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Note
On the Uniqueness of some Helical Flows of a Second Grade Fluid

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Summary

The uniqueness of some helical flows of a second grade fluid, between two infinite circular cylinders, is proved. Initially, the fluid is at rest and flow is produced by the motion of the cylinders. Finally, the special case of a flow in a circular cylinder is considered.

1. Introduction

For some time the special class of Rivlin-Ericksen fluids of second grade has been of interest to both the theorist and the experimentor in their efforts to understand the non-Newtonian behavior of incompressible, homogeneous fluids. Furthermore, the problem of finding of different motions in such fluids has been treated by a number of authors [1]–[4].

In this model the Cauchy stress $\boldsymbol{T}$ and the fluid motion are assumed to be related as follows

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu_0\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 \tag{1}$$

where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the first two Rivlin-Ericksen tensors, μ_0 is the viscosity, α_1 and α_2 are normal stress moduli, and $-p\boldsymbol{I}$ is the spherical stress due to constraint of incompressibility.

Of course, all motions are restricted to be isochoric, so that $\operatorname{div} \boldsymbol{v} = 0$ and, thus $\boldsymbol{A}_1$ is traceless.

The research reported here is devoted to the study of an helical flow of an homogeneous incompressible second grade fluid between two infinite concentric coaxial cylinders. The flow induced in fluid is considered when the cylinders rotate about their axis and slide in the direction of the same axis with prescribed velocities. Finally, the special case of the flow in a cylinder is considered. A motivation of the present analysis is the applicability of the flows in discussion to many technological problems.

2. Equations of Motion

In the following we use cylindrical co-ordinate (r, θ, z) and consider a flow having the contravariant velocity components [5]

$$\dot{r} = 0, \qquad \dot{\theta} = \omega(r, t), \qquad \dot{z} = u(r, t). \tag{2}$$

Such a flow is called *helical*, because, in general, the streamlines are helices. In [5, Sec. 123] it was proved that solving the dynamical equations, for this flow, reduces to finding solutions $\omega = \omega(r, t)$ and $u = u(r, t)$ of the next third-order linear partial differential equations[1]

$$\partial_r[r^3 \, \partial_r(\mu_0\omega + \alpha_1 \, \partial_t\omega)] = \varrho r^3 \, \partial_t\omega \tag{3}$$

$$\partial_r[r \, \partial_r(\mu_0 u + \alpha_1 \, \partial_t u)] = -ra(t) + \varrho r \, \partial_t u. \tag{4}$$

Once a solution pair $\omega(r, t)$, $u(r, t)$ has been found, all stress components can be determined by using (1), (2) and (112.9)—(112.10) from [5].

Remark 1. In the case when $\alpha_1 = 0$, corresponding to a Reiner-Rivlin fluid, the system (3)—(4) is identical with that resulting from the Navier-Stokes theory, so that the helical frows possible in a Reiner-Rivlin fluid of second gade are the same as those in the Navier-Stokes fluid. However, the surface tractions that must be applied in order to produce the flow will vary according to the value of α_2.

3. Statement of the Problem

We consider an unsteady helical flow between two infinite coaxial cylinders of radii R_1 and R_2 $(> R_1)$, which at $t = 0$ began to rotate about their axis with angular velocities $\Omega_1(t)$ and $\Omega_2(t)$ and to slide in the direction of the same axis with the velocities $U_1(t)$ and $U_2(t)$, respectively. Up to the moment $t = 0$ the whole system is assumed to be in rest.

Assuming that the fluid adheres to the walls and denoting by

$$\overline{\omega}(r, t) = r\omega(r, t) + \frac{r}{R_2 - R_1} \left[(r - R_2) \, \Omega_1(t) - (r - R_1) \, \Omega_2(t)\right] \tag{5}$$

and

$$\overline{u}(r, t) = u(r, t) + \frac{1}{R_2 - R_1} \left[(r - R_2) \, U_1(t) - (r - R_1) \, U_2(t)\right] \tag{6}$$

[1] The function $d(t)$ from [5] Eq. (123.17) must vanish. This ensure us that the radial stress T_{rr} from (112.9) is a single-valued function of position.

we attain to the next two problems with initial and boundary conditions

$$(\mu + \alpha\, \partial_t)\, L_n v_n(r, t) + a_n(r, t) = \partial_t v_n(r, t), \qquad t > 0 \tag{7.1}$$

$$v_n(r, 0) = V_n(r), \qquad r \in (R_1, R_2) \tag{7.2}$$

$$v_n(R_1, t) = v_n(R_2, t) = 0, \qquad t \geqq 0 \tag{7.3}$$

where

$$n = 0, 1, \qquad v_0 = \overline{u}, \qquad v_1 = \overline{\omega}, \qquad L_n = \partial_r{}^2 + \frac{1}{r}\,\partial_r - \frac{n}{r^2},$$

$$a_0(r, t) = \frac{a(t)}{\varrho} - \frac{\mu[U_1(t) - U_2(t)] + \alpha[U_1'(t) - U_2'(t)]}{r(R_2 - R_1)}$$

$$+ \frac{(r - R_2)\, U_1'(t) - (r - R_1)\, U_2'(t)}{R_2 - R_1},$$

$$a_1(r, t) = -\, 3\frac{\mu[\Omega_1(t) - \Omega_2(t)] + \alpha[\Omega_1'(t) - \Omega_2'(t)]}{r(R_2 - R_1)}$$

$$+ r\,\frac{(r - R_2)\, \Omega_1'(t) - (r - R_1)\, \Omega_2'(t)}{R_2 - R_1},$$

$$\mu = \mu_0/\varrho, \qquad \alpha = \alpha_1/\varrho,$$

$$V_0(r) = \frac{(r - R_2)\, U_1 - (r - R_1)\, U_2}{R_2 - R_1}$$

and

$$V_1(r) = \frac{(r - R_2)\, \Omega_1 - (r - R_1)\, \Omega_2}{R_2 - R_1}\, r$$

with $\Omega_n = \Omega_n(0)$ and $U_n = U_n(0)$.

4. The Uniqueness of the Solution

Theorem: If the problems (7) have continuous solutions on the domain $\{R_1 \leqq r \leqq R_2,\ t > 0\}$ then they are unique.

In order to prove this, we shall use, as in [6], an expansion theorem of Steklov [7, Ch. 4, § 1]. Let $v_n(r, t)$ be, continuous solutions of our problem, whose partial derivatives $\partial_r v_n$ and $\partial_r{}^2 v_n$ are piecewise continuous. For each $t > 0$, they can be written as Fourier-Bessel series absolutely and uniformly convergent in terms of the eigenfunctions

$$B_n(r r_{nm}) = A_n \left[J_n(r r_{nm}) - \frac{J_n(R_1 r_{nm})}{Y_n(R_1 r_{nm})}\, Y_n(r r_{nm}) \right] \tag{8}$$

of the problems

$$L_n v + \lambda v = 0, \qquad v(R_1) = v(R_2) = 0; \qquad n = 0, 1$$

i.e.,

$$v_n(r, t) = \sum_{m=1}^{\infty} \varphi_{nm}(t) \cdot B_n(rr_{nm}). \tag{9}$$

Here, $J_n(.)$ and $Y_n(.)$ denote Bessel functions in standard notations, r_{nm} are roots of the transcendental equations $B_n(R_2 r) = 0$ and the constants A_n are chosen so that the normalisation condition

$$\int_{R_1}^{R_2} r[B_n(rr_{nm})]^2 \, dr = 1$$

to be satisfied. Now, introducing (9) in (7.1), multiplying then by $rB_n(rr_{n_p})$ and integrating between the limits $r = R_1$ and $r = R_2$, we get the next linear differential equation in $\varphi_{nm}(.)$

$$\dot{\varphi}_{nm}(t) + b_{nm}\varphi_{nm}(t) = c_{nm}a_{nm}(t) \tag{10}$$

where $b_{nm} = \mu r_{nm}^2/(1 + \alpha r_{nm}^2)$, $c_{nm} = 1/(1 + \alpha r_{nm}^2)$ and $a_{nm}(.)$ are the finite Hankel transforms [8] of the functions $a_n(r, .)$.

The solutions of the Eq. (10) with initial conditions

$$\varphi_{nm}(0) = V_{nm} \qquad \left(V_{nm} = \int_{R_1}^{R_2} rV_n(r) \, B_n(rr_{nm}) \, dr\right)$$

together with (9) give us the unique solutions of (7)

$$v_n(r, t) = \sum_{m=1}^{\infty} \left[V_{nm} + c_{nm} \int_0^t a_{nm}(t) \cdot e^{b_{nm}t} \, dt \right] B_n(rr_{nm}) \cdot e^{-b_{nm}t}. \tag{11}$$

In the special case when the functions $\Omega_1(t)$, $\Omega_2(t)$, $U_1(t)$, $U_2(t)$ and $a(t)$ are constants and equal with Ω_1, Ω_2, U_1, U_2 and a, respectively, the relations (5), (6) and (11) lead to

$$\omega(r, t) = \frac{(r - R_1)\, \Omega_2 - (r - R_2)\, \Omega_1}{R_2 - R_1} + \frac{1}{r} \sum_{m=1}^{\infty} V_{1m} B_1(rr_{1m}) \, e^{-b_{1m}t}$$
$$+ \frac{3(\Omega_2 - \Omega_1)}{r(R_2 - R_1)} \sum_{m=1}^{\infty} \frac{A_m}{r_{1m}^2} B_1(rr_{1m}) \, (1 - e^{-b_{1m}t})$$

and

$$u(r, t) = \frac{(r - R_1)\, U_2 - (r - R_2)\, U_1}{R_2 - R_1} + \sum_{m=1}^{\infty} V_{0m} B_0(rr_{0m}) \, e^{-b_{0m}t}$$
$$+ \sum_{m=1}^{\infty} \left(\frac{a}{\mu_0} B_m + \frac{U_2 - U_1}{R_2 - R_1} C_m \right) B_0(rr_{0m}) \, (1 - e^{-b_{0m}t})/r_{0m}^2$$

where $A_m = \int_{R_1}^{R_2} B_1(rr_{1m})\,dr$, $B_m = \int_{R_1}^{R_2} rB_0(rr_{0m})\,dr$ and $C_m = \int_{R_1}^{R_2} B_0(rr_{0m})\,dr$ and the solution corresponding to the steady case appear as a limiting case when $t \Rightarrow \infty$.

Remark 2. If the motion is steady and the velocities of the two cylinders are same, i.e., $\Omega_1 = \Omega_2 = \Omega$ and $U_1 = U_2 = U$ and the pressure gradient which acts on the fluid parallel to the axis of the cylinders $a = 0$, then $\omega(r) = \Omega$ and $u(r) = U$ i.e., all fluid particles move with the same velocities Ω and U.

5. The Limiting Case $R_1 \Rightarrow 0$

Taking the limit of the Eq. (8) when $R_1 \Rightarrow 0$ we find the eigenfunctions corresponding to a helical flow through an infinite circular cylinder. Assuming that at the moment $t = 0$ the cylinder began to rotate about its axis with the angular velocity $\Omega(t)$ and to slide in the direction of the same axis with the velocity $U(t)$, and making the notations[2]

$$\overline{\omega}(r, t) = r\omega(r, t) - r\Omega(t) \qquad \text{and} \qquad \overline{u}(r, t) = u(r, t) - U(t) \tag{12}$$

we attain to the same Eq. (7.1) with

$$v_n(r, 0) = V_n(r), \qquad r \in [0, R); \qquad n = 0, 1$$

$$v_n(R, t) = 0, \qquad |v_n(0, t)| < \infty, \qquad t \geqq 0$$

and $a_0(t) = a(t)/\varrho - U'(t)$, $a_1(r, t) = -r\Omega'(t)$, $V_0(r) = -U$ and $V_1(r) = -r\Omega$.

For the present class of problems one obtains

$$\begin{aligned}\omega(r, t) = \Omega(t) + \frac{2\Omega}{r} \sum_{m=1}^{\infty} \frac{J_1(rr_{1m})}{r_{1m} J_1'(Rr_{1m})}\, e^{-b_{1m}t} \\ + 2 \sum_{m=1}^{\infty} \frac{J_1(rr_{1m})\, e^{-b_{1m}t}}{r_{1m}(1 + \alpha r_{1m}^2)\, J_1'(Rr_{1m})} \int_0^t \Omega'(t)\, e^{b_{1m}t}\,dt\end{aligned} \tag{13}$$

and

$$\begin{aligned}u(r, t) = U(t) + \frac{2U}{R} \sum_{m=1}^{\infty} \frac{J_0(rr_{0m})}{r_{0m} J_0'(Rr_{0m})}\, e^{-b_{0m}t} \\ + \frac{2}{R} \sum_{m=1}^{\infty} \frac{J_0(rr_{0m})\, e^{-b_{0m}t}}{r_{0m}(1 + \alpha r_{0m}^2)\, J_0'(Rr_{0m})} \int_0^t \left[U'(t) - \frac{a(t)}{\varrho}\right] e^{b_{0m}t}\,dt\end{aligned} \tag{14}$$

where r_{0m} and r_{1m} are roots of the equations $J_0(Rr) = 0$ and $J_1(Rr) = 0$, respectively.

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[2] These changes present an advantage over those resulting from (5), (6) for $U_1 = \Omega_1 = R_1 = 0$, $\Omega_2 = \Omega$, $U_2 = U$ and $R_2 = R$ the radius of the cylinder.

The solutions corresponding to the steady case

$$\omega(r) = \Omega \qquad \text{and} \qquad u(r) = U + \frac{a}{4\mu_0}(R^2 - r^2) \tag{15}$$

appear again making $\Omega(t) = \Omega$ and $t \Rightarrow \infty$ in these last relations.

Remark 3. Eqs. (13) and (14) are identical to (3.3) and (3.4) of Ref. [9] where the finite Hankel transforms had been used. They are in accordance with the results from [10] and [11].

The frictional couple per unit length of the inner or outer cylinder (in the first case) or of the cylinder $r = R$ (in the second case) (see [10]) can be easy estimate having in mind the expressions of $T_{r\theta} = r(\mu_0 + \alpha_1 \partial_t) \partial_r \omega$.

Remark 4. From the above results one can see that in steady flows the velocity field predict by the theory of fluids of second grade is identical with that from the Navier-Stokes theory, though of course the surface tractions that must be applied in order to produce the flow will vary according to the values of α_1 and α_2.

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