Acta Mechanica 65, 263–279 (1986)

ACTA MECHANICA
© by Springer-Verlag 1986

Plane Deformations of Nets with Bending Stiffness

By

W.-B. Wang and **A. C. Pipkin**, Providence, Rhode Island

(*Received December 8, 1985*)

Summary

The theory of nets with bending stiffness is meant to model some aspects of the mechanical behavior of cloth and cable networks. We use this theory to study the effect of small bending stiffness in plane deformations. It is shown that the theory of perfectly flexible networks gives an adequate approximation except at places where that theory predicts discontinuous changes in fiber directions, and at boundaries along fibers. In such regions singular perturbation methods can be used to analyze the stress and deformation.

1. Introduction

In a previous paper [1] we have formulated a theory of inextensible networks with bending stiffness, which is intended to model some aspects of the behavior of cloth in finite deformations. The theory of inextensible networks [2], [3], [4] was modified by attributing a bending stiffness to each fiber in the network. In the resulting theory the bending couples play no direct role. Their presence produces a modification in the stress-deformation relation for the material and in the boundary conditions needed in order to set a problem.

In the present paper we study the effect of small bending stiffness on plane deformations of inextensible networks. In plane deformations, boundary value problems are relatively easy to solve within the theory of perfectly flexible networks, so such deformations afford a convenient class for a first approach to the study of nets with bending stiffness.

Bending couples have the effect of introducing higher derivatives of the deformation into the stress-deformation relations [1], and the equilibrium equation is then two orders higher than it is in the theory of perfectly flexible networks. The derivatives of highest order are multiplied by a small stiffness coefficient. The higher-order derivatives in the equation make it necessary to pose additional boundary conditions, beyond the usual displacement or traction conditions. These additional conditions generally are not satisfied by solutions obtained from the theory of prefectly flexible networks. As we show in the present paper, the main effect of bending stiffness is to produce boundary layers at the edges of the sheet. The solution at interior points is adequately represented by the zero-bending-stiffness solution, provided that the latter is sufficiently smooth.

In Section 2 we outline Rivlin's [2] kinematic results concerning plane deformations, in vector notation [4]. The equilibrium equation and stress-deformation relations are given in Section 3. Boundary conditions are discussed at some length in Section 4. We restrict attention to cases in which boundary couples are not specified, except on pinned boundaries where a certain zero-couple condition is appropriate. On traction boundaries that cut across both families of fibers, the presence of boundary couples affects the prescribed traction boundary condition but no separate couple condition is required. The boundary conditions are discussed further in Section 5, in connection with a simple kinematically determinate example.

In Section 6 we find the general solution of the equilibrium equation by introducing a stress potential and then finding its form in terms of the deformation, following the pattern laid down by Rivlin [2] and used by Pipkin [4]. In Section 7 we use these results to set up a pair of differential-integral equations that can be used to determine the deformation in pure traction boundary-value problems. When the bending stiffness is zero, these become integral equations that can be solved by series or iteration methods [5], [6].

In Sections 8 and 9 we discuss examples in which the deformation is partly determined kinematically, so that only one differential equation needs to be solved. The examples in Section 8 show how boundary layers arise at boundaries where the zero-stiffness solution does not satisfy all of the boundary conditions. In Section 9 we discuss an example that shows how an interior discontinuity in the fiber direction, which is admissible in the zero-bending-stiffness theory, is smoothed out when the bending stiffness is not zero. There is again a boundary-layer type of effect, but with a transition layer on the interior of the sheet.

Ordinary and singular perturbation methods for the general traction boundary value problem are discussed briefly in Section 10.

2. Kinematics

We consider a plane sheet formed from two families of fibers that initially lie parallel to the x and y axes of a Cartesian coordinate system. We treat the sheet as a continuum, so that every line $x =$ constant or $y =$ constant in the initial domain D is regarded as a fiber. In a deformation, the particles of both fibers that initially passed through the point (x, y) go to the place $\boldsymbol{r}(x, y)$ in the same plane. The derivatives of $\boldsymbol{r}$ are denoted by

$$\boldsymbol{a} = \boldsymbol{r}_x \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{r}_y . \tag{2.1}$$

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are tangential to the deformed fibers, and because the fibers are assumed to be inextensible they are unit vectors:

$$\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{b} = 1. \tag{2.2}$$

By using (2.2) and the integrability condition for (2.1), Rivlin [2] showed that $\boldsymbol{a}$ and $\boldsymbol{b}$ are functions of only one variable each,

$$\boldsymbol{a} = \boldsymbol{a}(x), \qquad \boldsymbol{b} = \boldsymbol{b}(y), \tag{2.3}$$

and the deformation then has the form

$$\boldsymbol{r} = \boldsymbol{f}(x) + \boldsymbol{g}(y), \tag{2.4}$$

where $\boldsymbol{f}' = \boldsymbol{a}$ and $\boldsymbol{g}' = \boldsymbol{b}$.

Let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be unit vectors in the coordinate directions, and let $\boldsymbol{u}(\theta)$ be the unit vector defined by

$$\boldsymbol{u}(\theta) = \boldsymbol{i} \cos\theta + \boldsymbol{j} \sin\theta. \tag{2.5}$$

We can represent $\boldsymbol{a}$ and $\boldsymbol{b}$ in terms of their angles α and β by

$$\boldsymbol{a} = \boldsymbol{u}[\alpha(x)], \qquad \boldsymbol{b} = \boldsymbol{u}[\beta(y)]. \tag{2.6}$$

Then the curvature vectors for the deformed fibers are

$$\boldsymbol{a}'(x) = \boldsymbol{k} \times \boldsymbol{a}\alpha'(x), \qquad \boldsymbol{b}'(y) = \boldsymbol{k} \times \boldsymbol{b}\beta'(y). \tag{2.7}$$

The local distortion of the sheet is measured by the angle of shear γ defined by

$$\sin\gamma = \boldsymbol{a} \cdot \boldsymbol{b} = \cos\,[\beta(y) - \alpha(x)]. \tag{2.8}$$

The deformed area per unit undeformed area of an element of the sheet, which we denote by J, is given in various forms by

$$J = \cos\gamma = \boldsymbol{k} \cdot \boldsymbol{a} \times \boldsymbol{b} = \sin\,[\beta(y) - \alpha(x)]. \tag{2.9}$$

We restrict attention to deformations for which $J > 0$ everywhere. The possibility of folding [4], which would make $J < 0$, is ruled out because we intend to assign a non-zero bending stiffness to the sheet.

3. Stress and Equilibrium

Let $\boldsymbol{t}$ be the force per unit initial length exerted from right to left across a directed arc $d\boldsymbol{r} = \boldsymbol{a}dx + \boldsymbol{b}dy$, and let ds be the initial length of this arc. Then [1]

$$\boldsymbol{t}ds = \boldsymbol{t}_a dy - \boldsymbol{t}_b dx, \tag{3.1}$$

where the stress vectors $\boldsymbol{t}_a$ and $\boldsymbol{t}_b$ are independent of the direction of the arc. Equilibrium in the absence of body forces requires that [1]

$$\boldsymbol{t}_{a,x} + \boldsymbol{t}_{b,y} = \boldsymbol{0}. \tag{3.2}$$

Let $\boldsymbol{c}ds$ be the bending couple exerted across such an arc. Then

$$\boldsymbol{c}ds = \boldsymbol{c}_a dy - \boldsymbol{c}_b dx, \tag{3.3}$$

where the couple-stress vectors $\boldsymbol{c}_a$ and $\boldsymbol{c}_b$ are independent of the direction of the arc. Rotational equilibrium requires that [1]

$$(\boldsymbol{r} \times \boldsymbol{t}_a + \boldsymbol{c}_a)_x + (\boldsymbol{r} \times \boldsymbol{t}_b + \boldsymbol{c}_b)_y = \boldsymbol{0}. \tag{3.4}$$

We postulate that the couples $\boldsymbol{c}_a$ and $\boldsymbol{c}_b$ are proportional to the curvatures of the $\boldsymbol{a}$-fiber and $\boldsymbol{b}$-fiber, respectively:

$$\boldsymbol{c}_a = \Gamma \boldsymbol{a} \times \boldsymbol{a}_x, \qquad \boldsymbol{c}_b = \Gamma \boldsymbol{b} \times \boldsymbol{b}_y. \tag{3.5}$$

The bending stiffness coefficient Γ is a positive constant.

We require that the stress vectors $\boldsymbol{t}_a$ and $\boldsymbol{t}_b$ be of such forms that the rotational equilibrium equation (3.4) is satisfied identically whenever the translational equilibrium equation (3.2) is satisfied. This requirement leads to the result [1] that $\boldsymbol{t}_a$ and $\boldsymbol{t}_b$ must have the forms

$$\begin{aligned} \boldsymbol{t}_a &= T_a \boldsymbol{a} + S\boldsymbol{b} - \Gamma \boldsymbol{a}_{xx}, \\ \boldsymbol{t}_b &= T_b \boldsymbol{b} + S\boldsymbol{a} - \Gamma \boldsymbol{b}_{yy}. \end{aligned} \tag{3.6}$$

By using a work-energy relation we have shown [1] that T_a and T_b, which we call *fiber tensions,* are reactions to the constraints (2.2) of fiber inextensibility. They are primitive unknowns in any given problem. The shearing stress component S is a specified function of the angle of shear γ defined by (2.8). It is mathematically convenient and qualitatively reasonable to take S to be of the form [4]

$$S = G \tan \gamma = G\boldsymbol{a} \cdot \boldsymbol{b}/J, \tag{3.7}$$

where the shear modulus G is a positive constant. The terms involving Γ in (3.6) are analogous to the shearing stresses on the cross-section of a bent beam, which are necessarily present when the bending couple varies along the length of the beam. Here these reactions must be taken into account explicitly since we take the translational equilibrium equation (3.2) as the basic equation to be satisfied.

4. Boundary Conditions

Let C be the boundary of the region D, given parametrically in terms of arc length s by $x = x(s)$, $y = y(s)$. The derivatives $x'(s)$ and $y'(s)$ are the direction cosines of the tangent to C. Let X, Y, and Z denote the parts of C on which the following conditions are satisfied:

$$X\colon x' = 0; \qquad Y\colon y' = 0; \qquad Z\colon x'y' \neq 0. \tag{4.1}$$

That is, X consists of the parts of C that lie along fibers $x =$ constant, Y the parts that lie along fibers $y =$ constant, and Z the parts that cut across both families of fibers. Boundary conditions on Z are essentially different from those on X and Y.

Let C be divided into three parts C_p, C_c, and C_t that we call *pinned, clamped,* and *free* respectively. The deformation is specified on the pinned and clamped parts:

$$\boldsymbol{r}[x(s), y(s)] = \boldsymbol{r}_0(s) \quad \text{on} \quad C_p \quad \text{and} \quad C_c. \tag{4.2}$$

On ZC_p, the intersection of Z with C_p, the boundary condition (4.2) determines $\boldsymbol{a}$ and $\boldsymbol{b}$ uniquely in plane deformations with $J > 0$ [1], [2], [4]. On XC_p (or XC_c), (4.2) determines $\boldsymbol{b}$, and in fact kinematic admissibility of the boundary data on such a boundary is usually assured by specifying $\boldsymbol{b}$ and then using $\boldsymbol{r}_0' = \boldsymbol{b}y'$ to determine $\boldsymbol{r}_0$. Similarly, on YC_p or YC_c, (4.2) determines $\boldsymbol{a}$ but not $\boldsymbol{b}$.

We say that the boundary is *clamped* if the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ are specified on it, to the extent that they are not already determined by (4.2), and *pinned* if this extra information is not given. With the restriction to plane deformations, this distinction is relevant only on X and Y. A line $x = x_0$ in X is clamped if we specify the value of $\boldsymbol{a}(x_0)$ on it, or equivalently specify $\alpha(x_0)$, the angle of $\boldsymbol{a}$. Similarly, a line $y = y_0$ in Y is clamped if we specify the value of $\beta(y_0)$:

$$\alpha(x_0) = \alpha_0 \quad \text{on} \quad XC_c, \qquad \beta(y_0) = \beta_0 \quad \text{on} \quad YC_c. \tag{4.3}$$

On the free part C_t, neither $\boldsymbol{r}$, $\boldsymbol{a}$, nor $\boldsymbol{b}$ is specified.

We have used the minimum energy principle [1] to deduce the forms that traction and couple boundary conditions take when there is no energy associated with the boundary couples. In a certain sense these conditions refer to cases in which no couple is deliberately applied to the boundary. On a clamped part of the boundary no traction or couple condition can be specified. On a pinned part we can specify that there are no couples that would do work on a virtual displacement that leaves the boundary pinned. In plane deformations this condition is to a large extent satisfied identically because the couple vectors $\boldsymbol{c}_a$ and $\boldsymbol{c}_b$ are perpendicular to the plane of deformation. On Z there are no plane virtual displacements that leave the boundary fixed, so this kind of zero-couple condition is relevant only on X and Y. The same kind of condition can be specified on C_t:

$$\boldsymbol{c}_a = \boldsymbol{0} \quad \text{on} \quad X(C_p + C_t), \qquad \boldsymbol{c}_b = \boldsymbol{0} \quad \text{on} \quad Y(C_p + C_t). \tag{4.4}$$

With (2.7) and (3.5) we can put these conditions into the forms

$$\alpha'(x_0) = 0 \quad \text{on} \quad X(C_p + C_t), \qquad \beta'(y_0) = 0 \quad \text{on} \quad Y(C_p + C_t), \tag{4.5}$$

where x_0 and y_0 are boundary values in X and Y, respectively.

Traction boundary conditions can be specified on C_t. Let $\boldsymbol{T}(s)$ be the prescribed force per unit initial length, applied as a dead load along C_t. Of course, we specify that $\boldsymbol{T}$ has no component in the direction normal to the plane of deformation. Then on XC_t and YC_t the traction boundary conditions have the forms [1]

$$\boldsymbol{t}_a y' = \boldsymbol{T} + (F_b \boldsymbol{b})_y \quad \text{on} \quad XC_t, \tag{4.6.1}$$

$$-\boldsymbol{t}_b x' = \boldsymbol{T} + (F_a \boldsymbol{a})_x \quad \text{on} \quad YC_t. \tag{4.6.2}$$

These conditions involve unknown functions $F_b(y)$ and $F_a(x)$ that represent finite forces supported by the fibers that lie along the boundaries. Such singularities in boundary fibers are a well-known feature of solutions in theories that use the idealization of fiber inextensibility.

The traction condition on ZC_t is of a form that is far from obvious. In a virtual motion $\delta \boldsymbol{r}$, part of the couple distribution on the boundary performs work only when the boundary moves, so that $\delta \boldsymbol{r}'(s)$ is not zero. This part of the work is of the form $\boldsymbol{P} \cdot \delta \boldsymbol{r}'$, per unit initial length, where in plane deformations $\boldsymbol{P}$ has the form [1]

$$\boldsymbol{P} = (\Gamma/J)\,[\boldsymbol{a}(x)\,\beta'(y)\,(x'/y') + \boldsymbol{b}(y)\,\alpha'(x)\,(y'/x')]. \tag{4.7}$$

An integration by parts puts the work into the form of the work of a traction $-\boldsymbol{P}'$ on a displacement $\delta \boldsymbol{r}$. Then the traction boundary condition takes the form

$$\boldsymbol{t}_a y'(s) - \boldsymbol{t}_b x'(s) - \boldsymbol{P}'(s) = \boldsymbol{T}(s) \quad \text{on} \quad ZC_t. \tag{4.8}$$

Thus even though no couple distribution is specified on ZC_t there will generally be non-zero couples on such a boundary, but the couple distribution is of a type that is indistinguishable from a traction distribution. This is an effect that is familiar from classical plate theory (Love [8, Sec. 297]).

At corners on the boundary and at points where the type of boundary (X, Y or Z) changes, and at points where the applied tractions include a point force, traction boundary conditions specify a balance among certain finite forces. In a notation to cover all cases, the condition has the form

$$\boldsymbol{F} + \Delta(F_a \boldsymbol{a}) + \Delta(F_b \boldsymbol{b}) + \Delta \boldsymbol{P} = \boldsymbol{0}. \tag{4.9}$$

Here $\boldsymbol{F}$ is the applied point force. If an $\boldsymbol{a}$-line passes through the point, $\Delta(F_a \boldsymbol{a})$ is the sum of the two finite forces exerted by the singular fiber on the point; if the $\boldsymbol{a}$-line does not extend on both sides of the point, we conventionally set $F_a = 0$ on the side where there is no $\boldsymbol{a}$-line. The term $\Delta(F_b \boldsymbol{b})$ similarly accounts for the forces exerted by a singular $\boldsymbol{b}$-line on the point in question. The term $\Delta \boldsymbol{P}$ is the difference between the values of $\boldsymbol{P}$ on the two sides of the point; in evaluating this term we conventionally set $\boldsymbol{P} = \boldsymbol{0}$ on X and Y boundaries.

5. A Kinematically Determinate Example

As an exercise in the use of the boundary conditions we consider the deformation of a sheet bounded by the curves

$$X\colon x = 0; \qquad Y\colon y = H; \qquad Z\colon y = f(x), \tag{5.1}$$

where $f(x)$ increases monotonically from $f(0) = 0$ to $f(L) = H$. We suppose that the edge $x = 0$ is clamped in its initial position. Then $\boldsymbol{b}(y) = \boldsymbol{j}$ and $\alpha(0) = 0$, and since $\boldsymbol{b}$ is independent of x we then know $\boldsymbol{b}$ everywhere. The edge $y = H$ is drawn up onto a circular arc of radius R, in such a way that

$$\alpha(x) = x/R, \tag{5.2}$$

and pinned there. This determines $\boldsymbol{a} = \boldsymbol{u}(\alpha)$ everywhere. The pinning condition means that β is not specified but $\beta'(H) = 0$, and this is satisfied identically since $\beta = \pi/2$ everywhere. With $\boldsymbol{a}$ and $\boldsymbol{b}$ known, the deformation is determined by integration:

$$\boldsymbol{r}(x, y) = y\boldsymbol{j} - R\boldsymbol{k} \times [\boldsymbol{u}(\alpha) - \boldsymbol{i}]. \tag{5.3}$$

In order to determine the stress distribution it is necessary to specify the traction $\boldsymbol{T}(s)$ along the edge Z. In fact the plane deformation will be unstable unless the tractions on Z are rather large and in such directions as to keep the sheet stretched out. For our present purpose we suppose that some such traction distribution has been specified.

The couple distribution on Z can be computed immediately because the deformation is already known. From (3.5), $\boldsymbol{c}_a = \boldsymbol{k}\Gamma\alpha'$ and $\boldsymbol{c}_b = \boldsymbol{0}$, so the couple per unit initial length on the boundary is

$$\boldsymbol{c} = \boldsymbol{c}_a y' - \boldsymbol{c}_b x' = \boldsymbol{k}\Gamma\alpha'(x)\, y'(s). \tag{5.4}$$

However, it is not necessary to take any positive action to supply this distribution of couples. It is equivalent to a certain traction distribution, whose form is easy to compute although not intuitively obvious. From (4.7), $\boldsymbol{P}$ is of the form

$$\boldsymbol{P} = P(x)\boldsymbol{j}, \qquad P(x) = \Gamma\alpha'(x)\, f'(x)/\cos\alpha(x). \tag{5.5}$$

Then the traction boundary condition (4.8) takes the form

$$\boldsymbol{t}_a f'(x) - \boldsymbol{t}_b = \boldsymbol{T}/x'(s) + \boldsymbol{j}P'(x), \tag{5.6}$$

where we have divided by x' and have used $y'/x' = f'(x)$. Thus, the effect of the couple distribution on the boundary is to alter the apparent traction distribution.

To determine the effect of bending stiffness on the stress field we need not exhibit the complete solution. Let $\boldsymbol{t}_a^0$ and $\boldsymbol{t}_b^0$ be the stress vectors when $\Gamma = 0$, involving fiber tensions T_a^0 and T_b^0, and let

$$\Delta\boldsymbol{t}_a = \boldsymbol{t}_a - \boldsymbol{t}_a^0, \qquad \Delta\boldsymbol{t}_b = \boldsymbol{t}_b - \boldsymbol{t}_b^0. \tag{5.7}$$

Then $\Delta\boldsymbol{t}_a$ and $\Delta\boldsymbol{t}_b$ can be regarded as the stresses due to bending stiffness alone, in a sheet with no shear resistance and with no traction along the edge $y = f(x)$. On that edge they satisfy

$$\Delta\boldsymbol{t}_a f'(x) - \Delta\boldsymbol{t}_b = \boldsymbol{j}P'(x). \tag{5.8}$$

This is satisfied if

$$\Delta \boldsymbol{t}_a = \boldsymbol{0} \quad \text{and} \quad \Delta \boldsymbol{t}_b = -\boldsymbol{j} P'(x). \tag{5.9}$$

These stresses satisfy the equilibrium equation since $\Delta \boldsymbol{t}_b$ is independent of y. They are consistent with the stress-deformation relations (3.6) if

$$\Delta T_a = -\Gamma \alpha'^2 \quad \text{and} \quad \Delta T_b = -P'(x). \tag{5.10}$$

The change in T_a is merely a formal reallocation of fiber tension from the term $T_a{}^0 \boldsymbol{a}$ in $\boldsymbol{t}_a{}^0$ to the term $-\Gamma \boldsymbol{a}_{xx}$ in $\boldsymbol{t}_a$. For P' to be positive it is sufficient that $f' > 0$ and $f'' > 0$. If that is the case, the stress due to bending stiffness alone is a compressive stress $-P'$, constant along each $\boldsymbol{b}$-line.

This compressive stress in $\boldsymbol{b}$-lines means that the top edge $y = H$ is pressing against the circular support to which it is attached. From (4.6.2), assuming that $\Delta F_a = 0$, the reaction of the support on the sheet is

$$\Delta \boldsymbol{T} = \Delta \boldsymbol{t}_b = -\boldsymbol{j} P'(x). \tag{5.11}$$

In addition to this, there is a point force at the corner $x = L$, $y = H$. From (4.9), with $\Delta \boldsymbol{P} = -\boldsymbol{P}(L)$ since $\boldsymbol{P} = \boldsymbol{0}$ on $y = H$, the force at the corner is

$$\Delta \boldsymbol{F} = \boldsymbol{j} P(L), \tag{5.12}$$

with $P(L) > 0$. The corner is pulled into contact with the support by this force. The total force exerted on the sheet by the support from x to L is

$$\Delta \boldsymbol{F} + \int_x^L \Delta \boldsymbol{T} \, dx = \boldsymbol{j} P(x). \tag{5.13}$$

On the edge $x = 0$ there is a couple $\boldsymbol{c} = -\boldsymbol{k} \Gamma \alpha'$ per unit length. This couple is supplied by the gripping device that enforces the clamping condition there. Now, we would have deduced exactly the same deformation if we had specified that the edge $x = 0$ were pinned rather than clamped, but then the zero-couple condition $\alpha'(0) = 0$ would not be satisfied. The contradiction means that it is false to assume that the deformation is plane in such a case; if the edge $x = 0$ is merely pinned, and not clamped, the sheet cannot remain in its original plane.

There is a similar contradiction if the edge Z has a vertical part $x = L$. To look at this in more detail, let us suppose that $y = f(x)$ lies along $y = 0$ nearly all the way from $x = 0$ to $x = L$, and then rises nearly vertically (f' large) near $x = L$. From (5.5) we see that $P = 0$ along the horizontal part and that P is very large, proportional to f', along the nearly vertical part. Then the force (5.13) that the support must exert on the upper edge of the sheet in order to maintain the deformation approaches infinity as f' increases. In the limit in which the sheet has a vertical edge $x = L$, no finite force can maintain the sheet in the plane state that has been assumed.

6. Stress Potential

The equilibrium Eq. (3.2) is satisfied if $\boldsymbol{t}_a$ and $\boldsymbol{t}_b$ are related to a stress potential $\boldsymbol{F}$ by

$$\boldsymbol{t}_a = \boldsymbol{F}_y, \qquad \boldsymbol{t}_b = -\boldsymbol{F}_x, \qquad d\boldsymbol{F} = \boldsymbol{t}_a\, dy - \boldsymbol{t}_b\, dx. \tag{6.1.1, 2, 3}$$

To determine the relation of $\boldsymbol{F}$ to the deformation, we use the stress-deformation relations (3.6) in (6.1) and then take the inner products of (6.1.1) with $\boldsymbol{k} \times \boldsymbol{a}$ and (6.1.2) with $\boldsymbol{k} \times \boldsymbol{b}$. This yields

$$\boldsymbol{k} \times \boldsymbol{a} \cdot \boldsymbol{F}_y = G\boldsymbol{a} \cdot \boldsymbol{b} - \Gamma\boldsymbol{k} \cdot (\boldsymbol{a} \times \boldsymbol{a}_x)_x, \tag{6.2.1}$$

$$\boldsymbol{k} \times \boldsymbol{b} \cdot \boldsymbol{F}_x = G\boldsymbol{a} \cdot \boldsymbol{b} + \Gamma\boldsymbol{k} \cdot (\boldsymbol{b} \times \boldsymbol{b}_y)_y. \tag{6.2.2}$$

We have also used (2.9) and the special constitutive Eq. (3.7) for S. We now use (2.7) to simplify the terms involving Γ. Then, recalling that $\boldsymbol{a} = \boldsymbol{a}(x)$ and $\boldsymbol{b} = \boldsymbol{b}(y)$, and using (2.1), we can integrate (6.2.1) with respect to y and (6.2.2) with respect to x, to obtain

$$\begin{aligned} \boldsymbol{k} \times \boldsymbol{a} \cdot \boldsymbol{F} &= G\boldsymbol{a} \cdot \boldsymbol{r} - \Gamma y \alpha''(x) + M(x), \\ \boldsymbol{k} \times \boldsymbol{b} \cdot \boldsymbol{F} &= G\boldsymbol{r} \cdot \boldsymbol{b} + \Gamma \times \beta''(y) + N(y). \end{aligned} \tag{6.3}$$

Here M and N are as yet undetermined.

The expressions for $\boldsymbol{F}$ and $\boldsymbol{k} \times \boldsymbol{r}$ as linear combinations of the base vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are

$$\boldsymbol{F} = \boldsymbol{a}(\boldsymbol{F} \cdot \boldsymbol{b} \times \boldsymbol{k}/J) + \boldsymbol{b}(\boldsymbol{F} \cdot \boldsymbol{k} \times \boldsymbol{a}/J) \tag{6.4}$$

and

$$\boldsymbol{k} \times \boldsymbol{r} = -\boldsymbol{a}(\boldsymbol{r} \cdot \boldsymbol{b}/J) + \boldsymbol{b}(\boldsymbol{r} \cdot \boldsymbol{a}/J). \tag{6.5}$$

Here J is defined by (2.9). By using (6.3) in (6.4) and then using (6.5), we obtain the general solution for $\boldsymbol{F}$:

$$\begin{aligned} \boldsymbol{F} = {} & G\boldsymbol{k} \times \boldsymbol{r} + (1/J)\,[M(x)\,\boldsymbol{b}(y) - N(y)\,\boldsymbol{a}(x)] \\ & - (\Gamma/J)\,[x\boldsymbol{a}(x)\,\beta''(y) + y\boldsymbol{b}(y)\,\alpha''(x)]. \end{aligned} \tag{6.6}$$

When $\Gamma = 0$, this reduces to the form previously [4] found for perfectly flexible networks.

In cases in which the deformation is kinematically determinate, all of the quantities in (6.6) are known except M and N. Traction boundary conditions must be used to determine these functions, and for this purpose it is necessary that at least one end of each fiber be on the boundary C_t. In dealing with the boundary condition it is convenient to use an auxiliary function $\boldsymbol{F}_0(s)$ defined on C by

$$\boldsymbol{F}_0 = \boldsymbol{F} - \begin{cases} F_b \boldsymbol{b} y' & \text{on } X, \\ F_a \boldsymbol{a} x' & \text{on } Y, \\ \boldsymbol{0} & \text{on } Z. \end{cases} \tag{6.7}$$

Then the boundary conditions (4.6) and (4.8) can be written as

$$\boldsymbol{F}_0' = \boldsymbol{T} + \boldsymbol{P}', \tag{6.8}$$

with the understanding that $\boldsymbol{P} = \boldsymbol{0}$ on X and Y. We note that at discontinuities in $\boldsymbol{P}$, such as those that may occur in passing from an X interval to a Z interval, $d\boldsymbol{P} = \boldsymbol{P}'\,ds$ is to be interpreted as a finite difference. From (6.8),

$$\boldsymbol{F}_0 = \boldsymbol{P} + \int_0^s \boldsymbol{T}\,ds. \tag{6.9}$$

Thus when the deformation is known, so that $\boldsymbol{P}$ is known, $\boldsymbol{F}_0$ can be evaluated by integrating the specified boundary tractions. The origin $s = 0$ is an arbitrary point on C_t.

The purpose of introducing $\boldsymbol{F}_0$ is to eliminate the fiber tensions F_a and F_b that appear in the boundary conditions (4.6). From (6.7) we see that

$$\boldsymbol{k} \times \boldsymbol{a} \cdot (\boldsymbol{F} - \boldsymbol{F}_0) = 0 \quad \text{on } Y, \qquad \boldsymbol{k} \times \boldsymbol{b} \cdot (\boldsymbol{F} - \boldsymbol{F}_0) = 0 \quad \text{on } X, \tag{6.10}$$

and $\boldsymbol{F} = \boldsymbol{F}_0$ on Z. Then if (6.3) is evaluated at a boundary point, $\boldsymbol{F}$ can be replaced by $\boldsymbol{F}_0$, and we obtain

$$M(x) = \boldsymbol{k} \times \boldsymbol{a} \cdot \boldsymbol{F}_0 - G\boldsymbol{a} \cdot \boldsymbol{r} + \Gamma y \alpha''(x) \qquad \text{on } Y \text{ and } Z, \tag{6.11.1}$$

$$N(y) = \boldsymbol{k} \times \boldsymbol{b} \cdot \boldsymbol{F}_0 - G\boldsymbol{r} \cdot \boldsymbol{b} - \Gamma \times \beta''(y) \quad \text{on } X \text{ and } Z. \tag{6.11.2}$$

With M and N determined, $\boldsymbol{F}$ is known completely. The stress at interior points is found by using (6.1), and the forces F_a and F_b in boundary fibers are found by using (6.7).

The relations (6.11) remain valid even if the integral in (6.9) includes arcs on which $\boldsymbol{T}$ has not been specified. On such arcs (6.11) does not determine M and N because $\boldsymbol{F}_0$ is not known, but relations derived from (6.11) remain valid nonetheless.

7. Traction Boundary Value Problems

In the present section we consider pure traction boundary value problems, for which $C = C_t$, and show how to form equations that govern $\alpha(x)$ and $\beta(y)$. To simplify the notation we confine attention to cases in which no line $x =$ constant or $y=$ constant intersects C more than twice, so that each such fiber consists of one connected segment. We suppose that the boundary can be written as

$$y = y_+(x) \quad \text{and} \quad y = y_-(x), \quad \text{with} \quad y_+ \geqq y_-, \tag{7.1}$$

and also as

$$x = x_+(y) \quad \text{and} \quad x = x_-(y), \quad \text{with} \quad x_+ \geqq x_-, \tag{7.2}$$

plus lines $x=$ constant in the first case and lines $y=$ constant in the second.

The difference between the values of a function $f(x, y)$ at the two ends of a fiber is denoted by

$$\Delta f(x) = f(x, y_+(x)) - f(x, y_-(x)) \tag{7.3}$$

or

$$\Delta f(y) = f(x_+(y), y) - f(x_-(y), y), \tag{7.4}$$

and it is essential to show the argument of Δf in order to specify which difference is intended. In this notation $\Delta y(x)$ and $\Delta x(y)$ are the lengths of the fibers $x = $ constant and $y=$ constant, respectively. We use the same notation for differences of the values of the function $\boldsymbol{F}_0$, given by (6.9), at the two ends of a fiber.

The expression (6.11.1) gives two equations for $M(x)$, one for each end of the fiber $x=$ constant By subtracting one from the other we obtain

$$\Gamma \Delta y(x)\, \alpha''(x) = G\boldsymbol{a} \cdot \Delta \boldsymbol{r}(x) - \boldsymbol{k} \times \boldsymbol{a} \cdot \Delta \boldsymbol{F}_0(x). \tag{7.5}$$

From (6.11.2) we similarly obtain

$$\Gamma \Delta x(y)\, \beta''(y) = -G\boldsymbol{b} \cdot \Delta \boldsymbol{r}(y) + \boldsymbol{k} \times \boldsymbol{b} \cdot \Delta \boldsymbol{F}_0(y). \tag{7.6}$$

The differences $\Delta \boldsymbol{F}_0$ are, f·om (6 9),

$$\Delta \boldsymbol{F}_0(x) = \Delta \boldsymbol{P}(x) + \boldsymbol{F}_r(x), \qquad \Delta \boldsymbol{F}_0(y) = \Delta \boldsymbol{P}(y) - \boldsymbol{F}_u(y), \tag{7.7}$$

where $\boldsymbol{F}_r(x)$ is the total force on the boundary to the right of the line $x=$ constant and $\boldsymbol{F}_u(y)$ is the total force on the boundary above the line $y=$ constant. When (7.7) is used in (7.5) and (7.6) the terms involving $\Delta \boldsymbol{P}$ can be simplified. From (4.7) we find that

$$\boldsymbol{k} \times \boldsymbol{a} \cdot \boldsymbol{P} = \Gamma \alpha'(x)\,(y'/x'), \qquad \boldsymbol{k} \times \boldsymbol{b} \cdot \boldsymbol{P} = -\Gamma \beta'(y)\,(x'/y'), \tag{7.8}$$

where we have used (2.9). Now,

$$y'(s)/x'(s) = y_\pm'(x) \quad \text{and} \quad x'(s)/y'(s) = x_\pm'(y), \tag{7.9}$$

the subscript depending on the point at which $\boldsymbol{P}$ is being evaluated. Then

$$\boldsymbol{k} \times \boldsymbol{a} \cdot \Delta \boldsymbol{P}(x) = \Gamma \alpha'(x)\, \Delta y'(x), \quad \boldsymbol{k} \times \boldsymbol{b} \cdot \Delta \boldsymbol{P}(y) = -\Gamma \beta'(y)\, \Delta x'(y). \tag{7.10}$$

By using (7.7) in (7.5) and (7.6), then using (7.10), and finally combining terms that involve Γ, we obtain

$$\Gamma[\Delta y(x)\,\alpha'(x)]' = -\boldsymbol{k} \times \boldsymbol{a}(x) \cdot [\boldsymbol{F}_r(x) + G\Delta \boldsymbol{r}(x) \times \boldsymbol{k}] \tag{7.11}$$

and

$$\Gamma[\Delta x(y)\,\beta'(y)]' = \boldsymbol{k} \times \boldsymbol{b}(y) \cdot [\boldsymbol{F}_u(y) + G\boldsymbol{k} \times \Delta \boldsymbol{r}(y)]. \tag{7.12}$$

The differences $\Delta \boldsymbol{r}$ can be written as integrals:

$$\Delta \boldsymbol{r}(x) = \int_{y_-(x)}^{y_+(x)} \boldsymbol{b}(y)\,dy, \qquad \Delta \boldsymbol{r}(y) = \int_{x_-(y)}^{x_+(y)} \boldsymbol{a}(x)\,dx. \tag{7.13}$$

Then with (2.6), the pair of Eqs. (7.11) and (7.12) form a system of differential-integral equations for the fiber directions $\alpha(x)$ and $\beta(y)$.

When $\Gamma = 0$ they are integral equations that can be solved by series or iteration [5], [6]. When $G = 0$ as well, they are merely algebraic equations that can be solved immediately [2].

8. Ordinary and Singular Perturbations

To illustrate the use of ordinary and singular perturbation methods when the bending stiffness is small, we consider a class of examples in which the system (7.10), (7.11) reduces to only one equation. Let the sheet be bounded by the lines $x = 0$, $x = L$, and two smooth curves $y = y_\pm(x)$. Let us suppose that the edge $x = 0$ is clamped in its initial position, so that $\alpha(0) = 0$ and $\boldsymbol{b}(y) = \boldsymbol{j}$. Clamping the edge in any other position would similarly determine $\boldsymbol{b}(y)$ and $\alpha(0)$. A distribution of tractions $\boldsymbol{T}(y)$ with resultant $\boldsymbol{F}_r$ is prescribed on the edge $x = L$. The zero-couple condition (4.5) on that edge is $\alpha'(L) = 0$. The two edges $y = y_\pm(x)$ are left free from traction. Then the function $\boldsymbol{F}_r(x)$ in (7.11) has the constant value $\boldsymbol{F}_r$. The following analysis would be changed very little if non-zero tractions were prescribed on $y = y_\pm(x)$; $\boldsymbol{F}_r(x)$ would merely be non-constant.

The vector $\Delta \boldsymbol{r}(x)$ in (7.13) is equal to $\boldsymbol{j}\Delta y(x)$ because $\boldsymbol{b}(y) = \boldsymbol{j}$; with any other prescription of $\boldsymbol{b}$ it would still be the case that $\Delta \boldsymbol{r}(x)$ could be evaluated immediately. The relation (7.12) is not needed because $\beta(y)$ is known. From (7.11) we obtain

$$\Gamma[\Delta y(x)\,\alpha'(x)]' = -\boldsymbol{k} \times \boldsymbol{a} \cdot [\boldsymbol{F}_r + G\boldsymbol{i}\Delta y(x)], \tag{8.1}$$

an equation for the single unknown $\alpha(x)$. The boundary conditions are

$$\alpha(0) = 0, \qquad \alpha'(L) = 0. \tag{8.2.1, 2}$$

Let $\alpha_0(x)$ be the solution of (8.1) when $\Gamma = 0$, and let $\boldsymbol{a}_0$ be the corresponding value of $\boldsymbol{a}$. Then (8.1) yields

$$K(x)\,\boldsymbol{a}_0(x) = \boldsymbol{F}_r + G\boldsymbol{i}\Delta y(x), \tag{8.3}$$

where $K(x)$ is the magnitude of the right-hand member. With $\boldsymbol{F}_r = F_1\boldsymbol{i} + F_2\boldsymbol{j}$, we have

$$K(x) = [(F_1 + G\Delta y)^2 + F_2{}^2]^{1/2} \tag{8.4}$$

and

$$\tan \alpha_0(x) = F_2/[F_1 + G\Delta y(x)]. \tag{8.5}$$

By using (8.3) to define $\boldsymbol{a}_0$, we can rewrite (8.1) in the form

$$\Gamma[\Delta y(x)\, \alpha'(x)]' = K(x) \sin [\alpha(x) - \alpha_0(x)]. \tag{8.6}$$

Let us suppose that Γ is very small, in the sense that the left-hand member of (8.6) is small when evaluated with $\alpha = \alpha_0$. Then α is given to first order in Γ by

$$\alpha_1(x) = \alpha_0(x) + \Gamma[\Delta y(x)\, \alpha_0'(x)]'/K(x). \tag{8.7}$$

The approximations α_0 and α_1 cannot be valid near the boundaries unless they accidentally satisfy the boundary conditions (8.2), which were not used in determining them. Let h and h' be the length scales defined by

$$h = [\Gamma\Delta y(0)/K(0)]^{1/2}, \qquad h' = [\Gamma\Delta y(L)/K(L)]^{1/2}, \tag{8.8}$$

and let us suppose that $h/L \ll 1$ and $h'/L \ll 1$; here for the first time we define precisely what we mean in saying that Γ is small. Near the end $x = 0$ we introduce a stretched coordinate $\xi = x/h$ and write

$$\alpha = \alpha_0(h\xi) + \phi(\xi). \tag{8.9}$$

By using this in (8.6) and then taking the limit as h approaches zero with ξ fixed, we obtain

$$\phi''(\xi) = \sin \phi(\xi). \tag{8.10}$$

The boundary condition (8.2.1) yields $\phi(0) = -\alpha_0(0)$. The second condition is replaced by the requirement that for any fixed $x > 0$, $\alpha(x)$ approaches $\alpha_0(x)$ when Γ approaches zero. In terms of $\phi(\xi)$, this means that $\phi(\infty) = 0$. Then ϕ is determined by solving (8.10) with

$$\phi(0) = -\alpha_0(0), \qquad \phi(\infty) = 0. \tag{8.11}$$

Near the end $x = L$ we similarly introduce a stretched coordinate $\xi = (x - L)/h'$ and by a similar process again obtain an equation of the form (8.10) but now with boundary conditions

$$\phi'(0) = -h'\alpha_0'(L), \qquad \phi(-\infty) = 0. \tag{8.12.1, 2}$$

The factor h' is retained in (8.12.1) in order to obtain a non-trivial solution.

The solution of (8.10) that satisfies the boundary condition (8.11) is

$$\phi = -4 \arctan [e^{-\xi} \tan(\alpha_0/4)]. \tag{8.13}$$

If α_0 is small this is approximately

$$\phi \cong -\alpha_0(0) \exp(-x/h). \tag{8.14}$$

Thus, the correction to $\alpha_0(x)$ that is needed in order to satisfy the boundary condition (8.2.1) is negligible outside a thin layer near $x = 0$ whose thickness is of the order of h. Similarly, the correction near the end $x = L$ is qualitatively like

$$\phi \cong -h'\alpha_0'(L) \exp[(x - L)/h'], \tag{8.15}$$

negligible outside a layer whose thickness is of the order of h' and small even inside that layer; correcting the derivative requires only a very small correction to α itself.

9. Interior Transition Layer

The approximation (8.7) evidently cannot be valid near places where α_0 or even α_0' is discontinuous, since the supposedly small correction is infinite at such places. In the present section we consider some of the details of a specific problem in which the fiber direction is discontinuous in the zero-stiffness solution.

We consider a rectangular sheet with a straight central cut. The edges of the sheet are initially along the lines $x = -L'$, $x = 2L + L'$, and $y = \pm H$. The cut is along $y = 0$, $0 \leqq x \leqq 2L$. A uniform traction $\boldsymbol{T} = T_1\boldsymbol{i}$ is applied as a dead load along the edge $x = 2L + L'$ and an opposite traction $\boldsymbol{T} = -T_1\boldsymbol{i}$ is applied along $x = -L'$. Uniform tractions $\boldsymbol{T} = \pm T_2\boldsymbol{j}$ are applied along the edges $y = \pm H$, respectively. The two edges of the cut are left free from traction.

In the solution for a sheet with no bending stiffness the regions $x < 0$ and $x > 2L$ remain undistorted and the stress vectors in these regions are $\boldsymbol{t}_a^0 = T_1\boldsymbol{i}$ and $\boldsymbol{t}_b^0 = T_2\boldsymbol{j}$. In the region $0 \leqq y \leqq H$, $0 < x < 2L$ the vector $\boldsymbol{a}_0(x)$ has an expression of the form (8.3) with $\Delta y(x) = H$ and $\boldsymbol{F}_r$ a function of x given by

$$\boldsymbol{F}_r(x) = T_1H\boldsymbol{i} + T_2(L - x)\boldsymbol{j}. \tag{9.1}$$

Then

$$\tan\alpha_0(x) = T_2(L - x)/(T_1 + G)H \tag{9.2}$$

and

$$K(x) = [H^2(T_1 + G)^2 + T_2^2(L - x)^2]^{1/2}. \tag{9.3}$$

Because of the symmetry of the sheet and its loading we need not consider the region $y < 0$, nor even the region $x > L$. We see that $\alpha_0(x)$ is discontinuous across the line $x = 0$. There is accordingly a jump in shearing stress across this line, and for this reason the fiber $x = 0$ is singular, carrying a finite force

$$F_b(y) = T_2L(H - y)/H \qquad (y \geqq 0). \tag{9.4}$$

When $\Gamma \neq 0$ the regions $x < 0$ and $x > 2L$ are again undistorted and the stress is the same as in the solution for $\Gamma = 0$. In the region $y > 0$, $0 < x < L$, the equation for $\alpha(x)$ has a form similar to (8.6),

$$\Gamma H\alpha''(x) = K(x) \sin [\alpha(x) - \alpha_0(x)], \tag{9.5}$$

with α_0 and K given by (9.2) and (9.3). Continuity of $\alpha(x)$ across $x = 0$ is now required, so $\alpha(0) = 0$. For a condition at $x = L$ we use the symmetry condition $\alpha(L) = 0$:

$$\alpha(0) = \alpha(L) = 0. \tag{9.6}$$

Exactly as in Section 8 we can show that $\alpha(x)$ varies rapidly but continuously from its boundary value $\alpha(0) = 0$ to the value $\alpha_0(0)$ through a thin layer near the line $x = 0$. Then the shearing stress S is no longer discontinuous across $x = 0$, and this could lead one to guess that the fiber $x = 0$ might no longer be singular. However, this is not the case. We now show that the stress discontinuity is exactly as it was in the zero-stiffness solution, leading to the same fiber force $F_b(y)$ given in (9.4).

At $x = 0+$ we have $\boldsymbol{a} = \boldsymbol{i}$, $\boldsymbol{b} = \boldsymbol{j}$, and thus (3.6) gives

$$\boldsymbol{t}_a(0+, y) = T_a(0+, y)\, \boldsymbol{i} - \Gamma[\boldsymbol{j}\alpha''(0) - \boldsymbol{i}\alpha'(0)^2]. \tag{9.7}$$

We have used $S = 0$ since $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. The $\boldsymbol{i}$-component of $\boldsymbol{t}_a$ is continuous if

$$T_a(0+, y) = T_1 - \Gamma[\alpha'(0)]^2, \tag{9.8}$$

and the jump in $\boldsymbol{t}_a$ is then

$$\boldsymbol{t}_a(0+, y) - \boldsymbol{t}_a(0-, y) = -\boldsymbol{j}\Gamma\alpha''(0). \tag{9.9}$$

But from (9.5), with (9.2) and (9.3), we find that

$$\Gamma H\alpha''(0) = -T_2 L. \tag{9.10}$$

Then the jump in $\boldsymbol{t}_a$ is $\boldsymbol{j}T_2L/H$, and the fiber force required to equilibrate this is (9.4).

It should be pointed out that the present example is somewhat artificial in that the minimum-energy solution under the stated conditions would not be a plane deformation. As in the case of infinitesimal deformations [7] the fibers along the edges of the cut are singular and in compression, and this presumably means that the solution is unstable. However, this has nothing to do with our basic result: bending stiffness smooths the deformation but not the stress field.

10. Comments on the General Pure Traction Problem

In a pure traction boundary value problem the system of differential-integral equations (7.11), (7.12) must be solved simultaneously for $\alpha(x)$ and $\beta(y)$, subject to two boundary conditions on each of these unknowns. The boundary conditions on

$\alpha(x)$ will apply at the largest and smallest values of x on the sheet, and the conditions on $\beta(y)$ will similarly be applied at the extreme values of y. If the sheet is bounded by fibers at its extremes, the boundary conditions are the zero-couple conditions (4.5). If the extreme value of a coordinate is on an arc of type Z, the difference $\Delta y(x)$ or $\Delta x(y)$ vanishes at the endpoint, and the Eq. (7.11) or (7.12) is singular there. The boundary condition is then that the solution $\alpha(x)$ or $\beta(y)$ remains finite at the singular point.

For very small Γ, the zero-stiffness solution $\alpha_0(x)$, $\beta_0(y)$ will be acceptable throughout most of the sheet. This solution can be tested by using it to evaluate the terms involving Γ in (7.11) and (7.12). It can be accepted as a valid approximation except at places where it does not yield negligible values of these stiffness terms, and at extreme values of x or y at which it does not satisfy the boundary conditions.

At the particular values of x or y where the zero-stiffness solution is not valid by this test, we can use singular perturbation methods as in Sections 8 and 9. This is no more complicated than in the examples considered earlier, because in a transition-layer equation for $\alpha(x)$, say, β can be taken as having the known value $\beta_0(y)$, and so $\Delta \boldsymbol{r}(x)$ in (7.11) can be approximated by $\Delta \boldsymbol{r}_0(x)$ in the lowest order of approximation.

Boundary-layer effects occur only on boundaries that lie along fibers, because such effects are associated with particular values of x or y. On the boundary arcs of type Z that cut across both families of fibers the solution shows no special peculiarity. But even the boundary layers on X and Y are rather insignificant in pure traction problems, because the zero-couple condition requires only a weak correction of the type shown in the example (8.15).

Acknowledgment

The work described in this paper was supported by a grant DMS-8403196 from the National Science Foundation. We gratefully acknowledge this support.

References

[1] Wang, W.-B., Pipkin, A. C.: Inextensible networks with bending stiffness. Forthcoming.

[2] Rivlin, R. S.: Plane strain of net formed by inextensible cords. Arch. Rat'l. Mech. Anal. **4**, 951–74 (1955).

[3] Adkins, J. E.: Finite plane deformation of thin elastic sheets reinforced with inextensible cords. Phil. Trans. Roy. Soc. London **A 249**, 125–50 (1956).

[4] Pipkin, A. C.: Some developments in the theory of inextensible networks. Q. Appl. Math. **38**, 343–55 (1980).

[5] Pipkin, A. C.: Plane traction problems for inextensible networks. QJMAM **34**, 415–29 (1981).

[6] Pipkin, A. C.: Finite plane stress of stiff fibre-reinforced sheets. IMA J. Appl. Math. **27**, 195–209 (1981).
[7] England, A. H., Rogers, T. G.: Plane crack problems for ideal fibre-reinforced materials. QJMAM **26**, 303–20 (1973).
[8] Love, A. E. H.: A treatise on the mathematical theory of elasticity, 4th ed. New York: Dover 1944.

W.-B. Wang and A. C. Pipkin
Division of Applied Mathematics
Brown University
Providence, RI 02912