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Certain solutions of the equations of the planar motion of a second grade fluid for steady and unsteady cases

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Summary. Solutions for the equations of motion of an incompressible second grade fluid are derived by assuming certain conditions on the stream function. Exact solutions are obtained for a planar motion for both steady and unsteady cases.

1 Introduction

Known exact solutions of the Navier-Stokes equations are few in number. This is, in general, due to the non-linearities which occur in the inertial part of these equations. However, many flow situations of interest are such that a number of terms in the equations of motion either disappear automatically or may be neglected, and the resulting equations reduce to a form that can be readily solved.

By considering the vorticity to be a function of the stream function alone, Taylor [1] showed that the non-linearities are self-cancelling and obtained an exact solution which represents the decay of the double array of vortices. By taking the vorticity to be proportional to the stream function perturbed by a uniform stream, Kovasznay [2] also observed the similar cancellation of the non-linearities and found an exact solution which described the motion behind a two-dimensional grid. Wang [3] was also able to linearize the Navier-Stokes equations and showed that the results established in [1], [2] can be obtained from his findings as special cases. Recently, Lin and Tobak [4] and Hui [5] investigated similar flows where the non-linear terms vanish automatically.

In the case of the non-Newtonian fluids, namely the homogeneous incompressible Rivlin-Ericksen fluids of second grade [6], [7], it is found that the non-linearities occur not only in the inertial part but also in the viscosity part of the governing equations. As a result, the number of exact solutions becomes much smaller as compared to the exact solutions of Navier-Stokes equations. Rajagopal [8] observed that the non-linear convective terms which occur in the equations of motion of a second grade fluid also vanish for the specific problems studied by Taylor and Kovasnay as mentioned earlier. Rajagopal and Gupta [9] obtained a class of exact solutions to the equations of motion of a second grade fluid wherein the non-linearities are self cancelling though individually non-vanishing. They showed that these exact solutions form a subclass of the solution obtained by Wang [3] for the Navier-Stokes equations.

By assuming a certain form of the stream function, solutions for such fluids for the steady planar case were obtained by Kaloni and Huschlit [10], Siddiqui and Kaloni [11], and Siddiqui

[12]. Viscometric flows of such fluids have been studied by Markovitz and Coleman [13] and solutions to unsteady flows have been found by Ting [14] and Rajagopal [15].

The equations of motion for such fluids are, in general, one order higher than the Navier-Stokes equations and require additional boundary conditions over and above the boundary conditions used to solve the Navier-Stokes equations. However for special classes of solutions in unbounded domains, one may not need an additional boundary condition.

In the present paper, following Hui [5], we study the two-dimensional flow of a homogenous incompressible second grade fluid in which the vorticity is proportional to the stream function perturbed by a uniform stream and exhibit a class of exact solutions. In addition, on neglecting the inertial terms and assuming the Laplacian of vorticity to be proportional to the stream function perturbed by a uniform stream, we obtain another class of exact solutions in an unbounded domain which do not require an additional boundary condition.

2 Basic equations

The basic equations governing the motion of a second grade fluid are

$$\operatorname{div} \boldsymbol{v} = 0 \tag{1}$$

$$\begin{aligned}\varrho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\operatorname{grad} \boldsymbol{v})\,\boldsymbol{v}\right] &= -\operatorname{grad} p + \mu \nabla^2 \boldsymbol{v} + (\alpha_1 + \alpha_2) \operatorname{div} \boldsymbol{A}_1{}^2 \\ &\quad + \alpha_1 \left[\nabla^2 \boldsymbol{v}_t + \nabla^2(\nabla \times \boldsymbol{v}) + \operatorname{grad}\left\{(\boldsymbol{v} \cdot \nabla^2 \boldsymbol{v}) + \frac{1}{4}|\boldsymbol{A}_1|^2\right\}\right] + \varrho \boldsymbol{f}\end{aligned} \tag{2}$$

where $\boldsymbol{v}$ is the velocity vector, p the pressure, $\boldsymbol{A}_1$ the first Rivlin-Ericksen tensor given by

$$\boldsymbol{A}_1 = \operatorname{grad} \boldsymbol{v} + (\operatorname{grad} \boldsymbol{v})^T, \tag{3}$$

ϱ the constant density, $\boldsymbol{f}$ the body force, α_1 and α_2 the material constant and μ is the coefficient of viscosity. Here $\boldsymbol{v}_t = \frac{\partial \boldsymbol{v}}{\partial t}$, ∇^2 is the Laplacian operator and $|\boldsymbol{A}_1|$ denotes the usual norm of matrix $\boldsymbol{A}$ given by $\{\Sigma a_{ij}^2\}^{1/2}$.

If an incompressible fluid of second grade is to have motions which are compatible with thermodynamics in the sense of the Clausius-Duhem inequality and the condition that the Helmoltz free energy be a minimum when the fluid is at rest, then the following conditions must be satisfied [16]:

$$\mu \geqq 0, \quad \alpha_1 \geqq 0, \quad \alpha_1 + \alpha_2 = 0.$$

On the other hand, experimental results of tested fluids of second grade showed that $\alpha_1 < 0$ and $\alpha_1 + \alpha_2 \neq 0$ which contradicts the above conditions and imply that such fluids are unstable. This controversy is discussed in detail in [16] and [17]. However, in our paper we will discuss both cases, $\alpha_1 \geqq 0$ and $\alpha_1 < 0$.

Now let us consider the unsteady plane flow where $\boldsymbol{v}$ is represented by

$$\boldsymbol{v} = [u(x, y, t), v(x, y, t), 0] \tag{4}$$

and define the generalized pressure h and vorticity w functions as

$$h = \frac{\varrho}{2}(u^2 + v^2) + p - \left[\alpha_1(u\nabla^2 u + v\nabla^2 v) + \frac{1}{4}(3\alpha_1 + 2\alpha_2)|A_1|^2\right] \tag{5}$$

$$w = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{6}$$

Then using (3) we get

$$|A_1|^2 = 4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2.$$

Substituting (4), (5), and (6) into (1) and (2) and assuming that the body force is absent, the equations of motion become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

$$\begin{aligned} \frac{\partial h}{\partial x} + \varrho\left[\frac{\partial u}{\partial t} - vw\right] &= \mu\nabla^2 u + \alpha_1\nabla^2\frac{\partial u}{\partial t} - \alpha_1 v\nabla^2 w \\ \frac{\partial h}{\partial y} + \varrho\left[\frac{\partial v}{\partial t} + uw\right] &= \mu\nabla^2 v + \alpha_1\nabla^2\frac{\partial v}{\partial t} + \alpha_1 u\nabla^2 w. \end{aligned} \tag{8}$$

If we define the stream function $\psi(x, y, t)$ such that

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x},$$

Eq. (7) is satisfied identically and Eqs. (8) become

$$\begin{aligned} \frac{\partial h}{\partial x} + \varrho\left[\frac{\partial^2\psi}{\partial t\,\partial y} - \frac{\partial\psi}{\partial x}(\nabla^2\psi)\right] &= \mu\nabla^2\frac{\partial\psi}{\partial y} + \alpha_1\nabla^2\frac{\partial^2\psi}{\partial t\,\partial y} - \alpha_1\frac{\partial\psi}{\partial x}\nabla^4\psi \\ \frac{\partial h}{\partial y} + \varrho\left[-\frac{\partial^2\psi}{\partial t\,\partial x} - \frac{\partial\psi}{\partial y}(\nabla^2\psi)\right] &= -\mu\nabla^2\frac{\partial\psi}{\partial x} - \alpha_1\nabla^2\frac{\partial^2\psi}{\partial t\,\partial x} - \alpha_1\frac{\partial\psi}{\partial y}\nabla^4\psi. \end{aligned} \tag{9}$$

In eliminating the generalized pressure between the equations in (9), the above system reduces to a single partial differential equation:

$$\varrho\left[\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)}\right] = \mu\nabla^4\psi + \alpha_1\frac{\partial}{\partial t}[\nabla^4\psi] - \alpha_1\frac{\partial(\psi, \nabla^4\psi)}{\partial(x, y)}. \tag{10}$$

Equation (10) represents the motion of a second grade fluid when the motion is planar and the body force is null. We will solve Eq. (10) for some special cases.

3 General solutions of equations

Part I

In this Section we will consider the case of the motion when

$$\nabla^2\psi = K(\psi - Uy) \tag{11}$$

where K and U are constants. Using (11) one finds

$$\begin{aligned}\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} &= \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}(\nabla^2\psi) - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi)\\ &= \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}[K(\psi - Uy)] - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}[K(\psi - Uy)]\\ &= -KU\frac{\partial\psi}{\partial x},\end{aligned}$$

$$\nabla^4\psi = \nabla^2\nabla^2\psi = \nabla^2[K(\psi - Uy)] = K^2(\psi - Uy),$$

$$\frac{\partial}{\partial t}(\nabla^4\psi) = \frac{\partial}{\partial t}K^2(\psi - Uy) = K^2\frac{\partial\psi}{\partial t},$$

$$\frac{\partial}{\partial t}(\nabla^2\psi) = \frac{\partial}{\partial t}[K(\psi - Uy)] = K\frac{\partial\psi}{\partial t},$$

and

$$\frac{\partial(\psi, \nabla^4\psi)}{\partial(x, y)} = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^4\psi - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^4\psi) = -K^2U\frac{\partial\psi}{\partial x}.$$

Hence Eq. (10) subject to condition (11) becomes

$$(\varrho - \alpha_1 K)\frac{\partial\psi}{\partial t} + U(\varrho - \alpha_1 K)\frac{\partial\psi}{\partial x} = \mu K(\psi - Uy). \tag{12}$$

On setting $\Psi = \psi - Uy$, Eqs. (11) and (12) reduce to the system

$$\nabla^2\Psi = K\Psi. \tag{13}$$

$$(\varrho - \alpha_1 K)\frac{\partial\Psi}{\partial t} + U(\varrho - \alpha_1 K)\frac{\partial\Psi}{\partial x} = \mu K\Psi. \tag{14}$$

We note with interest that by setting $U = 0$ and $\alpha_1 = 0$ in the above system, we obtain Taylor's case [1]. Furthermore on setting $\alpha_1 = 0$ only, we recover Hui's case [5].

Steady flow

For the steady case Eqs. (13) and (14) become

$$\nabla^2\Psi = K\Psi \tag{15.1}$$

$$U(\varrho - \alpha_1 K)\frac{\partial\Psi}{\partial x} = \mu K\Psi. \tag{15.2}$$

We observe that if $U = 0$, we obtain the trivial solution $\Psi = 0$, which implies that $\psi = 0$. In the case when $\varrho - \alpha_1 K = 0$ it follows that $\Psi = 0$, or $K(\psi - Uy) = 0$, which implies that $\psi = Uy$.

We suppose in the following that $U(\varrho - \alpha_1 K) \neq 0$.

Then (15) becomes

$$\nabla^2\Psi = K\Psi$$
$$\frac{\partial\Psi}{\partial x} = \frac{\mu K\Psi}{U(\varrho - \alpha_1 K)}. \tag{16}$$

Solving the last equation using the product method we get

$$\Psi = F(y)\, e^{\frac{1}{U}\frac{\mu K x}{\varrho - \alpha_1 K}}$$

or

$$\Psi = F(y)\, e^{\frac{\lambda x}{U}} \quad \text{where} \quad \lambda = \frac{\mu K}{\varrho - \alpha_1 K}. \tag{17}$$

To find $F(y)$ we substitute (17) into (15.1) to get

$$F''(y) + \left[\frac{\lambda^2}{U^2} - K\right] F(y) = 0. \tag{18}$$

Upon solving Eq. (18) we have the following general solution for the stream function defined in the positive half space $x \geqq 0$:

(i) If $K < 0$ or $\dfrac{(2\varrho\alpha_1 U^2 + \mu^2) - [\mu^2(4\varrho\alpha_1 U^2 + \mu^2)]^{1/2}}{2U^2{\alpha_1}^2} < K$ $< \dfrac{(2\varrho\alpha_1 U^2 + \mu^2) + [\mu^2(4\varrho\alpha_1 U^2 + \mu^2)]^{1/2}}{2U^2{\alpha_1}^2}$ then

$$\psi = Uy + Ce^{\frac{\lambda x}{U}} \cos\left[\left(\frac{\lambda^2}{U^2} - K\right)^{1/2} y + A\right]. \tag{19}$$

These solutions for $K < 0$ represent a uniform stream plus a perturbation which decays when $\dfrac{\varrho}{K} < \alpha_1$, and grows when $\alpha_1 < \dfrac{\varrho}{K}$ and is periodic in y. For the other case they represent a uniform stream plus a perturbation which decays when $\alpha_1 > \dfrac{\varrho}{K}$ and grows when $\alpha_1 < \dfrac{\varrho}{K}$ and is periodic in y.

(ii) If $K = 0$ or $K = \dfrac{(2\varrho\alpha_1 U^2 + \mu^2) \pm [\mu^2(4\varrho\alpha_1 U^2 + \mu^2)]^{1/2}}{2U^2{\alpha_1}^2}$ then

$$\psi = Uy + (Ay + B)\, e^{\frac{\lambda x}{U}}. \tag{20}$$

For $K = 0$ these solutions represent a uniform stream for any α_1 and for the second case they represent a uniform stream with a perturbation which grows when $\dfrac{-\mu^2}{4\varrho U^2} < \alpha_1 < 0$ or $0 < \alpha_1 < \dfrac{\varrho}{K}$ and decays when $\alpha_1 > \dfrac{\varrho}{K}$ and is not periodic in y.

(iii) If $0 < K < \dfrac{(2\varrho\alpha_1 U^2 + \mu^2) - [\mu^2(4\varrho\alpha_1 U^2 + \mu^2)]^{1/2}}{2U^2{\alpha_1}^2}$ or $K > \dfrac{(2\varrho\alpha_1 U^2 + \mu^2) + [\mu^2(4\varrho\alpha_1 U + \mu^2)]^{1/2}}{2U^2{\alpha_1}^2}$ then

$$\psi = Uy + \left(Ae^{-\left(K - \frac{\lambda^2}{U^2}\right)^{1/2} y} + Be^{\left(K - \frac{\lambda^2}{U^2}\right)^{1/2} y}\right) e^{\frac{\lambda x}{U}}. \tag{21}$$

These solutions represent a uniform stream and a perturbation which decays if $\alpha_1 > \frac{\varrho}{K}$ and grows if $\alpha_1 < \frac{\varrho}{K}$.

Unsteady flows

A) Case 1

We will consider in the first part plane wave solutions of Eq. (14) of the form $\Psi = G(x, y)\, e^{mt}$ with $X = x - Ut$.

Equation (14) can be written as

$$\frac{\partial \Psi}{\partial t} + U \frac{\partial \Psi}{\partial x} = \lambda \Psi. \tag{22}$$

The case when $\varrho - \alpha_1 K = 0$ implies $\Psi = 0$, hence $\psi = Uy$.

Upon substituting $\Psi = G(x, y)\, e^{mt}$ into Eq. (22) we find that $m = \lambda$. Hence $\Psi = G(x, y)\, e^{\lambda t}$ satisfies Eq. (22). To find $G(x, y)$ we substitute $\Psi = G(x, y)\, e^{\lambda t}$ into Eq. (13) to get

$$\frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial y^2} = KG. \tag{23}$$

Plane wave solutions to Eq. (23) exist in the form

$$G = g(\xi) \quad \text{where} \quad \xi = X \cos\theta + y \sin\theta, \ -\Pi \leqq \theta < \Pi.$$

Substituting $G = g(\xi)$ into (23) we get

$$g''(\xi) = Kg(\xi)$$

or (24)

$$g'' - Kg = 0.$$

There are three cases

a) If $K = -k^2 < 0$, then the general solution of (24) is given by $g(\xi) = A(\theta) \cos k\,\{\xi + B(\theta)\}$ and the corresponding solution for the stream function is

$$\psi(x, y, t) = Uy + A(\theta)\, e^{\lambda t} \cos k\{(x - Ut) \cos\theta + y \sin\theta + B(\theta)\} \tag{25}$$

where $A(\theta)$ and $B(\theta)$ are real arbitrary constants depending on θ.

b) If $K = 0$, then the solution is

$$\psi = Uy + A(\theta)\,\{(x - Ut) \cos\theta + y \sin\theta\} + B(\theta). \tag{26}$$

c) If $K = k^2 > 0$, the general solution of (24) is

$$g(\xi) = A(\theta)\, e^{k\xi} + B(\theta)\, e^{-k\xi}$$

and the solution for the stream function is

$$\psi(x, y, t) = Uy + e^{\lambda t} A(\theta)\, e^{k[(x - Ut)\cos\theta + y \sin\theta]} + B(\theta)\, e^{-k[(x - Ut)\cos\theta + y\sin\theta]}. \tag{27}$$

We observe that when $K = 0$ the flow is irrotational and the solution is valid for both viscous and second grade fluid. For $K < 0$ and $\alpha_1 > 0$ the solution is exponentially decaying and reduces to

Hui's case, (Class A) [5]. For $K > \frac{\varrho}{\alpha_1} > 0$ the solution is exponentially decaying, but for $K > 0$ and $\alpha_1 = 0$, the solution is exponentially growing and can only have physical meaning in a finite time interval. It is interesting to observe that as α_1 increases the damping of the stream line functions is greater as expected for a second grade fluid when compared with viscous fluids.

B) Case 2

In this Section we will give another class of solutions to Eqs. (13) and (14) of the form

$$\Psi = H(X, y)\, e^{mx/U}$$

where $X = x - Ut$ and m and H are to be determined. Upon substituting $\Psi = H(X, y)\, e^{\frac{mx}{U}}$ into Eq. (14) we find that $m = \lambda$ where

$$\lambda = \frac{\mu K}{\varrho - \alpha_1 K}.$$

If $\alpha_1 = 0$, Eq. (14) reduces the Hui's case (Class B) [5]. To find H we substitute $\Psi = H(X, y)\, e^{\lambda t}$ into Eq. (13) to get

$$\frac{\partial^2 H}{\partial X^2} + \frac{2\lambda}{U}\frac{\partial H}{\partial X} + \left(\frac{\lambda^2}{U^2} - K\right) H + \frac{\partial^2 H}{\partial y^2} = 0. \tag{28}$$

Plane waves solutions to (28) exist in the form

$$H = h(\xi), \qquad \xi = X\cos\theta + y\sin\theta. \tag{29}$$

Substituting (29) into (28) we get

$$h''(\xi) + \frac{2\lambda}{U} h'(\xi)\cos\theta + \left(\frac{\lambda^2}{U^2} - K\right) h(\xi) = 0. \tag{30}$$

Solutions of (30) are of the form

$$h(\xi) = c e^{m\xi/U} \tag{31}$$

where c is an arbitrary constant and $m = -\lambda\cos\theta \pm [KU^2 - \lambda^2\sin^2\theta]^{1/2}$. Depending on the sign of $KU^2 - \lambda^2\sin^2\theta$, we have the following solutions:

a) If $K = k^2 \geqq \frac{\lambda^2}{U^2} > 0$, then

$$h(\xi) = e^{-(\lambda\cos\theta)\xi/U}[A(\theta)\, e^{\xi(KU^2 - \lambda^2\sin^2\theta)^{1/2}} + B(\theta)\, e^{-\xi(KU^2 - \lambda\sin^2\theta)^{1/2}}] \tag{32}$$

for $-\Pi \leqq \theta \leqq \Pi$ except when $K = \frac{\lambda^2}{U^2}$ and $\theta = \Pi/2$, then (30) reduces to $h''(\xi) = 0$ whose solution is $h = ay + b$. The latter solution is also obtained when $K = 0$.

b) If $K = k^2 < \frac{\lambda^2}{U^2}$, then for $|\theta| < \theta_0$ or $\Pi - \theta_0 < |\theta| \leqq \Pi$ where $\theta_0 = \sin^{-1}\frac{K^{1/2}}{\lambda} U$, the solution is given by

$$h = e^{-(\lambda\cos\theta)\xi/U}[A(\theta)\, e^{\xi(KU^2 - \lambda^2\sin^2\theta)^{1/2}} + B(\theta)\, e^{-\xi(KU^2 - \lambda^2\sin^2\theta)^{1/2}}] \tag{33}$$

and if $\theta_0 < |\theta| \leqq \Pi - \theta_0$, the solution is given by

$$h = A(\theta)\, e^{(-\lambda\cos\theta)\xi/U} \cos\left[\xi(\lambda^2 \sin^2\theta - KU^2)^{1/2} + B(\theta)\right]. \tag{34}$$

c) If $K = -k^2 < 0$, then

$$h = A(\theta)\, e^{(-\lambda\cos\theta)\xi/U} \cos\left[\xi(\lambda^2 \sin^2\theta - KU^2)^{1/2} + B(\theta)\right]. \tag{35}$$

Observe that the solutions $\Psi = h(\xi)\, e^{\frac{\lambda x}{U}}$ cannot be obtained from the solutions $\Psi = g(\xi)\, e^{\lambda t}$ and vice versa, except when $\theta = 0$. When $\theta = \frac{\Pi}{2}$, the plane wave solutions $\Psi = h(\xi)\, e^{\frac{\lambda x}{U}}$ reduce to the special case of the steady flow solutions discussed earlier.

Part II

We now consider the motion of a second grade fluid when $\nabla^4\psi = K(\psi - Uy)$ where K and U are real constants under the assumption that the motion is slow enough, so that we can neglect the inertia terms. Then Eq. (10) becomes

$$\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \nabla^4\psi - \alpha_1 \frac{\partial(\psi, \nabla^4\psi)}{\partial(x, y)} = 0.$$

Letting $\nabla^4\psi = K(\psi - Uy)$ and $\Psi = \psi - Uy$ we get

$$\nabla^4\Psi = K\Psi \tag{36}$$

$$\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \Psi = -\alpha_1 U \frac{\partial \Psi}{\partial x}. \tag{37}$$

We observe that when $\alpha_1 = 0$ then $\Psi = 0$ which implies that $\psi = Uy$. In the following we give solutions to equations (36) and (37).

Steady flows

For steady flows, the system (36), (37) reduces to

$$\left.\begin{aligned} U \frac{\partial \Psi}{\partial x} &= \frac{-\mu}{\alpha_1} \Psi \\ \nabla^4\Psi &= K\Psi. \end{aligned}\right\} \tag{38}$$

For $U = 0$, $\Psi = 0$, hence $\psi = 0$ which implies no flow. For $U \neq 0$

$$U \frac{\partial \Psi}{\partial x} = \frac{-\mu}{\alpha_1} \Psi$$

admits solutions of the form $\Psi = F(y)\, e^{\frac{-\mu x}{\alpha_1 U}}$.

Letting $\lambda = \frac{-\mu}{\alpha_1 U}$ and substituting $\Psi = F(y)\, e^{\lambda x}$ into equation $\nabla^4\Psi = K\Psi$, gives rise to the fourth order ordinary differential equation

$$F^{(\mathrm{IV})} + 2\lambda^2 F'' + (\lambda^4 - K)\, F = 0 \tag{39}$$

whose solutions are

a) If $K = 0$

$$F = A\cos(\lambda y + B) + Cy\cos(\lambda y + D) \tag{40}$$

where A, B, C, and D are arbitrary constants.

b) If $K = k^4 > 0$, then

i) For $k^2 = \lambda^2$, $F = A \cos[(\lambda^2 + k^2)^{1/2} y + B] + Cy + D$ (41)

ii) For $k^2 > \lambda^2$

$$F = A \cos[(\lambda^2 + k^2)^{1/2} y + B] + Ce^{(k^2 - \lambda^2)^{1/2} y} + De^{-(k^2 - \lambda^2)^{1/2} y} \tag{42}$$

iii) For $k^2 < \lambda^2$

$$F = A \cos[(\lambda^2 + k^2)^{1/2} y + B] + C \cos[(\lambda^2 - k^2)^{1/2} y + D]. \tag{43}$$

c) If $K = -k^4 < 0$, then

$$F = e^{(\lambda^4 + k^4)^{1/2}\left(\cos\frac{\theta_0}{2}\right)y} A \cos\left[(\lambda^4 + k^4)^{1/2}\left(\sin\frac{\theta_0}{2}\right) y + B\right] \tag{44}$$

$$+ Ce^{-(\lambda^4 + k^4)^{1/2}\left(\cos\frac{\theta_0}{2}\right)y} \cos\left[(\lambda^4 + k^4)^{1/2}\left(\sin\frac{\theta_0}{2}\right) y + D\right]$$

where $\theta_0 = \tan^{-1}\left(-\frac{k^2}{\lambda^2}\right)$. Hence

$$\psi = \Psi + Uy = Uy + F(y)\, e^{\lambda x} \tag{45}$$

where F is given by a), b) or c).

Unsteady flows

The Eqs. (36) and (37) of motion become

$$\nabla^4 \Psi = K\Psi \tag{46}$$

$$\frac{\partial \Psi}{\partial t} + U \frac{\partial \Psi}{\partial x} = -\delta\Psi \quad \text{where} \quad \delta = \frac{\mu}{\alpha_1}. \tag{47}$$

A) Case 1

We will consider solutions of (47) of the form $\Psi = G(X, y)\, e^{-\delta t}$ where $X = x - Ut$. Plane wave solutions of Eq. (47) exist in the form $G(X, y) = g(\xi)$ where $\xi = X \cos\theta + y \sin\theta$, $-\Pi \leqq \theta < \Pi$. Substituting $\Psi = g(\xi)\, e^{-\delta t}$ into Eq. (46) we get

$$g^{\mathrm{IV}}(\xi) - Kg(\xi) = 0. \tag{48}$$

Depending on the sign of K we obtain the following solutions:

a) If $K = 0$ then

$$\psi = Uy + e^{-\delta t}[A(\theta)\,(X\cos\theta + y\sin\theta)^3 + B(\theta)\,(X\cos\theta + y\sin\theta)^2$$

$$+ C(\theta)\,(X\cos\theta + y\sin\theta) + D(\theta)]. \tag{49}$$

b) If $K = k^4 > 0$ then

$$\psi = Uy + e^{-\delta t}[A(\theta)\, e^{k(X\cos\theta + y\sin\theta)} + B(\theta)\, e^{-k(X\cos\theta+y\sin\theta)}$$
$$+ C(\theta)\cos\{k(X\cos\theta + y\sin\theta) + B(\theta)\}]. \tag{50}$$

c) If $K = -k^4 < 0$ then

$$\psi = Uy + e^{-\delta t}\left[A(\theta)\, e^{\left(\cos\frac{\Pi}{8}\right)(X\cos\theta+y\sin\theta)}\cos\left\{\sin\frac{\Pi}{8}(X\cos\theta + y\sin\theta) + B(\theta)\right\}\right.$$
$$\left. + C(\theta)\, e^{-\sin\frac{\Pi}{8}(X\cos\theta+y\sin\theta)}\cos\left\{\cos\frac{\Pi}{8}(X\cos\theta + y\sin\theta) + D(\theta)\right\}\right]. \tag{51}$$

B) Case 2

We will now consider solutions of (47) of the form $\Psi = H(X, y)\, e^{-\delta x/U}$ where $X = x - Ut$ and $\delta = \frac{\mu}{\alpha_1}$.

Plane wave solutions of Eq. (47) exist in the form $H(X, y) = h(\xi)$ where

$\xi = X\cos\theta + y\sin\theta,\ -\Pi \leqq \theta < \Pi.$

Substituting $\Psi = h(\xi)\, e^{-\frac{\delta x}{U}}$ into Eq. (46) we get the fourth order homogeneous differential equation

$$h^{\mathrm{IV}} + 4\frac{\delta}{U}(\cos\theta)\, h''' + 2\frac{\delta^2}{U^2}(1 + 2\cos^2\theta)\, h'' + 4\frac{\delta^3}{U^3}(\cos\theta)\, h' + \left(\frac{\delta^4}{U^4} - K\right)h = 0. \tag{52}$$

The auxiliary equation associated with (52) has the following roots:

$$m = \frac{-\delta\cos\theta + [-\delta^2\sin^2\theta + (U^4K)^{1/2}]^{1/2}}{U}. \tag{53}$$

Depending on the sign of K we obtain the following solutions:

a) If $K = 0$, then

$$h = A(\theta)\, e^{-\frac{\delta}{U}(\cos\theta)(X\cos\theta+y\sin\theta)}\left\{\cos\left(\frac{\delta}{U}\sin\theta\right)(X\cos\theta + y\sin\theta) + B(\theta)\right\}$$
$$+ C(\theta)(X\cos\theta + y\sin\theta)\, e^{-\frac{\delta}{U}(\cos\theta)(X\cos\theta+y\sin\theta)}$$
$$\times\left\{\cos\left(\frac{\delta}{U}\sin\theta\right)(X\cos\theta + y\sin\theta) + D(\theta)\right\}. \tag{54}$$

b) If $K = k^4 > 0$, then for $|\theta| < \theta_0$ or $\Pi - \theta_0 < |\theta| \leqq \Pi$ where $\theta_0 = \sin^{-1}\frac{kU}{\delta}$

$$h = A(\theta)\, e^{\frac{-\delta\cos\theta(X_c\mathrm{os}\theta+y\sin\theta)}{U}}\cos\left\{\frac{(\delta^2\sin^2\theta + k^2U^2)^{1/2}(X\cos\theta + \sin\theta)}{U} + B(\theta)\right\}$$
$$+ C(\theta)\, e^{\frac{(-\delta\cos\theta + (k^2U^2-\delta^2\sin^2\theta)^{1/2})(X_c\mathrm{os}\theta+y\sin\theta)}{U}} + D(\theta)\, e^{\frac{(-\delta\cos\theta - (k^2U^2-\delta^2\sin^2\theta)^{1/2})(X_c\mathrm{os}\theta+y\sin\theta)}{U}} \tag{55}$$

and for $\theta_0 < |\theta| < \Pi - \theta_0$

$$h = A(\theta)\, e^{\frac{-\delta\cos\theta(X\cos\theta + y\sin\theta)}{U}} \cos\left\{\frac{(\delta^2 \sin^2\theta + k^2U^2)^{1/2}\,(X\cos\theta + y\sin\theta)}{U} + B(\theta)\right\}$$

$$+\, C(\theta)\, e^{\frac{-\delta\cos\theta}{U}(X\cos\theta + y\sin\theta)} \cos\left\{\frac{(\delta^2 \sin^2\theta - k^2U^2)^{1/2}\,(X\cos\theta + y\sin\theta)}{U} + D(\theta)\right\} \tag{56}$$

and for $\theta = \theta_0$

$$h = A(\theta)\, e^{-\delta\cos\theta(X\cos\theta + y\sin\theta)} \cos\left\{\frac{(\delta^2 \sin^2\theta + k^2U^2)^{1/2}\,(X\cos\theta + y\sin\theta)}{U} + B(\theta)\right\}$$

$$+\, [C(\theta) + D(\theta)\,(X\cos\theta + y\sin\theta)]\, e^{\frac{-\delta\cos\theta}{U}(X\cos\theta + y\sin\theta)}. \tag{57}$$

c) If $K = -k^4 < 0$ then

$$h = A(\theta)\, e^{\frac{-(\delta\cos\theta - r\cos\theta_0)(X\cos\theta + y\sin\theta)}{U}} \cos\left\{\frac{r\sin\theta_0(X\cos\theta + y\sin\theta)}{U} + B(\theta)\right\}$$

$$+\, C(\theta)\, e^{\frac{-(\delta\cos\theta - r\cos\theta_0)(X\cos\theta + y\sin\theta)}{U}} \cos\left\{\frac{r\sin\theta_0(X\cos\theta + y\sin\theta)}{U} + D(\theta)\right\} \tag{58}$$

where $r = (\delta^4 \sin^4\theta + U^4k^4)^{1/2}$ and $2\theta_0 = \tan^{-1}\dfrac{U^2k^2}{\delta^2\sin^2\theta}$. Finally the solutions to Eqs. (46) and (47) are given by

$$\psi = Uy + he^{-\frac{\delta x}{U}} \text{ where } h \text{ is given by a), b) or c).} \tag{59}$$

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References

[1] Taylor, G. T.: On the decay of vortices in a viscous fluid. Phil. Mag. **46**, 671–674 (1923).
[2] Kovasnay, L. I. G.: Laminar flow behind a two-dimensional grid. Proc. Cambridge Phil. Soc. **44**, 58–62 (1948).
[3] Wang, C. Y.: On a class of exact solutions of the Navier-Stokes equations. J. Appl. Mech. **33**, 696–698 (1966).
[4] Lin, S. P., Tobak, M.: Reversed flow above a plate with suction. AIAA J. **24**, 334–335 (1986).
[5] Hui, W. H.: Exact solutions of the unsteady two-dimensional Navier-Stokes equation. ZAMP **38**, 689–102 (1987).
[6] Rivlin, R. S., Ericksen, J. L.: Stress deformation relations for isotropic materials. J. Rat. Mech. Anal. **4**, 323–425 (1955)
[7] Coleman, B. D., Noll, W.: An approximation theorem for functionals with application in continuum mechanics. Arch. Rat. Mech. Anal. **6**, 355–370 (1960).
[8] Rajagopal, K. R.: On the decay of vortices in a second grade fluid. Meccanica, **9**, 185–188 (1980).
[9] Rajagopal, K. R., Gupta, A. S.: On a class of exact solutions to the equations of motion of a second grade fluid. Int. J. Engng. Sci. **19**, 1009–1014 (1981).

[10] Kaloni, P. N., Huschilt, K.: Semi-inverse solutions of non-Newtonian fluid. Int. J. Non-Linear Mech. **19**, 373–381 (1984).
[11] Siddiqui, A. M., Kaloni, P. N.: Certain-inverse solutions of a non-Newtonian fluid. Int. J. Non-Linear Mech. **21**, 459–473 (1986).
[12] Siddiqui, A. M.: Some more inverse solutions of a non-Newtonian fluid. Mech. Res. Comm. **17**, 157–163 (1990).
[13] Markovitz, H., Coleman, B. D.: Incompressible second order fluids. Adv. Appl. Mech. **8**, 69–101 (1964).
[14] Ting, T. W.: Certain non-steady flows of second order fluids. Arch. Rat. Mech. Anal. **14**, 1–26 (1963).
[15] Rajagopal, K. R.: A note on unsteady unidirectional flows of a non-Newtonian fluid. Int. J. Non-Linear Mech. **17**, 369–373 (1982).
[16] Dunn, J. E., Fosdick, R. L.: Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade. Arch. Rat. Mech. Anal. **3**, 191–252 (1974).
[17] Fosdick, R. L., Rajagopal, K. R.: Anomalous features in the model of second-order fluids. Arch. Rat. Mech. Anal. **70**, 145–152 (1979).

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