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Elastoplastic Cracks in Orthotropic Crystals Using Dislocation Layers

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Summary

The method of dislocation layers is used to study the stress-field created around an infinite row of collinear Griffith-type elastoplastic strip cracks in an orthotropic crystal loaded at infinity. Formal solutions are obtained in detail for the mode III antiplane shear case, leading to explicit expressions for the length of the plastic zones and the total plastic displacement at the crack-tips. Some representative numerical results are given. It is observed that the problems of a single, elastoplastic crack within a finite orthotropic plate and a finite plate containing a surface crack have solutions which are actually also provided by this analysis. The mode I and mode II analogous situations are briefly discussed.

1. Introduction

It is well-established (see, for example, Bilby and Eshelby [1], Lardner [2]) that certain strip-like cracks in elastic and elastoplastic media can be simulated by equivalent continuous distributions of straight dislocations. A particular advantage of this "dislocation layer method" over the more classical integral transform or complex potential function techniques which are often conveniently employed for studying elastic cracks (see, for example, Sneddon and Lowengrub [3], Sih [4]) is that it can be extended to provide some useful insight into elastoplastic crack problems. This stems from the fundamental work of Bilby, Cottrell and Swinden [5] who proposed the so-called BCS model for the plastic yield from a crack-tip.

Recently, there has been an increasing interest in seeking closed-form solutions for various cracked-strip problems, as illustrated by the work of Singh, Moodie and Haddow [6], Tait and Moodie [7], and Georgiadis and Theocaris [8] within the context of isotropic elasticity theory. The purpose of the present paper is to show that the dislocation layer method can be applied for studying the stress field created around an infinite row of collinear elastoplastic cracks in orthotropic crystals when subjected to applied tractions at infinity. This extends the analysis of Bilby, Cottrell, Smith and Swinden [9], who considered corresponding mode III situations within an isotropic material, and also the isotropic classical elastic mode I and II solutions which have been studied by various authors, as summarized in [3] and [4]. For the sake of brevity, we restrict our detailed discussion to mode III antiplane shear cracks and simply indicate, in Section 4, how the results appropriate for mode I and II loadings can be obtained. It is pointed out, in Section 4 also, that in fact our solution further corresponds to the problem of a single, elastoplastic crack within a finite orthotropic plate, or a finite plate containing a surface crack. To be able to tackle orthotropic problems in this way detailed formulae for the stress components around straight dislocations in such media are required. These are provided by the studies of Chou and his colleagues. Chou and Sha [10] have stated the relevant components for a glide edge dislocation, a climb edge dislocation and a screw dislocation. Their results generalize those given previously by Chou [11] for dislocations in a basal plane of a hexagonal crystal and by Chou, Garofalo and Whitmore [12] and Chou and Whitmore [13] for a cubic crystal.

The basic mode III problem is formulated in Section 2 before the derivation of its solution using dislocation layers is presented in Section 3. The length of the plastic zones and total plastic displacement created at the crack-tips are of interest. General explicit expressions are presented for these in Section 4. Low-stress approximations and representative numerical results are given. Some corresponding expressions for cracks in purely elastic orthotropic plates are deduced.

2. Basic Formulation

We consider an infinite, periodic row of plane, collinear, stationary, Griffith-type strip cracks each of width $2c$ in an infinite homogeneous crystal which is orthotropically symmetrical in its elastic response. We suppose that the material is initially everywhere at rest and stress-free in a natural reference state and situated so that its three mutually perpendicular planes of symmetry are the coordinate planes of a system of rectangular Cartesian coordinates x, y, z.

The cracks are assumed to be lying on the $y = 0$ plane and centred at $x = 0$, $\pm 2h$, $\pm 4h$, ..., so that they occupy the regions

$$I = \{(x, y, z)\colon 2nh - c < x < 2nh + c,\ y = 0,\ -\infty < z < \infty\} \tag{1}$$

of the $x-z$ plane, with $n = 0, \pm 1, \pm 2, \ldots$. We suppose that the medium is deformed in mode III, antiplane strain by stipulating that $\sigma_{yz} \to \sigma$ at infinity, where σ_{yz} is a component of the stress tensor referred to the x, y, z system of coordinates and σ is a prescribed constant. With respect to the x, y, z coordinate system, the relationship connecting the components of the stress and strain tensors, σ and ε respectively, for an orthotropic material can be written in the form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}, \tag{2}$$

where the c_{ij} denote the elastic constants referred to the chosen coordinate system. We further suppose that the material is elastoplastic with constant yield stress σ_1 ($> \sigma$) and that there is plastic flow at the tips of the cracks over plastic zones of length $a-c$ extending throughout the regions

$$I_p = \{(x, y, z)\colon c < |x - 2nh| < a,\ y = 0,\ -\infty < z < \infty\} \tag{3}$$

with $n = 0, \pm 1, \pm 2, \ldots$.

Hexagonal and cubic crystals are particularly important special classes of orthotropic crystals and it is worthwhile indicating here that Tupholme [14] has given explicit details of the substitutions which can be made into expression (2) to produce results applicable to them.

3. Solution

In order to discuss such mode III cracks, according to the general procedure of the dislocation layer method, we replace them by an equivalent continuous planar distribution of dislocations. To conform with the BCS model [5] these are not blocked at $x = 2nh - c$ and $x = 2nh + c$ ($n = 0, \pm 1, \pm 2, \ldots$), but are allowed to run into the material through the regions I_p, defined by Eq. (3), penetrating as far as the points $x = 2nh - a$ and $x = 2nh + a$ ($n = 0, \pm 1, \pm 2, \ldots$) beyond the crack-tip, to create plastic shear in these regions. For these shear cracks we utilize straight screw dislocations with their lines and Burgers vectors parallel to the z-axis. We suppose that such a dislocation corresponds to a displacement discontinuity given by

$$\boldsymbol{u}^{\mathrm{III}}(x, 0+) - \boldsymbol{u}^{\mathrm{III}}(x, 0-) = (0, 0, -b) \quad \text{for} \quad x > 0,$$

where b is a constant. Throughout a superfix III is attached to the displacement vector $\boldsymbol{u}$ and the components of the corresponding stress tensor. For a screw dislocation of this type situated at the origin, the stress field has non-zero components [10] given by

$$\left.\begin{aligned}\sigma_{xz}^{III}(x, y) &= -\frac{bK_S}{2\pi}\frac{\eta^2 y}{x^2+\eta^2 y^2},\\ \sigma_{yz}^{III}(x, y) &= \frac{bK_S}{2\pi}\frac{x}{x^2+\eta^2 y^2}\end{aligned}\right\} \tag{4}$$

where

$$K_s = (c_{44}c_{55})^{1/2}, \qquad \eta = (c_{55}/c_{44})^{1/2}. \tag{5}$$

The distribution of dislocations is odd in x; those to the right of each crack are positive and those to the left are negative. If the number of dislocations on $y = 0$ in the interval $(x, x + dx)$ is $f(x)\,dx$, then the components of the stress field created around the cracks can be calculated by direct substitution into the formula

$$\sigma_{ij}(x, y) = \int_{-\infty}^{\infty} \sigma_{ij}^{III}(x - x', y)\, f(x')\, dx', \tag{6}$$

with the appropriate σ_{ij}^{III} given by Eqs. (4) and (5).

The value of the density function $f(x)$ and the relationship between c and a are to be determined for various values of the prescribed physical constants σ and σ_1. This can be accomplished by constructing the singular integral equation which constitutes the equilibrium equation for the dislocations representing this row of cracks.

From Eq. (6) we see that the shear stress at a point on the x-axis due to all the other dislocations is

$$\sigma_{yz}(x, 0) = \int_{-\infty}^{\infty} \sigma_{yz}^{III}(x - x', 0)\, f(x')\, dx' = \frac{bK_S}{2\pi}\int_{-\infty}^{\infty} \frac{f(x')}{x - x'}\, dx', \tag{7}$$

where the integral here must be interpreted as a Cauchy principal value integral. Hence, since the resistance to the motion of the dislocations is taken to be σ_1 (the constant yield stress of the material) within the plastic zones, we must satisfy the equilibrium equation

$$\frac{bK_S}{2\pi}\int_{-\infty}^{\infty} \frac{f(x')}{x - x'}\, dx' = \begin{cases} -\sigma & \text{in } I, \\ -(\sigma - \sigma_1) & \text{in } I_p. \end{cases} \tag{8}$$

Using the observation that $f(x + 2nh) = f(x)$ and $f(x) = -f(-x)$ together with the result that

$$\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = -\frac{1}{2z^2} + \frac{\pi}{2z}\cot \pi z,$$

the left-hand side of this equation can be rewritten (cf. Liebfried [15, in German]) in the form

$$\frac{bK_S}{4h}\int_{-a}^{a}\frac{\cos(\pi x'/2h)}{\sin(\pi x/2h)-\sin(\pi x'/2h)}f(x')\,dx'.$$

Hence, by introducing the new variables defined by

$$\left.\begin{aligned} x_1 &= \sin(\pi x/2h), & a_1 &= \sin(\pi a/2h),\\ x_1' &= \sin(\pi x'/2h), & c_1 &= \sin(\pi c/2h), \end{aligned}\right\} \tag{9}$$

the integral equation for $f(x)$ becomes

$$\int_{-a_1}^{a_1}\frac{f_1(x_1')}{x_1-x_1'}\,dx_1' = \frac{2\pi}{bK_S}\begin{cases}-\sigma, & -c_1 < x_1 < c_1,\\ -(\sigma-\sigma_1), & c_1 < |x_1| < a_1,\end{cases} \tag{10}$$

with $f_1(x_1) = f_1\left[\sin\left[\frac{\pi x}{2h}\right]\right] \equiv f(x)$.

This is of precisely the same form as that obtained in the isotropic case originally by Bilby, Cottrell and Swinden [5] and subsequently by Bilby, Cottrell, Smith and Swinden [9], being of the well-known type discussed by, for example, Muskhelishvili [16] and Gakhov [17]. We can therefore take over the main results; referring the interested reader to these for more details of their derivation.

The appropriate solution can be conveniently written as

$$f_1(x_1) = f(x) = \frac{2\sigma_1}{\pi bK_S}\ln\left|\frac{x_1(a_1^2-c_1^2)^{1/2}+c_1(a_1^2-x_1^2)^{1/2}}{x_1(a_1^2-c_1^2)^{1/2}-c_1(a_1^2-x_1^2)^{1/2}}\right|. \tag{11}$$

We wish this to be bounded at both $x_1 = \pm a_1$ (i.e. $x = \pm a$), since there are no barriers there. This requirement will be met if and only if

$$\int_{-c_1}^{c_1}\frac{\sigma}{(a_1^2-x_1'^2)^{1/2}}\,dx_1' + \left\{\int_{-a_1}^{-c_1}+\int_{c_1}^{a_1}\right\}\frac{(\sigma-\sigma_1)}{(a_1^2-x_1'^2)^{1/2}}\,dx_1' = 0. \tag{12}$$

Substitution of the expression (11) into Eq. (6) leads to explicit representations of the stress components at any point in the medium.

4. Implications and Discussion

However, the two quantities which are usually of most interest are the maximum extent of the plastic zones in front of the cracks and the total plastic displacement, Φ_0^{III}, at the crack-tips. It can be shown that the condition (12) can be rewritten as

$$\frac{c_1}{a_1} \equiv \frac{\sin(\pi c/2h)}{\sin(\pi a/2h)} = \cos\left[\frac{\pi\sigma}{2\sigma_1}\right] \tag{13}$$

and it is this which determines the length, $a-c$, of the plastic zones in terms of the applied stress. Φ_0^{III} is the sum of the Burgers vectors of all the dislocations which have passed through a crack-tip and is thus given by

$$\begin{aligned}\Phi_0^{III} &= b\int_c^a f(x)\,dx = \frac{2bh}{\pi}\int_{c_1}^{a_1} \frac{f_1(x_1)}{(1-x_1^2)^{1/2}}\,dx_1 \\ &= \frac{4\sigma_1 h}{\pi^2 K_S}\tan\left[\frac{\pi a}{2h}\right]\int_0^{\theta_0} \frac{\sin\theta}{\{1+\sin^2\theta\tan^2(\pi a/2h)\}^{1/2}}\ln\left|\frac{\sin(\theta_0+\theta)}{\sin(\theta_0-\theta)}\right| d\theta\end{aligned} \tag{14}$$

with $\theta_0 = \pi\sigma/2\sigma_1$.

We observe that the condition (13) is identical to that given by Bilby, Cottrell, Smith and Swinden [9] for cracks within an isotropic medium, so that the length of the plastic zones is unaffected by the anisotropy of the material. On the other-hand, from Eq. (14), the ratio $\Phi_0^{III}/\Phi_i^{III}$ of the plastic displacement at a crack-tip in the orthotropic crystal to that in an isotropic material of shear modulus μ is seen to be given by

$$\Phi_0^{III}/\Phi_i^{III} = \mu K_s. \tag{15}$$

The Table indicates the range of values of K_s and μ/K_s for various materials. The values given for K_s are derived from the data of Chou [11] and Hearmon [18], and μ has been taken to be $7 \times 10^{10}\,\mathrm{Nm}^{-2}$, as being typical for an isotropic material. We see that μ/K_s varies significantly from one material to another, with the greatest plastic displacements occurring for CdS and Mg.

It is also of interest to record here the approximate results which are given in cases of small stress ($\sigma \ll \sigma_1$). From Eqs. (13) and (14) we find that then

$$\frac{a-c}{c} \approx \frac{\pi^2\sigma^2}{8\sigma_1^2}\frac{\tan(\pi c/2h)}{\pi c/2h}, \quad \Phi_0^{III} \approx \frac{\sigma^2}{K_S\sigma_1} h\tan(\pi c/2h). \tag{16}$$

Table. *Values of K_s (units 10^{10} Nm^{-2}) and μ/K_s for various materials*

Material	Be	C	Cd	CdS	Co	Mg	Zn	Zr
K_s	14.2	3.1	2.6	1.5	7.3	1.7	4.9	3.4
μ/K_s	0.5	2.2	2.7	4.6	1.0	4.2	1.4	2.1

We note that as $h \to \infty$ these reproduce the results expected from Tupholme [14] for a single crack in an infinite plate.

At this stage, it is instructive to point-out that, not only have we solved the problem formulated in Section 2 but moreover at the same time, we have actually provided the solution to two further situations. Firstly, since the distribution of dislocations which we have employed is antisymmetrical about the planes $x = \pm h$, it is clear, from a knowledge of the field of an image combination of such screw dislocations, that the stress components σ_{xz} vanish on these planes. The stress distribution obtained for the infinite row of cracks therefore also supplies that created around a single crack, $-c < x < c$, in a finite plate, $-h < x < h$, of orthotropic material whose surfaces $x = \pm h$ are stress-free. Secondly, since the distribution is odd about $x = 0$, it further represents the conditions in an orthotropic finite plate, $0 < x < h$, containing a surface crack, $0 < x < c$.

Bearing in mind the data presented in the Table, some consequences of these results and predictions for the theory of notch brittleness and high-stress fatigue can be deduced directly from those given by Bilby, Cottrell, Smith and Swinden [9].

Finally, we can deduce the solution for cracks in a purely elastic orthotropic plate, for example, by letting $\sigma_1 \to \infty$. In this limit we find from Eq. (11) that

$$f(x) = \frac{2\sigma}{K_S b} \frac{\sin(\pi x/2h)}{\{\sin^2(\pi c/2h) - \sin^2(\pi x/2h)\}^{1/2}},$$

and the corresponding shear stress directly in front of a crack can be shown from Eq. (7) to be

$$\sigma_{yz}(x, 0) = \frac{\sigma \sin(\pi x/2h)}{\{\sin^2(\pi x/2h) - \sin^2(\pi c/2h)\}^{1/2}}, \tag{17}$$

with resulting stress-intensity factor

$$K_{III} = \sigma \left\{\frac{h}{\pi} \tan\left[\frac{\pi c}{2h}\right]\right\}^{1/2}. \tag{18}$$

Combining Eqs. (16) and (18), it is therefore apparent that the low-stress approximations for elastoplastic cracks can in fact be neatly written in terms of the corresponding elastic stress-intensity factor as

$$a - c \approx \pi^2 K_{III}^2/4\sigma_1^2, \qquad \Phi_0^{III} \approx \pi K_{III}^2/K_s\sigma_1, \tag{19}$$

with the dependence on σ, c and h contained in K_{III}^2 and that upon σ_1 and K shown explicitly.

For the sake of brevity we have only studied mode III antiplane shear cracks here, but the same type of model could be applied to mode I and mode II situations in which the specified tractions are such that $\sigma_{yy} \to \sigma$ and $\sigma_{xy} \to \sigma$, respectively, at infinity, using straight edge dislocations whose fields are summarized by Tupholme [14]. The foregoing analysis for the infinite row of elastoplastic cracks would then again yield Eq. (15), for example, but with the constant K_e replaced by $K_n(1-\nu)$ or $K_e(1-\nu)$ for mode I or mode II, respectively, where

$$K_e = (\bar{c}_{12} + c_{12}) \left\{ \frac{c_{66}(\bar{c}_{12} - c_{12})}{c_{22}(\bar{c}_{12} + c_{12} + 2c_{66})} \right\}^{1/2}, \ \bar{c}_{12} = (c_{11}c_{22})^{1/2},$$

$$K_n = K_e(c_{22}/c_{11})^{1/2},$$

and ν is Poisson's ratio. These solutions however do not correspond to those for the analogous situation in a finite plate as did the mode III results, because the image constructions for edge dislocations do not make the planes $x = \pm nh$ stress-free. In the isotropic, purely elastic limit the results obtained can indeed be shown to reproduce those presented in Sneddon and Lowengrub [3] and Sih [4].

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