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On Large Strain Deformations of Shells

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With 4 Figures

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Summary

A nonlinear shell theory is derived for large strain — large bending deformations of shells composed of highly nonlinear materials. Expressions for the stress resultants and stress couples are presented. The equilibrium equations are obtained as weak solution of the stationary principle of total potential energy. A modified Kirchhoff hypothesis is used which accounts for thickness changes as well as for a shift in the location of the original midsurface of the shell. As example the eversion of a spherical shell is calculated numerically.

1. Introduction

To calculate large elastic deformations of shell structures general nonlinear shell theories had been derived and successfully applied during the past few years [1]. These theories are based on the assumption of small strains, while the rotations of shell material elements may be large or even unrestricted. But numerical applications showed that for various shell structures large rotations are often accompanied by large strains. The small strain assumption is also no more adequate for structures made of highly nonlinear materials.

To derive a nonlinear shell theory appropriate for large rotations and large strain deformations the distribution of the displacement field in the direction of the normals to the reference surface turn out to be of primary importance. This and associated effects can be taken into account by using a three-dimensional displacement field of the shell represented by a Taylor expansion with respect to the normal coordinate ξ (see [2] and literature cited therein). However, the corresponding local field equations are very complex and they have not found a general application in the analysis of engineering problems. More effective seems to be an approach applied by Libai and Simmonds [3] for the special case of cylindrical deformations of shells, which is a one-dimensional problem. The same approach has been used also by Taber [4] for axisymmetric deformations of shells of revolution, a one-dimensional problem as well. In both papers the considerations are based on the following three assumptions:

(I) material fibres normal to a reference surface in the initial configuration remain normal to it during the deformation;
(II) the deformation of the shell is isochoric (volume preserving);
(III) coordinate lines are principal directions of the deformations.

It should be noted that the assumption (III) is in contradiction to the assumptions (I) and (II), which will be shown in section 4.

In this paper the structure of the basic shell equations is studied for shells of arbitrary geometry undergoing large strain deformations. To obtain appropriate shell models, the first Kirchhoff hypothesis is introduced, which corresponds to assumption (I). On the other side we relax the second Kirchhoff hypotheses requiring that points on a given material normal do not change their distance from the shell reference surface during the deformation. With this model large extension and large bending of the shell can be taken into account. It includes the dependency of the shell thickness on the deformation process, the asymmetry by a shift of the original shell midsurface and also shear deformations.

In chapter 2 it is shown that whenever the assumption (I) is satisfied the three-dimensional deformation of the shell can be expressed in terms of the changes in the metric and curvature tensors of its reference surface and of an unspecified function $\zeta = \zeta(\theta^\alpha, \xi)$ characterizing the deformation in the direction of normals to the reference surface. Here θ^α, $\alpha = 1, 2$ are the curvilinear Gaussian coordinates of the initial reference surface. Assuming a particular form of this function $\zeta(\theta^\alpha, \xi)$ a wide class of two-dimensional kinematical models can be constructed, denoted here as generalized Kirchhoff-Love models.

In chapter 3 it is shown that for isochoric deformations (assumption (II)) a function $\zeta(\theta^\alpha, \xi)$ is uniquely determined by five surface invariants, two of them defining the initial geometry of the reference surface and three of them characterizing its deformation. From the obtained results it follows that the introduction of the assumption (I) and (II) does not exclude shear deformations. Thus in the particular cases of cylindrical shell deformations or axisymmetric deformations of shells of revolution the coordinate lines cannot be principal directions, which is outlined in section 4.

Assuming that the shell is made of an incompressible hyperelastic material the structure of a two-dimensional strain energy function is considered in chapter 5. Formulating the functional of total potential energy for large strain deformations of shells the local equilibrium equations are derived as weak solution of the stationary principle of total potential energy. Finally, as a numerical example the eversion of a spherical shell is calculated in chapter 6.

2. Generalized Kirchhoff-Love Models

Let us consider the deformation $\chi: \mathcal{P} \to \bar{\mathcal{P}}$ of a three-dimensional body, with its initial configuration $\mathcal{P} \subset \mathbb{E}^3$, a domain in the Euclidean space $\mathbb{E}^3$ and its deformed configuration $\bar{\mathcal{P}} \subset \mathbb{E}^3$. With $\{\theta^i\}$ we denote a material (convected) coordinate system of the body. Throughout this paper the usual summation convention is used, where Latin indices have the range 1, 2, 3 and Greek indices the range 1, 2. Then for any point $P \in \mathcal{P}$ the position vector, the base vectors and the covariant metric tensor are given by

$$\boldsymbol{p} = \boldsymbol{p}(\theta^i), \qquad \boldsymbol{g}_i = \boldsymbol{p}_{,i}, \qquad g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j. \tag{2.1}$$

The dual base vectors $\boldsymbol{g}^i$ and the contravariant metric tensor g^{ij} are defined by:

$$\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta_j^{\ i}, \qquad g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \qquad \boldsymbol{g}^i = g^{ij} \boldsymbol{g}_j. \tag{2.2.1, 2, 3}$$

Here $(\ )_{,i}$ indicates partial differentiation with respect to the associated coordinate direction θ_i and a dot denotes the inner product of two vectors. All quantities defined at points $\bar{P} = \chi(P) \in \bar{\mathcal{P}}$ of the deformed configuration will be marked by a bar, e.g. $\bar{\boldsymbol{p}}$, $\bar{\boldsymbol{g}}_i$, $\bar{g}_{ij}$, $\bar{\boldsymbol{g}}^i$, $\bar{g}^{ij}$ etc. (Fig. 1).

In this paper we restrict our considerations to the deformation of shells, three-dimensional bodies, which are "small" in one direction [2]. Then it is convenient to take $\{\theta^i\} = \{\theta^\alpha, \theta^3 = \xi\}$, $\xi \in [-h_0^-, +h_0^+]$ to be a normal coordinate system in the initial configuration $\mathcal{P}$ with an initial shell thickness $h_0 = h_0^- + h_0^+$.

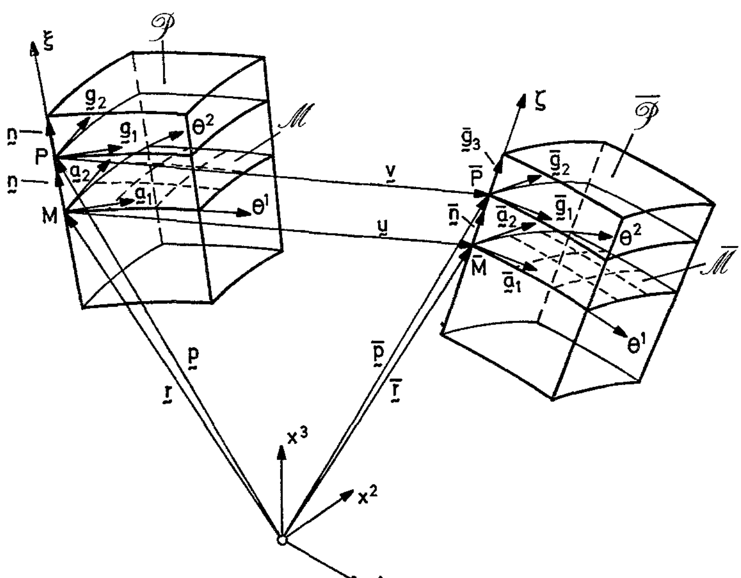

Fig. 1. Undeformed and deformed shell elements

In this case $\xi = 0$ defines a material surface $\mathcal{M}$ called a reference surface (Fig. 1). With each point $M \in \mathcal{M}$ we associate the position vector $\boldsymbol{r}$, base vectors $\boldsymbol{a}_\alpha$, surface metric tensor $a_{\alpha\beta}$, unit normal vector $\boldsymbol{a}_3 = \boldsymbol{n}$ and curvature tensor $b_{\alpha\beta}$:

$$\begin{gathered}\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha), \qquad \boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}, \qquad a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \\ \boldsymbol{n} = \frac{1}{2}\,\varepsilon^{\alpha\beta}\boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta, \qquad \varepsilon^{\alpha\beta} = (\boldsymbol{a}^\alpha \times \boldsymbol{a}^\beta)\cdot\boldsymbol{n}, \\ b_{\alpha\beta} = -\boldsymbol{n}_{,\alpha}\cdot\boldsymbol{a}_\beta = \boldsymbol{a}_{\alpha,\beta}\cdot\boldsymbol{n}.\end{gathered} \tag{2.3}$$

A cross indicates the usual vector product.

The dual base is defined by

$$\boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta = \delta_\beta^{\ \alpha}, \qquad a^{\alpha\beta} = \boldsymbol{a}^\alpha\cdot\boldsymbol{a}^\beta, \qquad \boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta, \qquad \boldsymbol{a}^3 = \boldsymbol{n}. \tag{2.4}$$

Mean curvature H and Gaussian curvature K are surface invariants of $\mathcal{M}$:

$$\begin{gathered}H = \frac{1}{2}\,a^{\alpha\beta}b_{\alpha\beta} = \frac{1}{2}\,b_\alpha^{\ \alpha} \\ K = \frac{1}{2}\,\varepsilon^{\alpha\lambda}\varepsilon_{\beta\mu}b_\alpha^{\ \beta}b^\mu_{\ \lambda} = \frac{b}{a}, \qquad a = \det a_{\alpha\beta}, \qquad b = \det b_{\alpha\beta}.\end{gathered} \tag{2.5}$$

All quantities referred to points $\overline{M} = \chi(M) \in \overline{\mathcal{M}}$ of the deformed shell reference surface are defined corresponding to (2.1)—(2.5) and will be marked by a bar, e.g. $\bar{\boldsymbol{r}}$, $\bar{\boldsymbol{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{\boldsymbol{n}}$, $\bar{b}_{\alpha\beta}$, etc. We emphasize that $\mathcal{M}$ need not be the geometric mid-surface of $\mathcal{P}$.

With the position vector $\boldsymbol{r}$ to a point $M \in \mathcal{M}$ of the reference surface and the unit normal vector $\boldsymbol{n}$ the position vector of any point $P \in \mathcal{P}$ of the shell space takes the form:

$$\boldsymbol{p}(\theta^\alpha, \xi) = \boldsymbol{r}(\theta^\alpha) + \xi\boldsymbol{n}(\theta^\alpha), \qquad \xi \in [-h_0^-, +h_0^+] \tag{2.6}$$

leading with (2.1) and (2.3) to the following representations of the base vectors and metric tensor components:

$$\begin{gathered}\boldsymbol{g}_\alpha = \mu_\alpha^{\ \beta}\boldsymbol{a}_\beta, \qquad \boldsymbol{g}_3 = \boldsymbol{n}, \\ g_{\alpha\beta} = \mu_\alpha^{\ \lambda}\mu_\beta^{\ \varkappa}a_{\lambda\varkappa}, \qquad g_{\alpha 3} = 0, \qquad g_{33} = 1, \\ \boldsymbol{g}^\alpha = (\mu^{-1})_\beta^{\ \alpha}\,\boldsymbol{a}^\beta, \qquad \boldsymbol{g}^3 = \boldsymbol{g}_3 = \boldsymbol{n}, \\ g^{\alpha\beta} = (\mu^{-1})_\lambda^{\ \alpha}\,(\mu^{-1})_\varkappa^{\ \beta}a^{\lambda\varkappa}, \qquad g^{\alpha 3} = 0, \qquad g^{33} = 1,\end{gathered} \tag{2.7}$$

where the shifter tensor $\mu_\alpha^{\ \beta}$ and its inverse $(\mu^{-1})_\alpha^{\ \beta}$ are given by

$$\mu_\alpha^{\ \beta} = \delta_\alpha^{\ \beta} - \xi b_\alpha^{\ \beta}, \qquad (\mu^{-1})_\lambda^{\ \alpha}\,\mu_\beta^{\ \lambda} = \delta_\beta^{\ \alpha}, \tag{2.8.1, 2}$$

$$(\mu^{-1})_\alpha^{\ \beta} = \frac{1}{\mu}\,[\delta_\alpha^{\ \beta} - \xi(2H\delta_\alpha^{\ \beta} - b_\alpha^{\ \beta})], \tag{2.8.3}$$

$$\mu = \det\mu_\alpha^{\ \beta} = \sqrt{\frac{g}{a}} = 1 - 2\xi H + \xi^2 K, \qquad g = \det g_{ij}. \tag{2.8.4, 5}$$

We turn now to the construction of a bidimensional kinematical model of a "thin" three-dimensional body, which should reflect the dominant behavior of this body. If large strain deformations are admitted in a shell structure made of a highly nonlinear material, the non-uniform distribution of transverse normal strains over the shell thickness with associated effects have to be taken into account. To avoid furthermore an excessive complexity of the resulting shell equations we introduce the assumption that material fibres normal to the reference surface in the initial configuration remain normal to it during the shell deformation (assumption (I)), while no restrictions are imposed on the shell deformation in the direction of the normals. Assumption (I) corresponds to the following constraint of the shell deformation:

$$[\bar{\boldsymbol{p}}(\theta^\lambda, \xi) - \bar{\boldsymbol{r}}(\theta^\lambda)] \cdot \bar{\boldsymbol{r}}_{,\alpha}(\theta^\lambda) = 0 \qquad \text{for } \xi \in [-h_0^-, +h_0^+]\,. \tag{2.9}$$

According to (2.9) the position vector of points $\bar{P} \in \bar{\mathcal{P}}$ in the deformed configuration of the shell must be of the form

$$\bar{\boldsymbol{p}}(\theta^\alpha, \xi) = \bar{\boldsymbol{r}}(\theta^\alpha) + \zeta(\theta^\alpha, \xi)\, \bar{\boldsymbol{n}}(\theta^\alpha), \tag{2.10}$$

where the function $\zeta = \zeta(\theta^\alpha, \xi)$ must satisfy the following condition

$$\zeta(\theta^\alpha, 0) = 0. \tag{2.11}$$

With (2.10) one gets the base vectors of points $\bar{P} \in \bar{\mathcal{P}}$ by differentiation according to (2.1):

$$\begin{gathered} \bar{\boldsymbol{g}}_\alpha = \bar{\mu}_\alpha{}^\beta \bar{\boldsymbol{a}}_\beta + \zeta_{,\alpha} \bar{\boldsymbol{n}}, \qquad \bar{\boldsymbol{g}}_3 = \zeta_{,\xi} \bar{\boldsymbol{n}}, \qquad (\;)_{,\xi} = \frac{\partial(\;)}{\partial \xi}, \\ \bar{\mu}_\alpha{}^\beta = \delta_\alpha{}^\beta - \zeta \bar{b}_\alpha{}^\beta. \end{gathered} \tag{2.12}$$

The corresponding metric tensor components are

$$\begin{gathered} \bar{g}_{\alpha\beta} = \bar{\mu}_\alpha{}^\lambda \bar{\mu}_\beta{}^\varkappa \bar{a}_{\lambda\varkappa} + \zeta_{,\alpha}\zeta_{,\beta} = \bar{a}_{\alpha\beta} - 2\zeta \bar{b}_{\alpha\beta} + \zeta^2 \bar{b}_\alpha{}^\lambda \bar{b}_{\lambda\beta} + \zeta_{,\alpha}\zeta_{,\beta}, \\ \bar{g}_{\alpha 3} = \zeta_{,\alpha}\zeta_{,\xi}, \qquad \bar{g}_{33} = (\zeta_{,\xi})^2. \end{gathered} \tag{2.13}$$

To obtain the reciprocal base vectors $\bar{\boldsymbol{g}}^i$ of $\bar{P} \in \bar{\mathcal{P}}$ we first have to determine the inverse $(\bar{\mu}^{-1})_\beta{}^\alpha$ of $\bar{\mu}_\beta{}^\alpha$ defined by (2.8.2) yielding

$$\begin{gathered} (\bar{\mu}^{-1})_\beta{}^\alpha = \frac{1}{\bar{\mu}}\,[\delta_\beta{}^\alpha - \zeta(2\bar{H}\delta_\beta{}^\alpha - \bar{b}_\beta{}^\alpha)], \\ \bar{\mu} = \det \bar{\mu}_\beta{}^\alpha = 1 - 2\zeta\bar{H} + \zeta^2 \bar{K}, \end{gathered} \tag{2.14}$$

where $\bar{H}$ and $\bar{K}$ denote the mean and Gaussian curvatures of the deformed reference surface $\bar{\mathcal{M}}$. A straightforward algebraic calculation lead to the reciprocal base vectors $\bar{\boldsymbol{g}}^i$ satisfying the system of Eq. (2.2.1):

$$\bar{\boldsymbol{g}}^\alpha = (\bar{\mu}^{-1})_\beta{}^\alpha \bar{\boldsymbol{a}}^\beta, \qquad \bar{\boldsymbol{g}}^3 = \frac{1}{\zeta_{,\xi}}\,[-\zeta_{,\alpha}(\bar{\mu}^{-1})_\beta{}^\alpha\, \bar{\boldsymbol{a}}^\beta + \bar{\boldsymbol{n}}]. \tag{2.15}$$

The corresponding metric tensor components are

$$\bar{g}^{\alpha\beta} = (\bar{\mu}^{-1})_\lambda{}^\alpha (\bar{\mu}^{-1})_\varkappa{}^\beta \bar{a}^{\lambda\varkappa}$$

$$= \frac{1}{\bar{\mu}^2} [\bar{a}^{\alpha\beta} - 2\zeta(2\bar{H}\bar{a}^{\alpha\beta} - \bar{b}^{\alpha\beta}) + \zeta^2(4\bar{H}^2\bar{a}^{\alpha\beta} - 4\bar{H}\bar{b}^{\alpha\beta} + \bar{b}_\lambda{}^\alpha \bar{b}^{\lambda\beta})], \tag{2.16}$$

$$\bar{g}^{\alpha 3} = -\frac{1}{\zeta_{,\xi}} \zeta_{,\beta} \bar{g}^{\alpha\beta}, \qquad \bar{g}^{33} = \left(\frac{1}{\zeta_{,\xi}}\right)^2 (\zeta_{,\alpha}\zeta_{,\beta} g^{\alpha\beta} + 1).$$

For the determinant $\bar{g} = \det \bar{g}_{ij}$ we can derive the following representation

$$\bar{g} = \det \bar{g}_{ij} = \bar{a}\bar{\mu}^2(\zeta_{,\xi})^2 = \bar{a}(1 - 2\zeta\bar{H} + \zeta^2\bar{K})^2 (\zeta_{,\xi})^2 \tag{2.17}$$

with $\bar{a} = \det \bar{a}_{\alpha\beta}$, where Cayley-Hamilton's theorem

$$\bar{b}_\alpha{}^\lambda \bar{b}_{\lambda\beta} = -\bar{K}\bar{a}_{\alpha\beta} + 2\bar{H}\bar{b}_{\alpha\beta} \tag{2.18}$$

is valid.

With the base vectors and the metric tensor of $P \in \mathcal{P}$ and $\bar{P} = \chi(P) \in \bar{\mathcal{P}}$ we introduce the deformation gradient tensor $\boldsymbol{F} = \nabla\chi$ in material coordinates

$$\boldsymbol{F} = \bar{\boldsymbol{g}}_i \otimes \boldsymbol{g}^i = \bar{g}_{ij}\bar{\boldsymbol{g}}^i \otimes \boldsymbol{g}^j \tag{2.19}$$

and the right Cauchy-Green strain tensor

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \bar{g}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j. \tag{2.20}$$

where $\otimes$ indicates the tensor product of two vectors.

With (2.7), (2.8) and (2.12)—(2.17) we can determine the principal invariants of the Cauchy-Green strain tensor

$$I_1(\boldsymbol{C}) = g^{ij}\bar{g}_{ij} = g^{\alpha\beta}\bar{g}_{\alpha\beta} + (\zeta_{,\xi})^2, \tag{2.21.1}$$

$$I_2(\boldsymbol{C}) = g_{ij}\bar{g}^{ij} I_3(\boldsymbol{C}) = \left\{\left[g_{\alpha\beta} + \left(\frac{1}{\zeta_{,\xi}}\right)^2 \zeta_{,\alpha}\zeta_{,\beta}\right]\bar{g}^{\alpha\beta} + \left(\frac{1}{\zeta_{,\xi}}\right)^2\right\} I_3(\boldsymbol{C}), \tag{2.21.2}$$

$$I_3(\boldsymbol{C}) = (\det \boldsymbol{F})^2 = \frac{\bar{g}}{g} = \frac{1}{\mu^2} j^2\bar{\mu}^2(\zeta_{,\xi})^2, \qquad j = \sqrt{\frac{\bar{a}}{a}}. \tag{2.21.3}$$

From (2.12)—(2.21) it follows that the three-dimensional deformation of the shell consistent within the constraint (2.9) is entirely specified by the changes of the metric and curvature tensors of its reference surface. For the determination of the function $\zeta = \zeta(\theta^\alpha, \xi)$ it is important to observe that according to (2.21.3) the third invariant I_3 of the right Cauchy-Green strain tensor does not depend on the derivatives of ζ with respect to the surface coordinates θ^α. Furthermore it is shown by (2.12)—(2.13) that the model includes transverse normal strains and transverse shear strains. The latter vanishes only on the reference surface according to (2.11).

To complete the given analysis the function $\zeta = \zeta(\theta^\alpha, \xi)$ must be specified. In the most general case we can use a representation of the form

$$\zeta(\theta^\alpha, \xi) = Z(\psi_K(\theta^\alpha), \xi), \qquad K = 1, 2, \ldots, M, \tag{2.22}$$

where ψ_K are additional independent kinematical variables apart from the position vector $\bar{\boldsymbol{r}}$ of the deformed reference surface. Postulating some forms of the function $Z(\theta^\alpha, \xi)$, which need not be a polynomial in ξ, a wide class of generalized Kirchhoff-Love type kinematical models for large strain deformations of shells may be constructed. In particular, for $\zeta = \xi$ the classical Kirchhoff-Love model [2] and for $\zeta = \psi_1\xi$ the model considered by Biricikoglu and Kalnins [6] are obtained. The latter is essentially equivalent to that of Naghdi [2], where the shell is modelled as a Cosserat surface with single director constrained to remain normal to it, which is commonly accepted in the classical membrane theory [5]. A generalization of these models had been considered by Chernykh [7], [8] introducing the function $\zeta = \psi_1\xi + \psi_2 \dfrac{\xi^2}{2}$. Using successively higher order polynomials the number of independent kinematical variables increases and consequently the corresponding field equations become more and more complex. Therefore it seems to be more effective to determine the function $\zeta = \zeta(\theta^\alpha, \xi)$ from additional constraints imposed on the strains or/and stresses in the shell. An approach of this type is considered in the next section.

3. Isochoric Deformation

In this section we introduce the additional assumption that the deformation of the shell is isochoric (volume preserving) satisfying the following constraint

$$\det \boldsymbol{F} = 1, \qquad \boldsymbol{F} = \nabla\chi, \tag{3.1}$$

which is generally assumed in large strain deformation problems.

Introducing (2.17) and (2.8.4) into (3.1) yields a first order differential equation for the function ζ:

$$(\bar{K}\zeta^2 - 2\bar{H}\zeta + 1)\,\frac{\partial\zeta}{\partial\xi} = j^{-1}(K\xi^2 - 2H\xi + 1). \tag{3.2}$$

With $\zeta = 0$ at the reference surface $\xi = 0$ Eq. (3.2) can be integrated and we obtain

$$\bar{K}\zeta^3 - 3\bar{H}\zeta^2 + 3\zeta = j^{-1}(K\xi^3 - 3H\xi^2 + 3\xi). \tag{3.3}$$

Let us first note that the general solution of the cubic Eq. (3.3) is of the form

$$\zeta(\theta^\alpha, \xi) = Z[H(\theta^\alpha), K(\theta^\alpha), j(\theta^\alpha), \bar{H}(\theta^\alpha), \bar{K}(\theta^\alpha), \xi] \tag{3.4}$$

leading to the statement that the function ζ is entirely determined by five invariants of the reference surface: H, K characterizing the initial geometry and

j, $\bar{H}$, $\bar{K}$ characterizing its deformation. With (3.4) and (2.13), (2.16) we obtain the result that within the constraints (2.9) and (3.1) the deformation of a shell as a three-dimensional body is completely determined by the kinematics of its reference surface in the undeformed and deformed configuration. However, the strain distribution over the shell thickness does not only depend on the changes of the metric and curvature tensors of the reference surface but also on their surface derivatives via the derivatives of the invariants of (3.4). This is a main difference to the classical nonlinear shell theories of Kirchhoff-Love type.

Let us first consider the solution of (3.3) for two special cases. If a shell of an arbitrary shape is deformed into a plate, then $\bar{K} = \bar{H} = 0$ and the solution of (3.3) is

$$\zeta(\theta^\alpha, \xi) = \frac{1}{3} j^{-1} \xi (K\xi^2 - 3H\xi + 3) \tag{3.5}$$

where for simplicity θ^α has been omitted in the arguments of the right hand side of (3.5).

If the reference surface of the deformed shell is of zero Gaussian curvature (cylindrical, conical shells), then $\bar{K} = 0$, $\bar{H} \neq 0$ and the solution is

$$\begin{gathered}\zeta(\theta^\alpha, \xi) = \frac{1}{2\bar{H}} \left[1 - \sqrt{ - \frac{4}{3} j^{-1} \bar{H} X(\xi)} \right], \\ X(\xi) = \xi(K\xi^2 - 3H\xi + 3), \qquad -\frac{3}{4} j \, |\bar{H}|^{-1} < X(\xi) < \frac{3}{4} j \, |\bar{H}|^{-1}, \\ \text{for } \xi \in [-h_0^-, +h_0^+].\end{gathered} \tag{3.6}$$

The general real solution of the third order Eq. (3.3) is

$$\begin{gathered}\zeta(\theta^\alpha, \xi) = \frac{1}{\bar{K}} \left(\bar{H} + (L + D^{1/2})^{1/3} + (L - D^{1/2})^{1/3} \right), \\ D = (\bar{K} - \bar{H}^2)^3 + L^2, \\ L = \bar{H}^3 - \frac{3}{2} \bar{H}\bar{K} + \frac{1}{2} j^{-1} \bar{K}^2 X(\xi), \\ X(\xi) = \xi(K\xi^2 - 3H\xi + 3)\end{gathered} \tag{3.7}$$

where again θ^α has been omitted in the arguments.

4. Cylindrical Deformation of Shells

Let us reduce the general results of Sect. 2 and 3 for the special case of cylindrical deformations of shells. An interesting study of this problem has been presented recently by Libai and Simmonds [3]. However, because of contradictory assumptions in their approach it is worthwile to reconsider this problem.

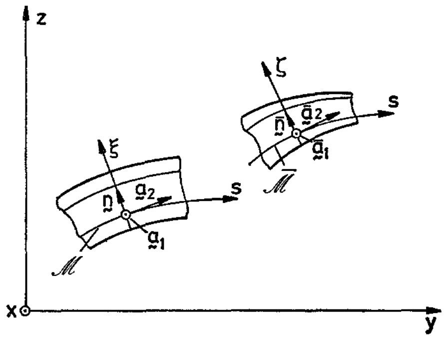

Fig. 2. Cylindrical deformation of shells — notations

By "cylindrical deformation" we mean the deformation of a cylindrical shell into another cylindrical shell such that the deformation in the direction of the generators of the cylinder, say x, consists of a uniform extension at the most. In this case it is convenient to take $\theta^1 = x$, $\theta^2 = s$, where s denotes the arc length (Fig. 2). Let σ and $\bar{\sigma}$ denote the curvatures of the reference surface in the undeformed and deformed configuration. Let $\lambda_x = \text{const.}$ and $\lambda = \lambda(s)$ be the (principal) stretches of the reference surface in x and s direction. With

$$H = \frac{1}{2}\,\sigma, \qquad \bar{H} = \frac{1}{2}\,\bar{\sigma}, \qquad K = \bar{K} = 0 \tag{4.1}$$

for cylindrical deformations the components of the metric tensors in the undeformed configuration (2.13) reduce to

$$\begin{aligned} &g_{11} = 1, \quad g_{22} = (1 - \xi\sigma)^2, \quad g_{33} = 1, \quad g_{12} = g_{13} = g_{23} = 0 \\ &g^{11} = 1, \quad g^{22} = (1 - \xi\sigma)^{-2}, \quad g^{33} = 1, \quad g^{12} = g^{13} = g^{23} = 0. \end{aligned} \tag{4.2}$$

For the deformed configuration we obtain from (2.13) and (2.16)

$$\begin{aligned} &\bar{g}_{11} = \lambda_x^2, \quad \bar{g}_{22} = \lambda^2(1 - \zeta\bar{\sigma})^2 + (\zeta_{,s})^2, \quad \bar{g}_{33} = (\zeta_{,\xi})^2, \\ &\bar{g}_{12} = \bar{g}_{13} = 0, \quad \bar{g}_{23} = \zeta_{,s}\zeta_{,\xi} \\ &\bar{g}^{11} = \lambda_x^{-2}, \quad \bar{g}^{22} = \lambda^{-2}(1 - \zeta\bar{\sigma})^{-2}, \quad \bar{g}^{33} = (\zeta_{,\xi})^{-2}\,[(\zeta_{,s})^2\,\lambda^{-2}(1 - \zeta\bar{\sigma})^{-2} + 1] \\ &\bar{g}^{12} = \bar{g}^{13} = 0, \quad \bar{g}^{23} = -(\zeta_{,\xi})^{-1}\,\zeta_{,s}\lambda^{-2}(1 - \zeta\bar{\sigma})^{-2} \end{aligned} \tag{4.3}$$

where $(\ )_{,s} = \dfrac{\partial(\ )}{\partial s}$ denotes differentiation with respect to the arc length s.

For isochoric cylindrical deformations result (3.6) leads to

$$\begin{aligned} &\zeta(s, \xi) = \frac{1}{\bar{\sigma}(s)} \left\{1 - \sqrt{1 - \varkappa(s)\left[\left(1 - \sigma(s)\,\xi\right)^2 - 1\right]}\right\}, \qquad \bar{\sigma} \neq 0 \\ &\varkappa(s) = -\frac{\bar{\sigma}(s)}{j(s)\,\sigma(s)}, \qquad j(s) = \lambda_x\lambda(s). \end{aligned} \tag{4.4}$$

To ensure that a cylindrical deformation of the shell consistent within the constraints (2.9) and (3.1) can take place the following inequality must be satisfied

$$1 - |\varkappa|^{-1} < (1 - \sigma\xi)^2 < 1 + |\varkappa|^{-1} \qquad \text{for } \xi \in [-h_0^-, +h_0^+]. \tag{4.5}$$

Introducing (4.4) into (4.3) we are able to determine the explicite form of the metric tensor in the deformed shell configuration and via this result also all other relevant kinematical quantities, what is not shown here.

Except for notations the formula (4.4) is identical with the result presented by Libai and Simmonds [3, Eq. (10)] by using a different approach. Besides the assumptions (2.9) and (3.1) it has been presumed there furthermore that the coordinate lines x, s, ξ are principal directions of the deformation. But from (4.3) it follows immediately that this is not true unless $\zeta_{,s}(s, \xi) \equiv 0$.

5. Large Strain Shell Theory

The previous sections are dealing with the kinematical aspects of shell deformations taking into account no particular material properties. In this chapter we consider the structure of local field equations assuming that the shell is made of an incompressible hyperelastic material. Then there exists a strain energy density per unit volume $W = W(\boldsymbol{C})$ as a function of the right Cauchy-Green strain tensor $\boldsymbol{C} = \bar{g}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$. For simplicity we presume that the material is homogeneous, but no restrictions will be made about the symmetry of the material.

A two-dimensional strain energy frunction Φ per unit area of the undeformed reference surface $\mathcal{M}$ is defined by

$$\iiint_{\mathcal{P}} W\, dV = \iint_{\mathcal{M}} \Phi\, dA, \qquad \Phi = \int_{-h_0^-}^{+h_0^+} W(\boldsymbol{C})\, \mu\, d\xi, \tag{5.1.1, 2}$$

where dV is a volume element of $\mathcal{P}$, dA a surface element of $\mathcal{M}$ and μ is give by (2.8.4). In view of the results obtained in the previous sections the components of $\mathbf{C}$ are known functions of the normal coordinate ξ. Consequently, performing in (5.1.2) the integration over the shell thickness it follows from (2.13), (2.16) and (3.4) that the strain energy density Φ is a function of the form

$$\Phi = \Phi(b_{\alpha\beta}, H_{,\beta}, K_{,\beta}, \bar{a}_{\alpha\beta}, \bar{b}_{\alpha\beta}, j_{,\beta}, \bar{H}_{,\beta}, \bar{K}_{,\beta}). \tag{5.2}$$

In the Kirchhoff-Love type nonlinear shell theory the conventional strain measures are the middle surface strain tensor $\gamma_{\alpha\beta}$ and the change of curvature tensor $\varkappa_{\alpha\beta}$ [1], [9]—[11]

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}), \qquad \varkappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}). \tag{5.3}$$

Besides the measures (5.3) we introduce for large strain deformations the following surface invariants

$$\lambda_\xi = j^{-1} = \sqrt{\frac{a}{\bar{a}}}, \qquad \varkappa_\xi = 2(\lambda_\xi \bar{H} - H), \qquad \tau_\xi = 2(\lambda_\xi{}^2 \bar{K} - K). \tag{5.4}$$

In the argument of the strain energy function Φ according to (5.2) we express the variables by the strain measures (5.3) and by the invariants (5.4) such that Φ is of the form

$$\Phi = \Phi(b_{\alpha\beta}, H_{,\beta}, K_{,\beta}, \gamma_{\alpha\beta}, \varkappa_{\alpha\beta}, \lambda_{\xi,\beta}, \varkappa_{\xi,\beta}, \tau_{\xi,\beta}). \tag{5.5}$$

The dependency of Φ on $b_{\alpha\beta}$, $H_{,\beta}$ and $K_{,\beta}$ is shown to underline that even for homogeneous material the two-dimensional strain energy function becomes inhomogeneous through these quantities.

Corresponding to the strain measures (5.3) and (5.4) we define work-conjugate stress measures by

$$n^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\Phi}{\partial\gamma_{\alpha\beta}} + \frac{\partial\Phi}{\partial\gamma_{\beta\alpha}}\right), \qquad m^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\Phi}{\partial\varkappa_{\alpha\beta}} + \frac{\partial\Phi}{\partial\varkappa_{\beta\alpha}}\right), \tag{5.6}$$

$$h^\alpha = \frac{\partial\Phi}{\partial\lambda_{\xi,\alpha}}, \qquad k^\alpha = \frac{\partial\Phi}{\partial\varkappa_{\xi,\alpha}}, \qquad l^\alpha = \frac{\partial\Phi}{\partial\tau_{\xi,\alpha}}, \tag{5.7}$$

where $n^{\alpha\beta}$ and $m^{\alpha\beta}$ are the stress resultant tensor and stress couple tensor o the Kirchhoff-Love type nonlinear shell theory for small strains and unrestricted rotations. The stress measures (5.7) are three additional stress variables, which we have to introduce taking into account large strain deformations.

The displacement vector from a point of the reference surface in the undeformed configuration $M \in \mathcal{M}$ to its image in the deformed configuration $\bar{M} \in \bar{\mathcal{M}}$ is

$$\boldsymbol{u}(\theta^\lambda) = \bar{\boldsymbol{r}} - \boldsymbol{r} = u^\lambda \boldsymbol{a}_\lambda + w\boldsymbol{n}. \tag{5.8}$$

With (5.8) the total potential shell energy can be defined by the functional

$$J(\boldsymbol{u}) = \iint_{\mathcal{M}} \Phi[\gamma_{\alpha\beta}(\boldsymbol{u}), \varkappa_{\alpha\beta}(\boldsymbol{u}), \lambda_{\xi,\alpha}(\boldsymbol{u}), \varkappa_{\xi,\alpha}(\boldsymbol{u}), \tau_{\xi,\alpha}(\boldsymbol{u})]\, dA - \iint_{\mathcal{M}} \boldsymbol{q}\cdot\boldsymbol{u}\, dA, \tag{5.9}$$

where the second term on the right side of (5.9) is the potential of the surface loads $\boldsymbol{q}(\theta^\alpha)$. For simplicity we assume homogeneous boundary conditions such that the contribution of given boundary forces to the potential energy (5.9) vanishes.

To derive the equilibrium equations by applying the stationary principle of total potential energy we determine the first Gâteaux differential

$$\begin{aligned} J^{(1)}(\boldsymbol{u}; \hat{\boldsymbol{u}}) = \iint_{\mathcal{M}} \{ & n^{\alpha\beta}(\boldsymbol{u})\, \gamma^{(1)}_{\alpha\beta}(\boldsymbol{u}; \hat{\boldsymbol{u}}) + m^{\alpha\beta}(\boldsymbol{u})\, \varkappa^{(1)}_{\alpha\beta}(\boldsymbol{u}; \hat{\boldsymbol{u}}) \\ & + h^\alpha(\boldsymbol{u})\, \lambda^{(1)}_{\xi,\alpha}(\boldsymbol{u}; \hat{\boldsymbol{u}}) + k^\alpha(\boldsymbol{u})\, \varkappa^{(1)}_{\xi,\alpha}(\boldsymbol{u}; \hat{\boldsymbol{u}}) \\ & + l^\alpha(\boldsymbol{u})\, \tau^{(1)}_{\xi,\alpha}(\boldsymbol{u}; \hat{\boldsymbol{u}})\}\, dA - \iint_{\mathcal{M}} \boldsymbol{q}\cdot\hat{\boldsymbol{u}}\, dA. \end{aligned} \tag{5.10}$$

It should be pointed out that the first Gâteaux differential of functions or functionals can be denoted also as their variation.

By partial integration and application of Gauss' divergence theorem we transform expression (5.10) into

$$\begin{aligned} J^{(1)}(\boldsymbol{u};\hat{\boldsymbol{u}}) = \iint_{\mathcal{M}} \{ & n^{\alpha\beta}(\boldsymbol{u})\,\gamma^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}) + m^{\alpha\beta}(\boldsymbol{u})\,\varkappa^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}) - h^{\alpha}|_{\alpha}\,(\boldsymbol{u})\,\lambda_{\xi}{}^{(1)}(\boldsymbol{u};\hat{\boldsymbol{u}}) \\ & - k^{\alpha}|_{\alpha}\,(\boldsymbol{u})\,\varkappa_{\xi}{}^{(1)}(\boldsymbol{u};\hat{\boldsymbol{u}}) - l^{\alpha}|_{\alpha}\,(\boldsymbol{u})\,\tau_{\xi}{}^{(1)}(\boldsymbol{u};\hat{\boldsymbol{u}})\}\,dA \\ & - \iint_{\mathcal{M}} \boldsymbol{q}\cdot\hat{\boldsymbol{u}}\,dA + \text{boundary term}. \end{aligned} \tag{5.11}$$

Next we have to determine the Gâteaux differentials of the invariants (5.4). Using the definitions

$$\begin{aligned} \frac{\bar{a}}{a} &= \frac{1}{2}\,\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}\bar{a}_{\alpha\beta}\bar{a}_{\lambda\mu}, \\ \bar{H} &= \frac{1}{2}\,\bar{a}^{\alpha\beta}\bar{b}_{\alpha\beta}, \\ \bar{K} &= \frac{1}{2}\,\bar{\varepsilon}^{\alpha\lambda}\bar{\varepsilon}_{\beta\mu}\bar{b}^{\beta}{}_{\alpha}\bar{b}_{\lambda}{}^{\mu}, \end{aligned} \tag{5.12}$$

we are able to prove the following formulas

$$\begin{aligned} \lambda_{\xi}{}^{(1)}(\boldsymbol{u};\hat{\boldsymbol{u}}) &= -\lambda_{\xi}(\boldsymbol{u})\,\bar{a}^{\alpha\beta}(\boldsymbol{u})\,\gamma^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}), \\ \varkappa_{\xi}{}^{(1)}(\boldsymbol{u};\hat{\boldsymbol{u}}) &= -2\lambda_{\xi}(\boldsymbol{u})\,[\bar{H}(\boldsymbol{u})\,\bar{a}^{\alpha\beta}(\boldsymbol{u}) + \bar{b}^{\alpha\beta}(\boldsymbol{u})]\,\gamma^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}) \\ &\quad - \lambda_{\xi}(\boldsymbol{u})\,\bar{a}^{\alpha\beta}(\boldsymbol{u})\,\varkappa^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}), \\ \tau_{\xi}{}^{(1)}(\boldsymbol{u};\hat{\boldsymbol{u}}) &= -8\lambda_{\xi}{}^{2}(\boldsymbol{u})\,\bar{K}(\boldsymbol{u})\,\bar{a}^{\alpha\beta}(\boldsymbol{u})\,\gamma^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}) \\ &\quad - 2\lambda_{\xi}{}^{2}(\boldsymbol{u})\,[2\bar{H}(\boldsymbol{u})\,\bar{a}^{\alpha\beta}(\boldsymbol{u}) - \bar{b}^{\alpha\beta}(\boldsymbol{u})]\,\varkappa^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}). \end{aligned} \tag{5.13}$$

According to (5.13) the Gâteaux differentials of the invariants (5.4) can be expressed as linear combinations of the Gâteaux differentials $\gamma^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}})$, $\varkappa^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}})$ of the strain measures (5.3).

Introducing (5.13) into (5.11) the differential of the total potential energy is obtained in the form

$$\begin{aligned} J^{(1)}(\boldsymbol{u};\hat{\boldsymbol{u}}) = \iint_{\mathcal{M}} & [N^{\alpha\beta}(\boldsymbol{u})\,\gamma^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}) + M^{\alpha\beta}(\boldsymbol{u})\,\varkappa^{(1)}_{\alpha\beta}(\boldsymbol{u};\hat{\boldsymbol{u}}) - \boldsymbol{q}\cdot\hat{\boldsymbol{u}}]\,dA \\ & + \text{boundary term} \end{aligned} \tag{5.14}$$

with a generalized stress resultant tensor $N^{\alpha\beta}$ and a generalized stress couple tensor $M^{\alpha\beta}$ defined by

$$\begin{aligned} N^{\alpha\beta} &= n^{\alpha\beta} + \lambda_{\xi}[\bar{a}^{\alpha\beta}h^{\mu}|_{\mu} + 2(\bar{H}\bar{a}^{\alpha\beta} + \bar{b}^{\alpha\beta})\,k^{\mu}|_{\mu} + 8\lambda_{\xi}\bar{K}\bar{a}^{\alpha\beta}\;l^{\mu}|_{\mu}], \\ M^{\alpha\beta} &= m^{\alpha\beta} + \lambda_{\xi}[\bar{a}^{\alpha\beta}k^{\mu}|_{\mu} + 2\lambda_{\xi}(2\bar{H}\bar{a}^{\alpha\beta} - \bar{b}^{\alpha\beta})\,l^{\mu}|_{\mu}]. \end{aligned} \tag{5.15}$$

The first Gâteaux differentials $\gamma^{(1)}_{\alpha\beta}(\boldsymbol{u}; \hat{\boldsymbol{u}})$ and $\varkappa^{(1)}_{\alpha\beta}(\boldsymbol{u}; \hat{\boldsymbol{u}})$ are presented in [11]. Following the procedure outlined there the energy differential (5.14) can be transformed into the equivalent expression

$$J^1(\boldsymbol{u}; \hat{\boldsymbol{u}}) = -\iint_{\mathscr{M}} \{[T^{\alpha\beta}(\boldsymbol{u})|_\beta - b_\beta{}^\alpha T^\beta(\boldsymbol{u}) + q^\alpha]\, \hat{u}_\alpha + [T^\beta(\boldsymbol{u})|_\beta + b_{\alpha\beta} T^{\alpha\beta}(\boldsymbol{u}) + q]\, \hat{w}\}\, dA + \text{boundary terms}, \tag{5.16}$$

which enables the derivation of the local equilibrium equations. The vector and tensor components T^α and $T^{\alpha\beta}$ are given in [1], [9]—[11] for the nonlinear shell theory undergoing small strains and unrestricted rotations. Only in the expressions given there the stress measures $N^{\alpha\beta}$ and $M^{\alpha\beta}$ have to be replaced by their generalizations according to (5.15) valid for large strain deformations.

We apply now the principle of stationary total potential energy stating that for arbitrary geometrically admissible superimposed deformations $\hat{\boldsymbol{u}}$ the first Gâteaux differential $J^{(1)}(\boldsymbol{u}; \hat{\boldsymbol{u}})$ vanishes

$$J^{(1)}(\boldsymbol{u}; \hat{\boldsymbol{u}}) = 0 \qquad \forall\, \hat{\boldsymbol{u}} \tag{5.17}$$

at the solution $\boldsymbol{u} = \bar{\boldsymbol{u}}$. Then (5.17) yields with (5.16) the three Lagrangean equilibrium equations for the large strain shell theory

$$\left.\begin{aligned} T^{\alpha\beta}(\boldsymbol{u})|_\beta - b_\beta{}^\alpha T^\beta(\boldsymbol{u}) + q^\alpha &= 0 \\ T^\beta(\boldsymbol{u})|_\beta + b_{\alpha\beta} T^{\alpha\beta}(\boldsymbol{u}) + q &= 0 \end{aligned}\right\} \text{ in } \mathscr{M}. \tag{5.18}$$

From (5.18) it follows that the equilibrium equations of the large strain shell theory are of similar structure as the equilibrium equations of the small strain shell theory with unrestricted rotations [1], [9]—[11]. Only according to (5.15) there are entering additional terms which are functions of the three invariants (5.4).

Up to here we have strictly avoided the introduction of additional assumptions concerning the magnitude of strains, rotations or other relevant parameters. It is obvious that whenever suitable restrictions are imposed on the magnitude of some parameters characterizing the geometry of the shell and/or its deformation a wide class of simplified shell models can be derived, which will be the subject of a forthcoming paper. To underline the importance of additional simplifying assumptions for the numerical applicability we introduce in the next section a Taylor expansion of the function $\zeta(\theta^\alpha, \xi)$.

6. Thin Shell Approximation with Example

Let $\varepsilon_h = \dfrac{h_0}{R}$ denote the thickness parameter of the shell with a maximum undeformed shell thickness h_0 and a minimum radius of curvature R of the undeformed reference surface $\mathscr{M}$. If $(\varepsilon_h)^N \ll 1$ for some positive integer numbers

N, the function $\zeta(\theta^\alpha, \xi)$ of (2.10) can be represented as Taylor expansion with respect to ξ

$$\zeta(\theta^\alpha, \xi) = \psi_1(\theta^\alpha)\, \xi + \frac{1}{2!}\, \psi_2(\theta^\alpha)\, \xi^2 + \dots + \frac{1}{(N-1)!}\, \psi_{N-1}(\theta^\alpha)\, \xi^{N-1} + O\big((\varepsilon_h)^N\big). \quad (6.1)$$

For isochoric deformations the coefficients of (6.1) have to be determined such that (3.3) is satisfied. Therefore we introduce (6.1) into (3.3) and differentiate successively with respect to ξ. Putting then $\xi = 0$ we obtain

$$\psi_1 = \lambda_\xi, \qquad \psi_2 = \lambda_\xi \varkappa_\xi, \qquad \psi_3 = -\lambda_\xi \tau_\xi + 3\lambda_\xi \varkappa_\xi(\varkappa_\xi + 2H), \dots \quad (6.2)$$

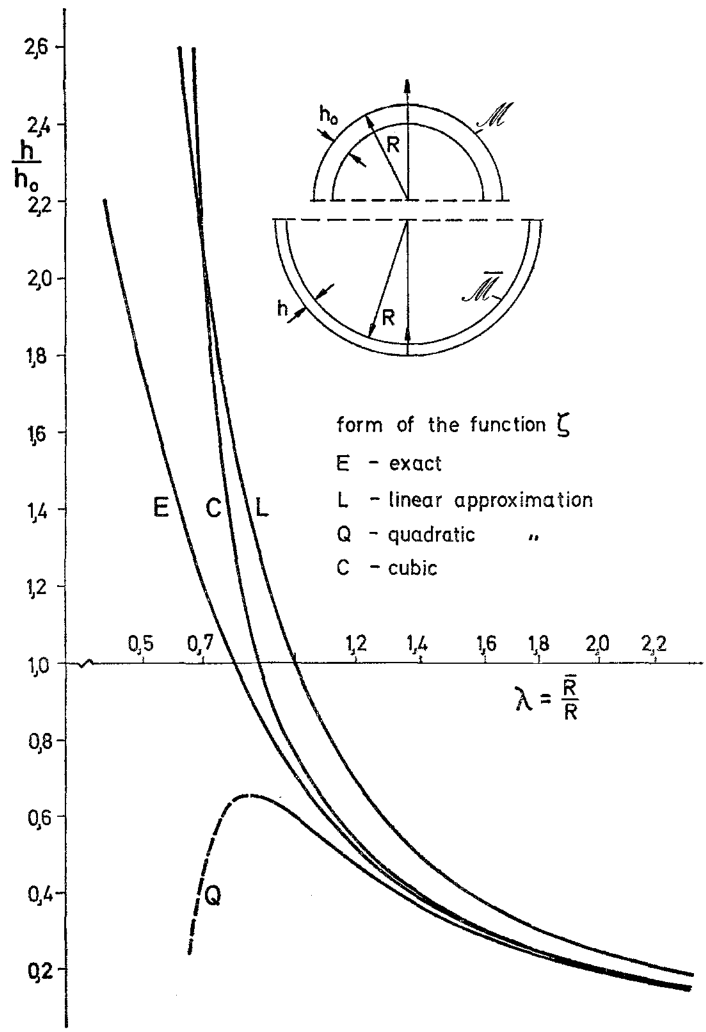

Fig. 3. Eversion of a spherical shell — change in shell thickness ($h_0/R = 0.2$)

and correspondingly the higher order coefficients $\psi_4, \ldots$, which are all functions of the three surface invariants $\lambda_\xi, \varkappa_\xi, \tau_\xi$.

With the Taylor expansion (6.1) and the known coefficients (6.2) the deformation of the shell can be represented in the form of a power series with respect to ξ.

As numerical application we consider the eversion of a spherical shell, for which a detailed description is given in [5]. If R and $\bar{R}$ denote the radii of the reference surface in the initial and deformed configuration, respectively, we have

$$H = -R^{-1}, \qquad K = R^{-2}, \qquad \bar{H} = \bar{R}^{-1}, \qquad \bar{K} = \bar{R}^{-2}, \qquad j = \lambda^2 = \left(\frac{\bar{R}}{R}\right)^2, \tag{6.3}$$

where λ represents the stretch of the reference surface. For simplicity we choose the outer shell surface in the initial configuration as reference surface (Fig. 3).

Inserting (6.3) into the exact function ζ according to (3.7) and (3.8) one gets

$$\frac{\zeta(\hat{\xi})}{h_0} = \frac{\lambda}{\varepsilon_h}\left\{1 - \left(1 - \frac{1}{\lambda^3}\left[(1 + \varepsilon_h\hat{\xi})^3 - 1\right]\right)^{1/3}\right\}, \tag{6.4}$$

with $\varepsilon_h = \dfrac{h_0}{R}$ and $\hat{\xi} = \dfrac{\xi}{h_0} \in [-1, 0]$. With (6.4) we calculate the change in the shell thickness h/h_0 and the transverse normal strains Λ_ξ yielding

$$\frac{h}{h_0} = \frac{1}{h_0}[\zeta(0) - \zeta(-1)] = -\frac{\lambda}{\varepsilon_h}\left\{1 - \left(1 - \frac{1}{\lambda^3}\left[(1 - \varepsilon_h)^3 - 1\right]\right)^{1/3}\right\}, \tag{6.5}$$

$$\Lambda_\xi(\hat{\xi}) = \frac{\partial\zeta}{\partial\xi} = \frac{(1 + \varepsilon_h\hat{\xi})^2}{[1 + \lambda^3 - (1 + \varepsilon_h\hat{\xi})^3]^{2/3}}. \tag{6.6}$$

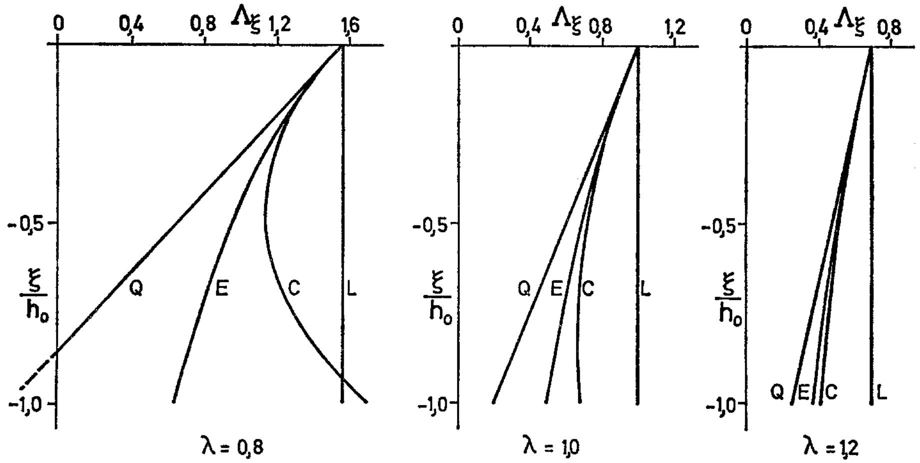

Fig. 4. Eversion of a spherical shell — distribution of transverse normal strains over shell thickness ($h_0/R = 0.2$)

The results are represented in Fig. 3 and 4, where they are indicated by E (exact). To compare them with those, which can be obtained by using for ζ the first terms of the Taylor expansion (6.1) we introduce (6.3) into (6.1) and (6.2), respectively. Figs. 3 and 4 show the results due to a linear (L), quadratic (Q) and cubic (C) approximation.

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