$L^{(i)}U_\xi + \left(\lambda^{(1)}\lambda^{(i)}/(\lambda^{(1)} - \lambda^{(i)})\right) x_\xi L^{(i)}U_{r'} + \left(\lambda^{(1)}/(\lambda^{(1)} - \lambda^{(i)})\right) x_\xi L^{(i)}B = 0 \quad (i - \text{unsummed}), \tag{9}$$

where $x_\xi = 1/\xi_x$ is the Jacobian of the transformation and the index i takes values 1, 2, 3 and 4.

Across the wavefront $\xi = 0$, U and $U_{r'}$ are continuous and have their subscripts -0 values whilst U_ξ and x_ξ are discontinuous. In view of (7) and the flow conditions ahead, the evaluation of (9) at the rear side of $\xi = 0$ for $i = 2, 3$ and 4 yields:

$$v_\xi = \left((M_0^2 - 1)^{1/2}/\varrho_0 u_0\right) p_\xi, \tag{10}$$

$$\varrho_\xi = (1/a_0^2)\, p_\xi, \tag{11}$$

$$u_\xi = -(1/\varrho_0 u_0)\, p_\xi. \tag{12}$$

We now set $i = 1$ in Eq. (9), differentiate the resulting equation with respect to ξ, and then evaluate it at the rear of $\xi = 0$, we get

$$a_0^2(M_0^2 - 1)^{1/2}\, p_{\xi r'} + \varrho_0 a_0^2 u_0 v_{\xi r'} + (m/r')\, \varrho_0 a_0^2 u_0 v_\xi + u_0(\gamma - 1)\, F_\xi = 0. \tag{13}$$

In view of the equation of state $p = \varrho RT$, Eq. (5) on differentiating with respect to ξ and evaluating it at the rear of $\xi = 0$ yields:

$$F_\xi = \left(16\sigma\alpha T_b^4(\gamma - 1)/\varrho_0 a_0^2\right) p_\xi. \tag{14}$$

Equation (13), in view of (10) and (14), becomes

$$p_{\xi r'} + p_\xi\{(m/2r') + \Lambda\} = 0, \tag{15}$$

where $\Lambda = \{8(\gamma - 1)\, M_0\alpha/\beta(M_0^2 - 1)^{1/2}\} > 0$ is a measure of importance of thermal radiation with $\beta = \{(\gamma - 1)\, \sigma T_b^4/\varrho_0 a_0^3\}^{-1}$, the Boltzmann number, representing the rate of convective energy flux.

Also, along $\xi =$ constant we have

$$x_{r'} = (u^2 - a^2)/\{uv + a^2(M^2 - 1)^{1/2}\},$$

which, when differentiated with respect to ξ and evaluated at the rear of $\xi = 0$, yields on using the foregoing result that

$$x_{\xi r'} = -\frac{M_0^2(\gamma+1)}{2\varrho_0 a_0^2(M_0^2-1)^{1/2}}\,(r_0/r')^{m/2}\exp\big(A(r_0-r')\big)\,p_{\xi 0}. \tag{16}$$

Equations (15) and (16) are the required transport equations for the discontinuities p_ξ and x_ξ, which we have been seeking.

4. Steepening of Waves

Equation (15) on integration yields:

$$p_\xi = p_{\xi 0}(r_0/r')^{m/2}\exp\big(A(r_0-r')\big), \tag{17}$$

where $p_{\xi 0} = \lim\limits_{r'\to r_0} p_\xi$, taken along $\xi = 0$.

Substituting (17) into (16) and integrating we get:

$$x_\xi = 1 - \frac{M_0^2(\gamma+1)\,r_0^{m/2}e^{Ar_0}p_{\xi 0}}{2\varrho_0 a_0^2(M_0^2-1)^{1/2}}\int\limits_{r_0}^{r} s^{-m/2}\exp(-As)\,ds, \tag{18}$$

where we have used the fact that $x_{\xi 0} = x_\xi|_{\xi=0-} = x_\xi|_{\xi=0+} = 1$; this follows from the boundary condition (8.2). Let $r = R(x)$ be the equation of the body contour with tangent, being parallel to the velocity of the stream line, at the leading body edge. We, thus, have $dr/dx = v/u$, which on differentiating with respect to ξ and evaluating it at the rear of $\xi = 0$ yields

$$v_{\xi 0} = u_0 R_0'', \tag{19}$$

where R_0'' is the body curvature at the tip.

By virtue of Eqs. (10) and (19), Eq. (18) reduces to the following form:

$$x_\xi = 1 - \frac{(\gamma+1)\,M_0^4 e^{Ar_0}r_0^{m/2}R_0''}{2(M_0^2-1)}\int\limits_{r_0}^{r} s^{-m/2}\exp(-As)\,ds. \tag{20}$$

The left hand side of Eq. (20) is the Jacobian of coordinate transformation in the region immediately behind $\xi = 0$, so that if for some $r = r_s$ this Jacobian vanishes, the neighbouring characteristics of the family $\xi =$ constant must intersect on the wavefront $\xi = 0$ and a strong discontinuity known as shock wave then occurs in the solution vector U. This will be the case if U_ξ is finite at $r = r_s$ as $x_\xi = 0$, for then, just behind the wavefront $\xi = 0$, $U_x = U_\xi/x_\xi$ becomes infinite; this describes the phenomenon of the steepening of the wavefront. Significance of the result (20) for plane ($m = 0$) and axisymmetric ($m = 1$) flow configurations is discussed in the following section.

5. Results and Discussion

In this section we shall deal with a supersonic flow past a plane beak ($m = 0$) and a sharp edged ring ($m = 1$). The described phenomenon is sketched in Fig. 1.

In case of a plane beak ($m = 0$) with body contour $r = R_b(x)$, the initial disturbance is released by a sharpe edge of the contour with a vanishing small initial tangent (beak). For the present case, Eq. (20) becomes

$$x_\xi = 1 - \left(R_b''(0)/d\right)\{1 - \exp\left[-\Lambda(r - r_0)\right]\}, \tag{21}$$

where $d = 2\Lambda(M_0^2 - 1)\{(\gamma + 1) M_0^4\}^{-1} > 0$, and $R_b''(0)$ is the radius of curvature of the body shape at the tip where the body contour begins to bend.

As mentioned earlier, the formation of shock is characterized by the vanishing of the Jacobian x_ξ, i.e., when the characteristics begin to coalesce. Since $\Lambda > 0$, it is evident from (21) that the Jacobian can vanish on the leading wavefront for $r > r_0$ only when $R_b''(0) > 0$ (which corresponds to the situation when the body shape has a compressive corner at $x = 0$) with $R_b''(0) > d$. For $R_b''(0) \leqq d$, the Jacobian remains positive for finite $r > r_0$ and consequently a shock will not form on the leading wavefront. Thus, the parameter d represents a critical level such that when this level is exceeded by the radius of curvature $R_b''(0)$ at the body tip, a shock will form at a finite distance away from the body. This is in contrast with the corresponding case of a non-radiating gas, where one always finds a shock after a finite length of run, no matter how small the initial body curvature may be. It may be recalled that at the wavehead $\xi = 0$, v_x and v_ξ are related according to $v_x = v_\xi/x_\xi$; it is therefore immaterial whether we seek an expression for v_ξ or v_x at the wavehead. Since v_x has a slightly more direct physical interpretation we shall opt to work in terms of that quantity and note from

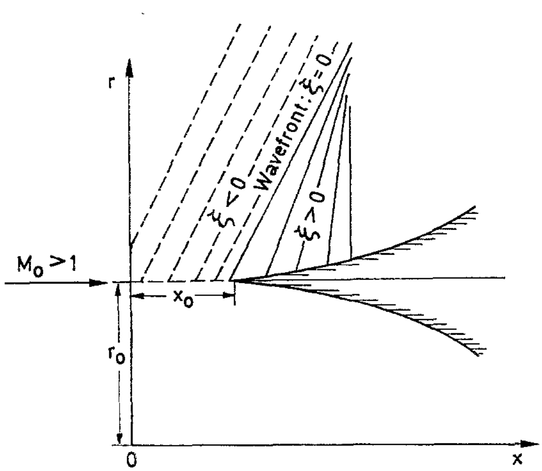

Fig. 1. Flow field and convergence of characteristics for a plane and axisymmetric supersonic flow

Eqs. (10), (17) and (21) that for the plane ($m = 0$) case

$$v_x = \frac{a_0 M_0 R_b''(0) \exp[-\Lambda(r - r_0)]}{1 - \left(R_b''(0)/d\right)\{1 - \exp[-\Lambda(r - r_0)]\}}. \tag{22}$$

A criterion for the tendency of the wave to steepen or to flatten follows from (22). It is evident from (22) that when $R_b''(0)$ is positive and has a magnitude greater than d, there exists a finite length of run, r_s, given by

$$r_s = r_0 + \Lambda^{-1} \ln\left\{R_b''(0)/\left(R_b''(0) - d\right)\right\}, \tag{23}$$

such that at $r = r_s$, the denominator of (22) becomes zero whereas numerator remains finite, i.e. the velocity gradient at the wavehead becomes unbounded at $r = r_s$, thus signifying the steepening of the wave into a shock wave; the coincidence of this behaviour with the vanishing of the Jacobian x_ξ is clear from (21). In the event that $R_b''(0) \leqq d$ the wave is still compressive but the steepening of the velocity gradient does not occur. On the contrary v_x either diminishes out along the wavehead or it remains stationary according as $R_b''(0) < d$ or $R_b''(0) = d$ respectively, and no shock wave will ever form on the leading wavehead $\xi = 0$. Moreover, when $R_b''(0) < 0$, which corresponds to the situation when the body shape has an expansive corner at $x = 0$, then for $|R_b''(0)| \gg \Lambda$, Eq. (22) implies that

$$v_x = -\{u_0(M_0^2 - 1)\,\Lambda \exp[-\Lambda(r - r_0)]\}/\left\{(\gamma + 1)\,M_0^4\left(1 - \exp[-\Lambda(r - r_0)]\right)\right\},$$

which is an expression for the velocity gradient at the head of a Prandtl-Meyer expansion flow.

For an axisymmetric case ($m = 1$), we consider a ring shaped body $r = R_s(x)$ with sharp edged inlet releasing the initial disturbance which runs both inwards and outwards along characteristic lines. Equation (20) describes both the phenomena as can be easily understood. For $r > r_0$, a similar behaviour occurs as in the case of a plane flow; indeed, for $m = 1$, Eq. (20) can be written as

$$x_\xi = 1 - R_s''(0)\left\{1 - \frac{\operatorname{erfc}(\Lambda r)^{1/2}}{\operatorname{erfc}(\Lambda r_0)^{1/2}}\right\} A_0, \tag{24}$$

where $A_0 = [(\gamma + 1)\,M_0^4 e^{\Lambda r_0} r_0^{1/2} \pi^{1/2} \operatorname{erfc}(\Lambda r_0)^{1/2}]/\left(2\Lambda^{1/2}(M_0^2 - 1)\right)$ and $\operatorname{erfc}(x) = \left(2/\sqrt{\pi}\right)\int_x^\infty \exp(-t^2)\,dt$. It may be noted that for $r > r_0$ the entity within the curly bracket in (24) is always positive and less than unity. Thus the left hand side of (24) will vanish, leading to the formation of shock, provided $R_s''(0)$ is positive and exceeds the critical value A_0^{-1}; when $R_s''(0) \leqq A_0^{-1}$ the Jacobian x_ξ is always positive and hence no shock wave will ever form on the leading wavehead. We, thus, infer that the shock formation will take place only when the radius of curvature at the body tip exceeds the critical level $1/A_0$, and consequently for the ordinate $r = r_s$ of the beginning of shock, we get

$$\operatorname{erfc}(\Lambda r_s)^{1/2} = \left[1 - \left(A_0 R_s''(0)\right)^{-1}\right] \operatorname{erfc}(\Lambda r_0)^{1/2}. \tag{25}$$

We notice from (23) and (25) that the shock formation distance r_s depends on the upstream flow Mach number M_0, the initial body curvature k being either $\left(R_b''(0)\right)^{-1}$ or $\left(R_s''(0)\right)^{-1}$, and the parameter Λ, which represents the importance of thermal radiation. The dependence of r_0 on k and Λ is quite straight forward in the sense that both $\partial r_s/\partial k$ and $\partial r_s/\partial \Lambda$ are positive, which means that an increase in the initial body curvature or a decrease in the Boltzmann number both cause the shock formation distance to increase. However the dependence of r_s on M_0 is somewhat less straightforward; indeed, we find that as the velocity of the approaching flow increases beyond sonic speed, the shock formation distance first increases and then decreases in the supersonic region exhibiting a unique maximum. It can be seen from (23) that for $M_0 \sim 1$, the shock formation distance r_s is given by $r_s \sim r_0 + 4(M_0 - 1)/\{(\gamma + 1)\, R_b''(0)\}$, whereas for $M_0 \gg 1$, $r_s \sim 2/\{(\gamma + 1)\, R_b''(0)\, M_0^2\}$. Here we also notice from (23) and (25) that the length of run required up to the shock in axisymmetric case is larger than in the case of a plane flow under the same initial conditions.

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