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Further Properties of the Falkner-Skan Equation

By

P. Brodie and **W. H. H. Banks**, Bristol, England

With 1 Figure

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Summary

We present some analytical perturbation results from which the skin-friction is accurately predicted over the range $1/2 < \beta < \infty$, where β is the usual Falkner-Skan parameter. The first eigenvalue for various β in the range $2 < \beta < \infty$ is also calculated numerically and certain properties noted. We find, analytically, the whole set of eigensolutions for the case when β is large.

1. Introduction

The Falkner-Skan one-parameter family of solutions of the boundary-layer equations has proved to be very useful in the interpretation of fluid flows at moderately high Reynolds numbers, and various properties of this family have been discussed extensively in the literature for all relevant values of the parameter β.

The purpose of this paper is twofold. In section 2 we present a perturbation analysis about a known analytical solution of the Falkner-Skan equation which corresponds to β infinite. This section is analytical and the results are found to be very useful in accurately predicting the skin-friction in the range $1/2 < \beta < \infty$. In the third section the numerical calculation of the eigenvalues by Chen and Libby [1] for $\beta \leqq 2$ is extended to the range $2 < \beta < \infty$, thereby confirming the spatial stability of such flows. Guided by the numerical results, we analytically consider the eigenvalue problem for large β, and for this special case we are able to derive the complete set of eigensolutions.

2. The Falkner-Skan Equation

The boundary-layer equations are

$$uu_x + vu_y = UU_x + \nu u_{yy}, \tag{1}$$

$$u_x + v_y = 0, \tag{2}$$

and the relevant boundary conditions for the class of flows of interest are

$$u = v = 0, \quad \text{on} \quad y = 0, \qquad u \to U(x) \quad \text{as} \quad y \to \infty. \tag{3}$$

The 'initial' condition at some station $x = x_0$ will not concern us in this investigation.

The Falkner-Skan similarity solutions are obtained if we assume $U(x) = ax^m$ and write

$$\psi = \left[\frac{2\nu x U(x)}{(1+m)}\right]^{1/2} F(\eta) \tag{4}$$

where $\eta = \{[(1+m)\,U(x)]/(2\nu x)\}^{1/2}\,y$, $u = \psi_y$ and $v = -\psi_x$. Substituting into Eq. (1) we obtain the well-known Falkner-Skan equation,

$$F''' + FF'' + \beta(1 - F'^2) = 0, \tag{5}$$

where dashes imply differentiation with respect to η, $\beta = 2m/(1+m)$ and the boundary conditions are

$$F(0) = F'(0) = 0, \qquad F'(\eta) \to 1 \quad \text{as} \quad \eta \to \infty. \tag{6}$$

It may be opportune to note here that the range $2 < \beta < \infty$ corresponds to $-\infty < m < -1$ and for such values we require $a < 0$, i.e. the flow velocity is in the direction of x-decreasing.

The aspect we are concerned with in this section is the behaviour of the solutions for large β ($m \approx -1$). It is known that the analytical solution for $m = -1$ can be derived from (5) and (6) by writing

$$F(\eta) = \beta^{-1/2} f(z) \tag{7}$$

where $z = \beta^{1/2}\eta$ and then taking the limit $\beta \to \infty$. On substituting (7) into (5) and (6) we find that

$$f''' + \beta^{-1} ff'' + 1 - f'^2 = 0, \tag{8}$$

where dashes here imply differentiation with respect to z, and the boundary conditions become

$$f(0) = f'(0) = 0, \qquad f'(z) \to 1 \quad \text{as} \quad z \to \infty. \tag{9}$$

We now look for a solution to (8) subject to (9) by writing

$$f(z) = f_0(z) + \beta^{-1} f_1(z) + \cdots \tag{10}$$

and find

$$f_0''' + 1 - f_0'^2 = 0, \tag{11}$$

$$f_1''' + f_0 f_0'' - 2f_0' f_1' = 0, \tag{12}$$

on equating terms of $O(1)$ and $O(\beta^{-1})$ respectively. The boundary conditions become

$$f_0(0) = f_0'(0) = 0, \qquad f_0'(z) \to 1 \quad \text{as} \quad z \to \infty, \tag{13}$$

$$f_1(0) = f_1'(0) = 0, \qquad f_1'(z) \to 0 \quad \text{as} \quad z \to \infty. \tag{14}$$

The solution of (11) subject to (13) is well-known and can be written

$$f_0(z) = z + 2\sqrt{3} - 3\sqrt{2}\tanh\left(z/\sqrt{2} + c\right), \tag{15}$$

where $\tanh c = \sqrt{2}/\sqrt{3}$. A property which will be useful later is $f_0''(0) = 2/\sqrt{3} = 1.154701$ which is related to the skin-friction. With $f_0(z)$ known it is possible to solve (12) for $f_1(z)$ subject to (14) and after some algebra we find that

$$\begin{aligned} f_1(\theta) = {} & \frac{-\sqrt{2}\left(\theta - c + \sqrt{6}\right)}{5S^2} + \left(\frac{3\sqrt{2}\,S^2}{2} - \frac{12\sqrt{2}}{5}\right)\int_c^\theta yT(y)\,dy - \frac{6\sqrt{2}}{5} \\ & \cdot\left(\sqrt{6} - c\right)\log S + \frac{161\sqrt{2}\,\theta}{40} + \frac{7\sqrt{2}\,T}{20S^2} - \frac{167\sqrt{2}\,T}{140} - \frac{13\sqrt{2}\,\theta T^2}{40} \\ & + \frac{K_2T^2}{2} + \left(K_1 - \frac{8\sqrt{2}}{5}\log S\right)\left(\frac{3\theta}{4} + \frac{T}{8S^2} - \frac{15\theta S^2}{16} - \frac{15T}{16}\right) + K_3 \end{aligned} \tag{16}$$

where $\theta = z/\sqrt{2} + c$, $S = \operatorname{sech}\theta$, $T = \tanh\theta$ and the constants K_i $(i = 1, 2, 3)$ are given by

$$K_1 = \left(2\sqrt{2}/5\right)\left[4\left(\log 2 + \sqrt{6} - c\right) - 7\right],$$

$$K_2 = \left[K_1 - \left(4\sqrt{2}/5\right)\log 3\right]\left(15c/8 + 9\sqrt{6}/8\right) + 11\sqrt{3}/2 + 13\sqrt{2}\,c/20,$$

$$K_3 = \sqrt{2}\,c(\log 3 - 457) - K_2/3 + 3\sqrt{6}\,K_1/16 - 197\sqrt{3}/60 - \left(9\sqrt{3}\log 3\right)/10.$$

These lead to the numerical values

$$\begin{aligned} c &= 1.146216, \\ K_1 &= 0.557587, \\ K_2 &= 1.748655, \\ K_3 &= 2.112126, \end{aligned}$$

and also to $f_1''(0) = 0.074614$. A numerical integration of (11), (12) subject to (13), (14) confirmed these results.

The skin friction for these flows is given by

$$\left(\mu\frac{\partial u}{\partial y}\right)_0 = \left[\frac{(1 + m)\,\varrho\mu U^3}{2x}\right]^{1/2} F''(0).$$

The results using just two terms of the expansion in (10) give

$$F''(0) = \beta^{1/2}[f_0''(0) + \beta^{-1}f_1''(0)] \tag{17}$$

and in table 1 a comparison is given with the exact numerical results, also obtained in this study, using Eq. (8)[1]. As will be noted, the large-β results from (17) are sufficient to predict the skin-friction to within 0.6% over the range $1/2 < \beta < \infty$.

Table 1

β	$F''(0)$ numerical	$F''(0)$ analytical
100	11.55447	11.55447
50	8.17553	8.17552
20	5.18072	5.18066
10	3.67523	3.67508
5	2.61578	2.61536
2	1.68722	1.68575
1	1.23259	1.22932
0.5	0.92768	0.92202

3. Perturbations about Falkner-Skan Solutions

In this section we examine the spatial stability of the Falkner-Skan flows for $2 < \beta < \infty$ by use of a perturbation procedure. We generalise (4) by writing

$$\psi = \left[\frac{2\nu x U(x)}{(1+m)}\right]^{1/2} \mathcal{F}(x, \eta)$$

where η, u, v are as defined in section 1, and on substituting into (1) we get

$$\mathcal{F}_{\eta\eta\eta} + \mathcal{F}\mathcal{F}_{\eta\eta} + \beta(1 - \mathcal{F}_\eta{}^2) = (2 - \beta)\, x(\mathcal{F}_\eta \mathcal{F}_{\eta x} - \mathcal{F}_x \mathcal{F}_{\eta\eta}). \tag{18}$$

Writing $\mathcal{F}(x, \eta) = F(\eta) + F_1(x, \eta)$, where $F(\eta)$ is the Falkner-Skan function in section 1, we obtain, by linearising, the following equation for F_1:

$$F_{1\eta\eta\eta} + FF_{1\eta\eta} - 2\beta F'F_{1\eta} + F''F_1 = (2 - \beta)\, x(F'F_{1\eta x} - F_{1x}F''). \tag{19}$$

A separable solution in the form $F_1 = X(x)\, H(\eta)$ is possible provided $X(x) = x^{\lambda/(\beta-2}$ and

$$H''' + FH'' + (\lambda - 2\beta)\, F'H' + (1 - \lambda)\, F''H = 0. \tag{20}$$

[1] The numerical integration of (5) subject to (6) was straight-forward for $\beta \leqq 20$ but, because the range of integration necessary decreases as β increases, it was found to be essential to use the formulation in (8) and (9).

This equation, which is subject to the boundary conditions

$$H(0) = H'(0) = 0, \qquad H'(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \tag{21}$$

is precisely the same as that arising in the work of Chen and Libby [1] who confined attention to the range $-0.1988 \leqq \beta \leqq 2$ because they posed their problem in terms of Görtler variables. We should note that, since $U(x) < 0$ for the range $2 < \beta < \infty$, we expect the similarity solution posed in (5) and (6) to be valid as $x \to 0$ and therefore for spatial stability we require $\lambda > 0$.

The eigenvalue problem defined in (20) and (21) has been solved numerically, with normalisation $H''(0) = 1$, for various values of β and the results of the first eigenvalues are presented in table 2. For values of β in excess of 20 we have had to use the large-β formulation exemplified by (7) by necessity since the Falkner-Skan function was found only in this way. It will be convenient to note the form of the equation used for $\beta \gg 1$: it is

$$h''' + \beta^{-1} f h'' + (\lambda\beta^{-1} - 2) f'h' + \beta^{-1}(1 - \lambda) f''h = 0 \tag{22}$$

where $h(z) = \beta H(\eta)$, $z = \beta^{1/2}\eta$, $f(z) = \beta^{1/2} F(\eta)$ as introduced earlier and dashes imply differentiation with respect to z. The boundary conditions are

$$h(0) = h'(0) = 0, \qquad h'(z) \to 0 \quad \text{as} \quad z \to \infty, \tag{23}$$

with normalisation $h''(0) = 1$.

However, there are two points of interest that we have noted regarding the numerical results in this section. First, it is clear that $\lambda_1\beta^{-1} \to 2$ as $\beta \to \infty$ and on closer inspection of the results we find $\lambda_1 \sim 2\beta + 2$ for large β. Secondly, when the properties of the numerical results of the eigenfunctions were examined it was found that the large-β transformation referred to above was inappropriate for β of order 400, since the range of integration found to be necessary was increasing with increasing β. Again, closer examination revealed that the natural structure of the eigenfunction was more aptly described in terms of $H(\eta)$, the original variables. Consequently, the formulation in (22) is not appropriate for the direct numerical evaluation of the eigenvalues for $\beta \gg 1$.

Table 2

β	λ/β	$2(1 + \beta^{-1})$
400	2.00507	2.005
200	2.01016	2.01
100	2.02043	2.02
50	2.04117	2.04
20	2.10427	2.1
10	2.21077	2.2
5	2.42516	2.4
2	3.06566	3

In view of the above, including the fact that from (15) we are able to obtain simple expressions for F, F' and F'' which are correct to $O(\beta^{-1/2})$ providing $\eta\beta^{1/2} \gg 1$, the form of the 'outer' structure of $H(\eta)$ can be determined for large β by noting that Eq. (20) can be approximated by

$$H_0''' + \eta H_0'' + (\lambda - 2\beta)\, H_0' = 0 \tag{24}$$

except when $\eta = O(\beta^{-1/2})$. The general solution of (24) is

$$H_0' = e^{-\eta^2/4}[AU(a, \eta) + BV(a, \eta)], \tag{25}$$

where U, V are the parabolic cylinder functions, $a = 1/2 + 2\beta - \lambda$ and A, B are constants. The term $V(a, \eta)$ gives rise to an algebraic variation in H_0' as $\eta \to \infty$ and so we take $B = 0$.

The next stage in this approach would be to obtain an inner solution valid for $\eta = O(\beta^{-1/2})$ from (22) and subject to $h(0) = h'(0) = 0$ and a certain matching condition as $z \to \infty$. However, it transpires that the essential properties of all the eigenfunctions, including the eigenvalues, are obtainable from (24) by noting that, from the properties of $H_0(\eta)$ and the envisaged form of the inner structure, we must impose the condition $H_0'(0) = 0$. Since $U(a, 0) = \sqrt{\pi} \Big/ \left[2^{\frac{1}{2}a+\frac{1}{4}}\,\Gamma\left(\frac{3}{4} + \frac{1}{2}a\right)\right]$ we require the zeros of $1/\Gamma(x)$ and hence deduce that

$$\lambda_n = 2\beta + 2n, \tag{26}$$

where $n = 1, 2, 3, \ldots$ It follows that the complete set of eigenvalues, corresponding to $\beta \gg 1$, is given by (26) and we note that the result, $\lambda_1 = 2\beta + 2$, for the first eigenvalue is in agreement with the numerical results referred to above.

Further, with $\lambda_1 = 2\beta + 2$ we find from (25) (with $B = 0$) that the corresponding first eigenfunction is $H_{01}' = A_1 \eta e^{-\eta^2/2}$. The form of the higher-order eigenfunc-

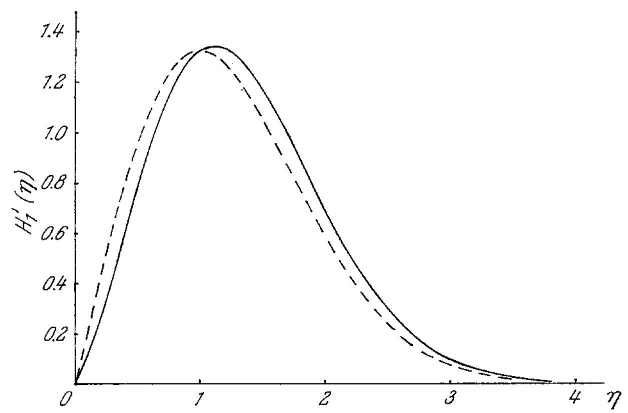

Fig. 1. The first eigenfunction $H_1'(\eta)$ for $\beta = 100$. The asymptotic result $H_{01}'(\eta) = A_1 \eta e^{-\eta^2/2}$ with $A_1 = 2.2$ is also shown by a dashed line

tions can then readily be found, and the nth-order one, H'_{0n}, is related to $(d^{2n}/d\eta^{2n})\, H'_{01}$, so that $H'_{02} = A_2\eta(3 - \eta^2)\, e^{-\eta^2/2}$ for example. The first eigenfunction for $\beta = 100$ is plotted, using the numerical results, in Fig. 1, and we also show the variation of H'_{01} with $A_1 = 2.2$ for comparison. Bearing in mind that the error involved in finding the term H'_{01} is $O(\beta^{-1/2})$ the agreement is very satisfactory.

Finally, we note that for $2 < \beta < \infty$ the flows are spatially stable, although since the x-variation of the disturbance is given by $X(x) = x^p$ where $p = \lambda/(\beta - 2)$ we observe that, for large β, the set of values for the exponent p bunch together, and, in the limit $\beta \to \infty$, collapse to the single value $p = 2$.

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Reference

[1] Chen, K. K., Libby, P. A.: Boundary layers with small departures from the Falkner-Skan profile. J. Fluid Mech. **33**, 273—282 (1968).

P. Brodie and W. H. H. Banks
School of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
England