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Crack Propagation in a Laminated Composite Material Modeled by a Two-Dimensional Mixture Theory

By

Y. Benveniste and **J. Aboudi**, Tel-Aviv, Israel

With 4 Figures

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Summary — Zusammenfassung

Crack Propagation in a Laminated Composite Material Modeled by a Two-Dimensional Mixture Theory. A two-dimensional mixture theory is developed for wave-propagation in a laminated composite material. The formulated theory is applied for the treatment of a semi-infinite crack propagating in the composite in mode-I and II types of motion. The maximum velocity of the crack tip is determined by the minimum value of the dispersion curves of the relevant generalized Rayleigh waves in the medium. A numerical procedure is applied for the determination of the dynamic stress fields induced by the propagating crack in a laminated composite made of glass-epoxy layers.

Rißausbreitung in einem geschichteten kompositen Material, beschrieben durch eine zweidimensionale Mischungstheorie. Eine zweidimensionale Mischungstheorie wird für die Wellenausbreitung in einem geschichteten kompositen Material entwickelt. Die formulierte Theorie wird für die Behandlung einer halbunendlichen Rißausbreitung im Komposit in Mode-I und II Bewegungsformen angewendet. Das Geschwindigkeitsmaximum der Rißspitze wird durch den minimalen Wert der Dispersionskurve der entsprechenden verallgemeinerten Rayleigh-Wellen im Medium bestimmt. Ein numerisches Verfahren wird angewendet für die Bestimmung der dynamischen Spannungsfelder hervorgerufen durch den ausbreitenden Riß in einem geschichteten Komposit aus Glas-Epoxy-Lagen.

Introduction

There are several papers concerning the propagation of cracks in homogeneous media. A recent review by Achenbach [1] presents the mathematical methods for the analysis of elasto-dynamic stress fields near propagating cracks as well as a list of references for several investigations related to the subject.

The problem of crack propagation in composite materials is much more complicated than the corresponding problem in homogeneous media due to the existence of constituent interfaces. Accordingly, there are only some works which treat crack problems in a composite body made of two or three layers only; see for example Aboudi [2] and the references cited there for a crack propagating along the interface between two dissimilar half-spaces, and Atkinson [3] for a crack propagating across the interface. The problem of the diffraction of anti-plane harmonic shear waves by a crack in a layered composite made of a single layer bonded to two identical half-spaces has been treated in references [4] and [5].

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By modeling the multiphase medium while taking into account its microstructure it is possible to treat crack propagation in such media. In this paper we develop a *two-dimensional* mixture theory for a bi-laminated composite medium, which, by incorporating two displacement components makes possible to treat the three-dimensional elasto-dynamic problem of a semi-infinite crack propagating in the direction of the laminates while intersecting all the interfaces simultaneously. The microstructure effects in this mixture model are taken into account by allowing every constituent to have its own motion while interacting with the other.

In Section I we present the mixture theory which, by means of an averaging process, replaces the bi-laminated composite material in its three-dimensional motion by a two-dimensional binary mixture model. The resulting system of four dynamic equations of motion are coupled by means of terms which express the interaction between the constituents. A similar two-dimensional mixture theory for wave propagation from a cylindrical cavity in a laminated medium has been formulated in [6]. For other forms of mixture theories see a recent review by Atkin and Craine [7].

In Section II of this paper the problem of a semi-infinite crack propagating in the direction of the layering is formulated in the framework of the developed mixture theory for mode-I as well as for mode-II type of crack motion. In treating such a problem of crack propagation the question of the velocity of the crack tip has to be investigated. For cracks propagating in a homogeneous medium in-plane motion, the Rayleigh wave speed forms the upper limit for the velocity of the crack [8]; for an interfacial crack this limit is determined by the smallest value of the Rayleigh waves in the two media, [9], [10]. The situation is much more complicated in the case of a composite material containing many interfaces. In this paper we propose to determine the maximum crack velocity by the minimum value of the dispersion curves of the relevant generalized Rayleigh waves in the laminated medium. The dispersion curves of these Rayleigh waves are developed in the framework of the formulated mixture theory and they are given for glass-epoxy laminates for the case of two reinforcement ratios.

The method of solution is numerical and is essentially similar to a previous numerical treatment of crack propagation along the interface of two dissimilar media [2].

Results are given for the normal stress in every constituent as well as for the overall average normal stress, for the case of a crack propagating in the mode-I type of motion in a glass-epoxy laminated medium. The stress intensity factor can be extracted by using the similar method exployed in [2].

Two-Dimensional Mixture Theory

Consider a periodic array of two alternating isotropic linearly elastic layers of widths $2h_1$ and $2h_2$ respectively. Let z_α be a local coordinate measured from the midplane of each layer (see. Fig. 1a)[1].

[1] In the sequel the superscript or the subscript α will take the values 1 and 2 and will indicate that the quantities belong to either one of the constituents.

The equations of motion in each layer are given by:

$$\sigma_{ij,j}^{(\alpha)} = \varrho_\alpha \frac{\partial^2}{\partial t^2} u_i^{(\alpha)}, \qquad i = x, y, z \tag{1—3}$$

where the Eqs. (1—3) refer to the equations of motion in the x, y and z directions respectively and $\phi_{,j}$ refer to the derivatives with respect to x and y for $j = x$ and y, and to derivative with respect to z_α for $j = z$. In these equations $\sigma_{ij}^{(\alpha)}$, $u_j^{(\alpha)}$ are the stresses and displacements respectively, ϱ_α is the density of each constituent, and t is the time.

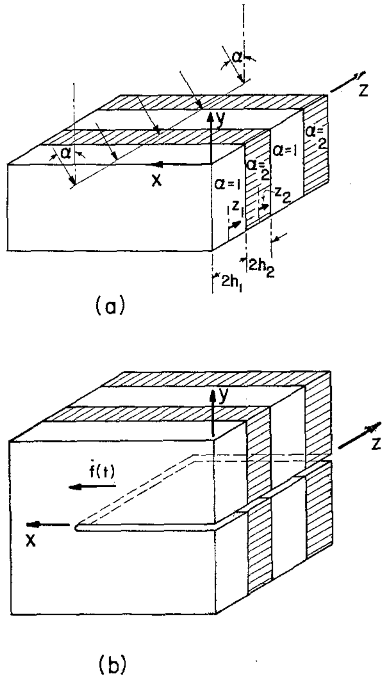

Fig 1. a) A laminated half-space subjected to a oblique line load. b) A semi-infinite plane crack propagating in a laminated composite medium in the positive x-direction

The constitutive equations are

$$\sigma_{ij}^{(\alpha)} = \delta_{ij}\lambda_\alpha e^{(\alpha)} + \mu_\alpha(u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)}) \tag{4—9}$$

where these equations refer to $\sigma_{xx}^{(\alpha)}$, $\sigma_{yy}^{(\alpha)}$, $\sigma_{zz}^{(\alpha)}$, $\sigma_{xy}^{(\alpha)}$, $\sigma_{xz}^{(\alpha)}$, $\sigma_{yz}^{(\alpha)}$ respectively, δ_{ij} is the Kronecker delta,

$$e^{(\alpha)} = u_{i,i}^{(\alpha)}, \tag{10}$$

and λ_α and μ_α are the Lamé parameters of the constituents.

Let us now consider loading situations of the composite medium in which the $u_z^{(\alpha)}$ displacements are anti-symmetric and the $u_x^{(\alpha)}$ and $u_y^{(\alpha)}$ displacements are symmetric with respect to the midplanes of the layers. Note that this motion is quasi-plane in the sense that the averaged motion in the z-direction is vanishing.

Such a motion can be caused, for instance, when a laminated half-space, with the surface perpendicular to the layering, (see Fig. 1a) is impacted obliquely by an uniform infinite line load perpendicular to the interfaces of the layers.

We will now develop a microstructure theory which will model the laminated composite in this quasi-plane motion. Let us define the average quantity

$$\bar{\psi}^{(\alpha)}(x, y, t) = (1/h_\alpha) \int_0^{h_\alpha} \psi^{(\alpha)}(x, y, z_\alpha, t)\, dz_\alpha \tag{11}$$

and the partial stresses and densities as:

$$\sigma_{\beta\gamma}^{(\alpha p)} = n_\alpha \bar{\sigma}_{\beta\gamma}^{(\alpha)}, \qquad \beta, \gamma = x, y, \qquad \varrho_{\alpha p} = n_\alpha \varrho_\alpha \tag{12}$$

where

$$n_\alpha = h_\alpha / h \tag{13}$$

with

$$h = h_1 + h_2.$$

We note that the stresses $\sigma_{xx}^{(\alpha)}$, $\sigma_{yy}^{(\alpha)}$, $\sigma_{xy}^{(\alpha)}$, $\sigma_{zz}^{(\alpha)}$ are symmetric and the stresses $\sigma_{xz}^{(\alpha)}$, $\sigma_{yz}^{(\alpha)}$ are anti-symmetric with respect to the midplanes, this being due to the stated symmetry and anti-symmetry of the displacements. Taking the average of Eqs. (1) and (2) according to (11), using the continuity conditions of the stresses $\sigma_{xz}^{(\alpha)}$ and $\sigma_{yz}^{(\alpha)}$ at the interfaces and their anti-symmetry properties we obtain the following equations of motion:

$$\frac{\partial}{\partial x} \sigma_{xx}^{(\alpha p)} + \frac{\partial}{\partial y} \sigma_{xy}^{(\alpha p)} - \varrho_{\alpha p} \frac{\partial^2}{\partial t^2} \bar{u}_x^{(\alpha)} = q_\alpha R(x, y, t) \tag{14}$$

$$\frac{\partial}{\partial x} \sigma_{xy}^{(\alpha p)} + \frac{\partial}{\partial y} \sigma_{yy}^{(\alpha p)} - \varrho_{\alpha p} \frac{\partial^2}{\partial t^2} \bar{u}_y^{(\alpha)} = q_\alpha P(x, y, t) \tag{15}$$

where

$$[R(x, y, t)]\, h = \sigma_{xz}^{(1)}(x, y, -h_1, t) = \sigma_{xz}^{(2)}(x, y, h_2, t) = -\sigma_{xz}^{(1)}(x, y, h_1, t) \tag{16}$$

$$[P(x, y, t)]\, h = \sigma_{yz}^{(2)}(x, y, h_2, t) = \sigma_{yz}^{(1)}(x, y, -h_1, t) = -\sigma_{yz}^{(1)}(x, y, h_1, t) \tag{17}$$

and

$$q_1 = 1, \qquad q_2 = -1. \tag{18}$$

As it is seen, Eqs. (14) and (15) are in standard binary mixture form with R and P being the interaction terms. By means of an asymptotic method we will now develop constitutive equations relating the partial stresses $\sigma_{xx}^{(\alpha p)}$, $\sigma_{yy}^{(\alpha p)}$, $\sigma_{xy}^{(\alpha p)}$ to the average displacement gradients and furthermore expressions relating P and R to the mixture variables (average displacements and partial stresses) will be derived.

Let us average Eqs. (4—7) according to (11). We get:

$$\left[(\bar{\sigma}_{xx}^{(\alpha)}/\lambda_\alpha) - (E_\alpha/\lambda_\alpha)\frac{\partial}{\partial x}\bar{u}_x^{(\alpha)} - \frac{\partial}{\partial y}\bar{u}_y^{(\alpha)}\right] n_\alpha = -q_\alpha S(x, y, t) \tag{19}$$

$$\left[(\bar{\sigma}_{yy}^{(\alpha)}/\lambda_\alpha) - (E_\alpha/\lambda_\alpha)\frac{\partial}{\partial y}\bar{u}_y^{(\alpha)} - \frac{\partial}{\partial x}\bar{u}_x^{(\alpha)}\right] n_\alpha = -q_\alpha S(x, y, t) \tag{20}$$

$$\left[(\sigma_{zz}^{(\alpha)}/E_\alpha) - (\lambda_\alpha/E_\alpha)\left(\frac{\partial}{\partial x}\bar{u}_x^{(\alpha)} + \frac{\partial}{\partial y}\bar{u}_y^{(\alpha)}\right)\right] n_\alpha = -q_\alpha S(x, y, t) \tag{21}$$

$$\sigma_{xy}^{(\alpha p)} = n_\alpha \mu_\alpha \left(\frac{\partial}{\partial y}\bar{u}_x^{(\alpha)} + \frac{\partial}{\partial x}\bar{u}_y^{(\alpha)}\right) \tag{22}$$

where

$$[S(x, y, t)]\, h = u_x^{(2)}(x, y, h_2, t) = u_z^{(1)}(x, y, -h_1, t) = -u_z^{(1)}(x, y, h_1, t) \tag{23}$$

and

$$E_\alpha = \lambda_\alpha + 2\mu_\alpha . \tag{24}$$

Note that the anti-symmetry properties of $u_z^{(\alpha)}$ have been employed in Eq. (23).

Obtaining an expression for $S(x, y, t)$ in terms of the mixture variables will be done by means of an asymptotic method and while neglecting terms of order ε^2 where $\varepsilon = (h_1 + h_2)/l$ with l being a characteristic wave length. If the displacements are expanded in terms of z_α while utilizing their symmetry properties and the obtained series are introduced in Eq. (6) the following expansion for $\sigma_{zz}^{(\alpha)}$ is obtained (see [11]):

$$\sigma_{zz}^{(\alpha)} = \bar{\sigma}_{zz}^{(\alpha)}[1 + 0(\varepsilon^2)]. \tag{25}$$

The continuity of $\sigma_{zz}^{(\alpha)}$ across the interfaces, up to the order of approximation we are concerned in this paper, then implies:

$$\bar{\sigma}_{zz}^{(1)} \cong \bar{\sigma}_{zz}^{(2)}. \tag{26}$$

Using Eq. (26) together with (21) provides an equation for $S(x, y, t)$ in terms of the mixture variables:

$$S(x, y, t) = (\lambda_1/E)\left(\frac{\partial}{\partial x}\bar{u}_x^{(1)} + \frac{\partial}{\partial y}\bar{u}_y^{(1)}\right) - (\lambda_2/E)\left(\frac{\partial}{\partial x}\bar{u}_x^{(2)} + \frac{\partial}{\partial y}\bar{u}_y^{(2)}\right) \tag{27}$$

where

$$E = (E_1/n_1) + (E_2/n_2).$$

Substituting Eq. (27) in Eqs. (19) and (20) and using the definition of the partial stresses give:

$$\sigma_{xx}^{(1p)} = c_{11}\frac{\partial}{\partial x}\bar{u}_x^{(1)} + c_{12}\left(\frac{\partial}{\partial x}\bar{u}_x^{(2)} + \frac{\partial}{\partial y}\bar{u}_y^{(2)}\right) + d_1\frac{\partial}{\partial y}\bar{u}_y^{(1)}, \tag{28}$$

$$\sigma_{xx}^{(2p)} = c_{22}\frac{\partial}{\partial x}\bar{u}_x^{(2)} + c_{12}\left(\frac{\partial}{\partial x}\bar{u}_x^{(1)} + \frac{\partial}{\partial y}\bar{u}_y^{(1)}\right) + d_2\frac{\partial}{\partial y}\bar{u}_y^{(2)}, \tag{29}$$

$$\sigma_{yy}^{(1p)} = c_{11}\frac{\partial}{\partial y}\bar{u}_y^{(1)} + c_{12}\left(\frac{\partial}{\partial y}\bar{u}_y^{(2)} + \frac{\partial}{\partial x}\bar{u}_x^{(2)}\right) + d_1\frac{\partial}{\partial x}\bar{u}_x^{(1)}, \tag{30}$$

$$\sigma_{yy}^{(2p)} = c_{22}\frac{\partial}{\partial y}\bar{u}_y^{(2)} + c_{12}\left(\frac{\partial}{\partial y}\bar{u}_y^{(1)} + \frac{\partial}{\partial x}\bar{u}_x^{(1)}\right) + d_2\frac{\partial}{\partial x}\bar{u}_x^{(2)}, \tag{31}$$

where

$$\left.\begin{aligned} c_{\alpha\alpha} &= [n_\alpha E_\alpha - (\lambda_\alpha^2/E)] \\ c_{\alpha\beta} &= \lambda_\alpha \lambda_\beta/E \\ d_\alpha &= [n_\alpha \lambda_\alpha - (\lambda_\alpha^2/E)] \end{aligned}\right\} \text{ with } \alpha, \beta = 1, 2 \text{ and } \alpha \neq \beta. \tag{32}$$

Eqs. (28)—(31) together with Eqs. (22) constitute the mixture constitutive equations.

The expressions relating P and R to the mixture variables will be obtained by using the procedure followed in [11]. (See this reference for details.) Let us first obtain the expression for R. Multiplying Eq. (8) by z_α, expanding in powers of z_α and intergrating by parts gives:

$$u_x^{(\alpha)}(x, y, h_\alpha, t) - \bar{u}_x^{(\alpha)} + (h_\alpha/3)\left[\frac{\partial}{\partial x} u_z^{(\alpha)}(x, y, h_\alpha, t) - (1/\mu_\alpha)\, \sigma_{xz}^{(\alpha)}(x, y, h_\alpha, t)\right] = 0. \tag{33}$$

When the definition of the interaction terms R and S in Eqs. (16) and (23) respectively are used in (33) we obtain:

$$u_x^{(1)}(x, y, h_1, t) - \bar{u}_x^{(1)} + (h_1 h/3)\left[-\frac{\partial}{\partial x} S + (1/\mu_1)\, R\right] = 0, \tag{34}$$

$$u_x^{(2)}(x, y, h_2, t) - \bar{u}_x^{(2)} + (h_2 h/3)\left[\frac{\partial}{\partial x} S - (1/\mu_2)\, R\right] = 0. \tag{35}$$

Employing the continuity condition of the displacements $u_x^{(\alpha)}$ across the interfaces and subtracting Eq. (35) from (34) furnishes:

$$\begin{aligned} R = (K/h^2)\,(\bar{u}_x^{(1)} - \bar{u}_x^{(2)}) + M\Bigg[(\lambda_1/E)\left(\frac{\partial^2}{\partial x^2}\bar{u}_x^{(1)} + \frac{\partial^2}{\partial x\,\partial y}\bar{u}_y^{(1)}\right) \\ - (\lambda_2/E)\left(\frac{\partial^2}{\partial x^2}\bar{u}_x^{(2)} + \frac{\partial^2}{\partial x\,\partial y}\bar{u}_y^{(2)}\right)\Bigg] \end{aligned} \tag{36}$$

where

$$\begin{aligned} K &= [(n_1/3\mu_1) + (n_2/3\mu_2)]^{-1}, \\ M &= K/3. \end{aligned} \tag{37}$$

The derivation of the expression for interaction term P can be carried out by applying the same steps this time to Eq. (7). For the sake of brievity we only give the result:

$$\begin{aligned} P = (K/h^2)\,(\bar{u}_y^{(1)} - \bar{u}_y^{(2)}) + M[(\lambda_1/E)\left(\frac{\partial^2}{\partial y^2}\bar{u}_y^{(1)} + \frac{\partial^2}{\partial x\,\partial y}\bar{u}_x^{(1)}\right) \\ - (\lambda_2/E)\left(\frac{\partial^2}{\partial y^2}\bar{u}_y^{(2)} + \frac{\partial^2}{\partial x\,\partial y}\bar{u}_x^{(2)}\right). \end{aligned} \tag{38}$$

Substitution of the constitutive Eqs. (19), (20), (22) and the Eqs. (36) and (38) into the equations of motion (14) and (15) give the following displacements equations of motion:

$$\frac{\partial^2}{\partial t^2} \boldsymbol{U} = \boldsymbol{A} \frac{\partial^2}{\partial x^2} \boldsymbol{U} + \boldsymbol{B} \frac{\partial^2}{\partial y^2} \boldsymbol{U} + \boldsymbol{C} \frac{\partial^2}{\partial x \, \partial y} \boldsymbol{U} + \boldsymbol{D}\boldsymbol{U} \tag{39}$$

where the vector $\boldsymbol{U}$, and the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are given in Appendix I.

It can readily be checked that the two-dimensional mixture equations of motion which describe the quasi-plane motion of the composite decouple into two sets of equations in the special cases in which there's no dependance either on x or on y. When $\frac{\partial}{\partial y} = 0$ for instance, the first set of equations involve only $\overline{u}_x^{(1)}$ and $\overline{u}_x^{(2)}$ and they model the effectively one-dimensional quasi-longitudinal wave propagation in a laminated composite. These equations have been obtained in Ref. [11]. The second set of equations on the other hand involve only $u_y^{(1)}$ and $u_y^{(2)}$ and they describe the horizontally polarized shear motions (SH) of the laminated medium. These equations are again one-dimensional and have been given in [12]. It should be noticed that the special case of a homogeneous isotropic material can be obtained by choosing equal material constants and densities.

Crack Propagation in a Laminated Composite

In this paper we will use the developed mixture theory in order to treat the problem of crack propagation in a laminated composite. Let us consider a semi-infinite plane crack in the xz-plane ($y = 0$) propagating in the positive x-direction (see Fig. 1b). The location of the crack tip relative to the origin of the stationary coordinates is described by the function $x = f(t)$ which is an arbitrary function of time subject to the condition that $\frac{d^2}{dt^2} f(t)$ exists.

Let us treat the following two modes of crack propagation: in the first mode the surface of the crack is subjected to a tensile loading in the y-direction whereas in the second mode there's a shear loading in the x-direction. In both cases the loading is uniform in the z-direction and the medium is assumed to be initially at rest. The appropriate boundary conditions on the plane of the crack $y = 0$ are:

$$\left.\begin{aligned} &\sigma_{yx}^{(\alpha)}(x, \pm 0, z_\alpha, t) = 0, \quad \sigma_{yz}^{(\alpha)}(x, \pm 0, z_\alpha, t) = 0 && -\infty < x < \infty \\ &\sigma_{yy}^{(\alpha)}(x, \pm 0, z_\alpha, t) = g_1(x, t) && -\infty < x \leqq f(t) \\ &u_y^{(\alpha)}(x, 0, z_\alpha, t) = 0 && x > f(t) \end{aligned}\right\} \tag{40}$$

for mode-I type of motion, and

$$\left.\begin{aligned} &\sigma_{yy}^{(\alpha)}(x, \pm 0, z_\alpha, t) = 0, \quad \sigma_{yz}^{(\alpha)}(x, \pm 0, z_\alpha, t) = 0 && -\infty < x < \infty \\ &\sigma_{yx}^{(\alpha)}(x, \pm 0, z_\alpha, t) = g_2(x, t) && -\infty < x \leqq f(t) \\ &u_x^{(\alpha)}(x, 0, z_\alpha, t) = 0 && x > f(t) \end{aligned}\right\} \tag{41}$$

for mode-II. In the above equations $g_1(x, t)$, $g_2(x, t)$ are prescribed loading functions which determine the driving mechanism of the crack.

It is clear that with the above loading conditions the displacements $u_z^{(\alpha)}$ are anti-symmetric and the displacements $u_x^{(\alpha)}$, $u_y^{(\alpha)}$ are symmetric with respect to the midplanes of the layers and the developed mixture theory can be implemented.

In the framework of the two-dimensional mixture theory, the three-dimensional motion induced by the boundary conditions (40) and (41) reduces to a two-dimensional one due to the averaging process. The appropriate boundary conditions will be as those in Eqs. (40) and (41), where this time the partial stresses and the average displacements will be involved.

In order to implement the above theory for crack propagation in a laminated composite, the extension velocity of the tip of the crack $\dot{f}(t) = \frac{d}{dt} f(t)$ needs to be prescribed. For the case of crack propagation in an isotropic medium the limiting velocity that the crack tip can attain is obtained through energy considerations [8]. It turns out that the speed of Rayleigh waves is the largest velocity with which the crack can propagate.

For a crack propagating along the interface of two semi-infinite homogeneous media, the maximum velocity of the crack tip should be smaller than the Rayleigh wave speeds in the two half-spaces (see Refs. [9], [10]). For the present case of a crack propagating in a laminated composite material an exact derivation for the maximum speed of the crack seems to be very difficult and to the knowledge of the authors no such attempt appears in the literature. We propose in this paper to determine the upper limit of the crack velocity in the composite by the minimum value of the dispersion curves for the Rayleigh waves propagating in the laminated half-space. The relevant Rayleigh waves are those which propagate in the half-space $y \geqq 0$ in the x-direction (see Fig. 1b). Those Rayleigh waves are three-dimensional in nature and consequently their analysis based on exact elasticity should be extremely difficult. Obviously in the framework of the mixture theory those waves are independent of the z-coordinate and we are able to treat them as two-dimensional disturbances. Our motivation to bound the crack velocity by the minimum of the dispersion curves is based on the well known phenomenon that the Rayleigh wave speed for the problem of a moving load on a half-space forms the critical velocity at which resonance phenomena occur [13]. In the case of a moving load on a layered half-space the Rayleigh waves become generalized Rayleigh waves whose speed depend on the frequency and consequently the resonance effects occur at the minimum point of the dispersion curves, see Achenbach, et al. [14], for a moving load on a specific type of a layered half-space.

Based on these arguments we will construct now the frequency equation for the generalized Rayleigh waves for our laminated half-space as described above. A plane harmonic wave propagating in the x-direction and decaying in the y-direction is given by:

$$\boldsymbol{U} = \boldsymbol{L}(y) \exp\left[ik(x - ct)\right], \tag{42}$$

where k is the wave number, c is the phase velocity and $\boldsymbol{L}(y)$ is the amplitude function. Substituting Eq. (42) in (39) we obtain:

$$\boldsymbol{B} \frac{\partial^2}{\partial y^2} \boldsymbol{L} + ik\boldsymbol{C} \frac{\partial}{\partial y} \boldsymbol{L} + \boldsymbol{J} = 0, \tag{43}$$

where

$$J = D - k^2 A + k^2 c^2 I \quad \text{and } I \text{ is the unit matrix.}$$

Substituting in Eq. (43) a solution of the form

$$L = G \exp(\lambda y)$$

we obtain the equation

$$FG \exp(\lambda y) = 0 \tag{44}$$

with

$$F = B^2 \lambda^2 + ik\lambda C + J. \tag{45}$$

For a non-trivial solution we get the characteristic equation

$$\det F = 0. \tag{46}$$

This equation turns out to be a fourth order algebraic equation in λ^2 whose negative real roots λ should be selected in order to obtain a decaying surface wave. The condition for the existence of Rayleigh waves for a given k is the existence of four real and negative roots λ. If Eq. (46) has four negative real roots λ, then the solution for L is given by:

$$L = \sum_{j=1}^{4} G_1^{(j)} Q^{(j)} \exp(\lambda_j y) \tag{47}$$

with

$$Q^{(j)} = \begin{bmatrix} 1 \\ iQ_2^{(j)} \\ Q_3^{(j)} \\ iQ_4^{(j)} \end{bmatrix}$$

and where $Q_m^{(j)}$ $(m = 2, 3, 4)$ are complicated expressions of the roots λ_j $(j = 1, 2, 3, 4)$.

Note that in Eq. (47) there are four unknowns $G_1^{(j)}$ $(j = 1, 2, 3, 4)$ to be determined by the four boundary conditions $\sigma_{yy}^{(\alpha p)} = 0$, $\sigma_{yx}^{(\alpha p)} = 0$ $(\alpha = 1, 2)$ at the free surface $y = 0$. The condition for the nontrivial solution of $G_1^{(j)}$ furnishes the desired frequency equation:

$$\det(\Delta) = 0, \tag{48}$$

where the explicit form of the matrix Δ is given in the Appendix II. This frequency equation determines the phase velocity c for a given frequency k. In Fig. 2 the phase velocity c/c_g with $c_g = (\mu_1/\varrho_1)^{1/2}$ (the shear wave speed in medium 1), is plotted versus the non-dimensional wave number kh for the two reinforcement ratios $h_1/h = 0.3$ and $h_1/h = 0.8$ for a glass-epoxy laminated composite whose material constants are given by:

$\lambda_1 = 1.99 \times 10^{11}$ dynes/cm^2 $\quad \mu_1 = 3.03 \times 10^{11}$ dynes/cm^2 $\quad \varrho_1 = 2.54$ gm/cm^3

$\lambda_2 = 2.96 \times 10^{10}$ dynes/cm^2 $\quad \mu_2 = 1.24 \times 10^{10}$ dynes/cm^2 $\quad \varrho_2 = 1.18$ gm/cm^3

where the subscript "1" stands for glass and the subscript "2" for epoxy.

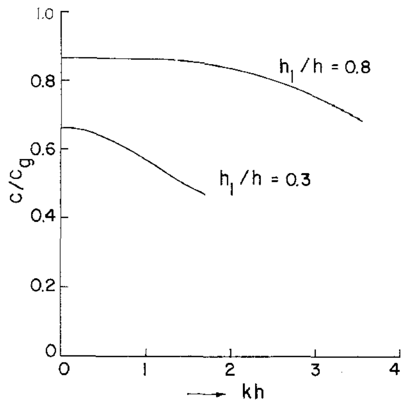

Fig. 2. The lowest mode of the dispersion curves of the generalized Rayleigh waves in a laminates half-space made of glass-epoxy layers with reinforcement ratios $h_1/h = 0.3$ and $h_1/h = 0.8$

Method of Solution

The method of solution of the system of equations of motion (39) together with their boundary conditions is numerical and is basically the same as that described previously in Ref. [2] where the problem of a crack propagating along the interfaces of two dissimilar half-spaces in contract was treated.

Let (ξ, η, ζ) be a system of moving coordinates whose origin is located at the tip of the crack. The moving and stationary coordinates are related by:

$$\left.\begin{aligned} \xi &= x - f(t) \\ \eta &= y \\ \zeta &= z \end{aligned}\right\} \tag{49}$$

The numerical treatment is applied to the transformed equations of motion in the moving coordinate system. In this formulation the moving tip of the crack appears always at $\xi = 0$. In terms of the moving coordinate system the equations of motion (39) take the form:

$$\frac{\partial^2}{\partial t^2} \boldsymbol{U} = \boldsymbol{A}' \frac{\partial^2}{\partial \xi^2} \boldsymbol{U} + \boldsymbol{B} \frac{\partial^2}{\partial \eta^2} \boldsymbol{U} + \boldsymbol{C} \frac{\partial^2}{\partial \xi\, \partial \eta} \boldsymbol{U} + \boldsymbol{D}\boldsymbol{U} + 2\dot{f}(t) \frac{\partial^2}{\partial \xi\, \partial t} \boldsymbol{U} + \ddot{f}(t) \frac{\partial}{\partial \xi} \boldsymbol{U} \tag{50}$$

where $\boldsymbol{A}' = \boldsymbol{A} - \dot{f}(t)^2 \boldsymbol{I}$ and $\boldsymbol{I}$ is the unit matrix.

As in [2] the resulting finite difference numerical scheme is implicit (in the sense that more than one grid point at the advanced time level is involved) and of three level (so that it is possible to compute the displacement vector at the advanced time from its values at the previous steps) and its accuracy is of second order. The boundary conditions for the average displacement and partial stresses are imposed implicitly as in [2] yielding a system of algebraic equations in the unknown displacements at the boundary, and the Gauss-Seidel iteration procedure was employed for their solution, see [2] for details. The reliability of the numerical procedure was assessed in [2] by comparison of the numerical solutions obtained in several situations in which some analytical results are known, and satisfactory agreement was obtained.

Results

In this section results are given for mode-I type of crack propagation in a laminated composite mode of glass-epoxy layers whose material constants were given previously. The loading function in (40) is given by:

$$g_1(x, t) = \sigma_0 H(x) \tag{51}$$

with $H(x)$ being the Heaviside step function. We choose to exhibit the partial stresses in each constituent as well as the overall average stress $\bar{\sigma}_{yy} = \sigma_{yy}^{(1p)} + \sigma_{yy}^{(2p)}$ at the surface of the crack $y = 0$. In every case the velocity of the propagating crack is chosen to be constant and given by $\dot{f}(t)/c_e = 0.6$ where c_e is the shear wave speed in the epoxy. In terms of the shear wave speed in the glass as exhibited in Fig. 2 this value corresponds to $\dot{f}(t)/c_g = 0.178$ which is smaller than the smallest value of the phase velocities which appear in this figure.

In Fig. 3 the normal stresses at the plane of the crack $y = 0$ are given versus the distance x at two different times $\tau = C_g t/h = 0.5$, 1 where $C_g = [(\lambda_1 + 2\mu_1)/\varrho_1]^{1/2}$ which is the longitudinal wave speed in the glass. In this figure the reinforcement ratio is $h_1/h = 0.3$. In Fig. 4 the same results are given, this time for the reinforcement ratio $h_1/h = 0.8$.

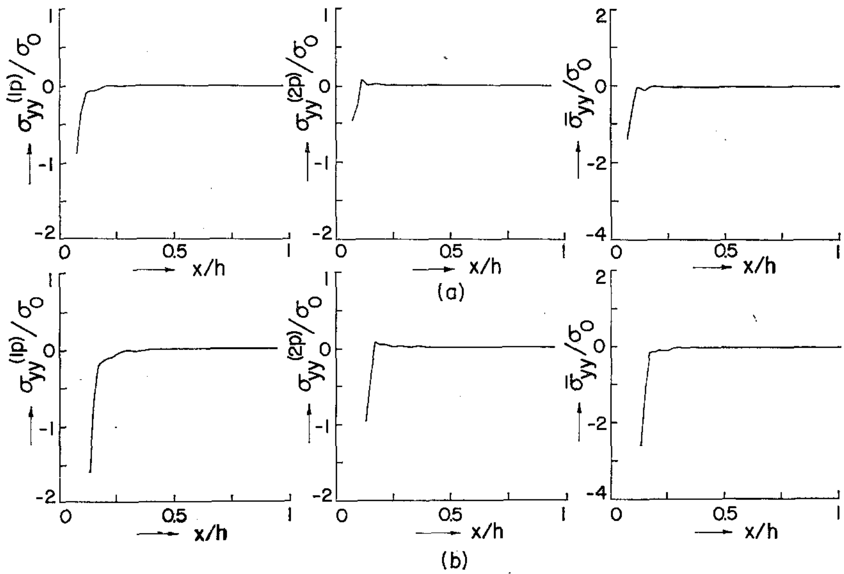

Fig. 3. The partial stresses $\sigma_{yy}^{(1p)}$, $\sigma_{yy}^{(2p)}$ and the overall average stress $\bar{\sigma}_{yy}$ versus the distance x/h at the plane of the crack $y = 0$ when a) $\tau = 0.5$ and b) $\tau = 1.0$. The composite is made of glass-epoxy layers with reinforcement ratio $h_1/h = 0.3$

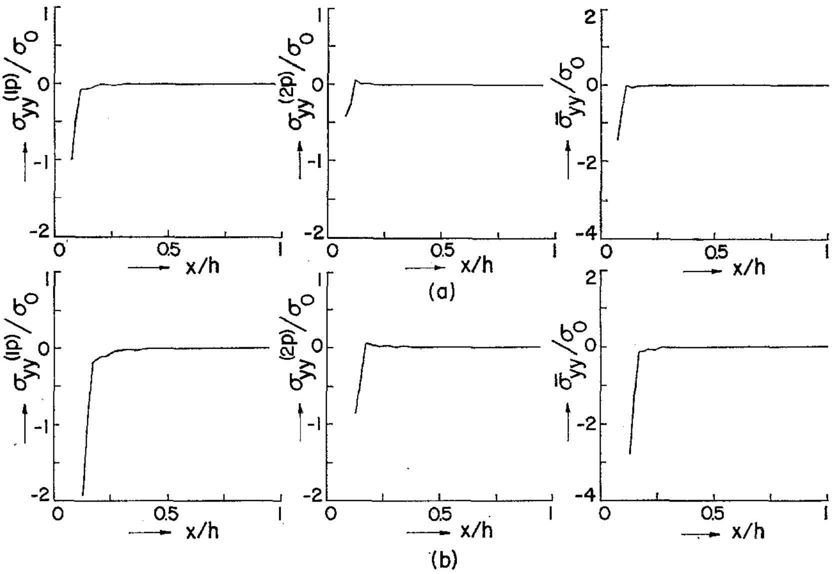

Fig. 4. Same as Fig. 3 for the reinforcement ratio $h_1/h = 0.8$

It is known that one of the most important concepts in fracture mechanics is the stress intensity factor which determines the intensity of the singularity of the stresses at the tip of the crack. It was shown in [2] that the stress intensity factor can be extracted from the numerical solution for the stresses at the tip of the crack, once the power of the singularity is known. The power of the singularity of the stress field in the immediate vicinity of the tip of the crack can be extracted from the equations of motion by assuming that the displacements there, are proportional to r^q where r is the radial distance in a polar coordinate system centered at the tip of the crack and q is an index which is to be determined by the relevant boundary conditions [15]. By substituting the assumed form of the near-tip displacement field in the equations of motion it is found that in the limit of $r \to 0$, the power of the singularity is determined by the highest order derivatives in the equations of motion. For a crack in a homogeneous medium the power of the singularity of the stresses is equal to 1/2. As it can be obtained from the mixture equations of motion (39) in our present problem this power turns out to be also 1/2.

Conclusion

A two-dimensional mixture theory has been formulated and applied to investigate crack propagation on a laminated composite medium. The obtained theory enables the treatment of crack motion in mode-I and mode-II type of motion (Eqs. (40—41)). For mode-III type of motion where the applied stresses are tangential at the interfaces of the crack such that the σ_{yz} stresses are prescribed this time, a similar mixture theory, although much more complicated, can be formulated and applied as well.

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Appendix I

The elements of the vector $\boldsymbol{U}$ and the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ which appear in the mixture equations of motion (39) are given by:

$$\boldsymbol{U} = \begin{bmatrix} \bar{u}_x^{(1)} \\ \bar{u}_y^{(1)} \\ \bar{u}_x^{(2)} \\ \bar{u}_y^{(2)} \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} c_1/\varrho_{1p} & 0 & c_2/\varrho_{1p} & 0 \\ 0 & c_3/\varrho_{1p} & 0 & 0 \\ c_7/\varrho_{2p} & 0 & c_6/\varrho_{2p} & 0 \\ 0 & 0 & 0 & c_8/\varrho_{2p} \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} c_3/\varrho_{1p} & 0 & 0 & 0 \\ 0 & c_1/\varrho_{1p} & 0 & c_2/\varrho_{1p} \\ 0 & 0 & c_8/\varrho_{2p} & 0 \\ 0 & c_7/\varrho_{2p} & 0 & c_6/\varrho_{2p} \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} 0 & c_4/\varrho_{1p} & 0 & c_2/\varrho_{1p} \\ c_4/\varrho_{1p} & 0 & c_2/\varrho_{1p} & 0 \\ 0 & c_7/\varrho_{2p} & 0 & c_9/\varrho_{2p} \\ c_7/\varrho_{2p} & 0 & c_9/\varrho_{2p} & 0 \end{bmatrix},$$

$$\boldsymbol{D} = \begin{bmatrix} -c_5/\varrho_{1p} & 0 & c_5/\varrho_{1p} & 0 \\ 0 & -c_5/\varrho_{1p} & 0 & c_5/\varrho_{1p} \\ c_5/\varrho_{2p} & 0 & -c_5/\varrho_{2p} & 0 \\ 0 & c_5/\varrho_{2p} & 0 & -c_5/\varrho_{2p} \end{bmatrix}.$$

with

$$c_1 = c_{11} - (M\lambda_1/E) \qquad c_2 = c_{12} + (M\lambda_2/E) \qquad c_3 = n_1\mu_1$$
$$c_4 = d_1 + n_1\mu_1 - (M\lambda_1/E) \qquad c_5 = K/h^2 \qquad c_6 = c_{22} - (M\lambda_2/E)$$
$$c_7 = c_{12} + (M\lambda_1/E) \qquad c_8 = n_2\mu_2 \qquad c_9 = d_2 + n_2\mu_2 - (M\lambda_2/E)$$

Appendix II

The elements of the four by four matrix Δ in the frequency Eq. (48) are:

$$\begin{aligned}
\Delta_{11} &= c_{11}\lambda_1 Q_2^{(1)} + c_{12}\lambda_1 Q_4^{(1)} + c_{12}kQ_3^{(1)} + d_1 k \\
\Delta_{12} &= c_{11}\lambda_2 Q_2^{(2)} + c_{12}\lambda_2 Q_4^{(2)} + c_{12}kQ_3^{(2)} + d_1 k \\
\Delta_{13} &= c_{11}\lambda_3 Q_2^{(3)} + c_{12}\lambda_3 Q_4^{(3)} + c_{12}kQ_3^{(3)} + d_1 k \\
\Delta_{14} &= c_{11}\lambda_4 Q_2^{(4)} + c_{12}\lambda_4 Q_4^{(4)} + c_{12}kQ_3^{(4)} + d_1 k \\
\Delta_{21} &= c_{22}\lambda_1 Q_4^{(1)} + c_{12}\lambda_1 Q_2^{(1)} + d_2 kQ_3^{(1)} + c_{12}k \\
\Delta_{22} &= c_{22}\lambda_2 Q_4^{(2)} + c_{12}\lambda_2 Q_2^{(2)} + d_2 kQ_3^{(2)} + c_{12}k \\
\Delta_{23} &= c_{22}\lambda_3 Q_4^{(3)} + c_{12}\lambda_3 Q_2^{(3)} + d_2 kQ_3^{(3)} + c_{12}k \\
\Delta_{24} &= c_{22}\lambda_4 Q_4^{(4)} + c_{12}\lambda_4 Q_2^{(4)} + d_2 kQ_3^{(4)} + c_{12}k
\end{aligned} \qquad \text{(A-1)}$$

$$\begin{aligned}
\Delta_{31} &= \lambda_1 - kQ_2^{(1)} \\
\Delta_{32} &= \lambda_2 - kQ_2^{(2)} \\
\Delta_{33} &= \lambda_3 - kQ_2^{(3)} \\
\Delta_{34} &= \lambda_4 - kQ_2^{(4)} \\
\Delta_{41} &= \lambda_1 Q_3^{(1)} - Q_4^{(1)}k \\
\Delta_{42} &= \lambda_2 Q_3^{(2)} - Q_4^{(2)}k \\
\Delta_{43} &= \lambda_3 Q_3^{(3)} - Q_4^{(3)}k \\
\Delta_{44} &= \lambda_4 Q_3^{(4)} - Q_4^{(4)}k.
\end{aligned} \tag{A-1}$$

In these expressions $Q_m^{(j)}$ are of the form

$$Q_m^{(j)} = N_m^{(j)}/p^{(j)} \qquad j = 1, 2, 3, 4 \qquad m = 2, 3, 4 \tag{A-2}$$

where

$$\left.\begin{aligned}
N_2^{(j)} &= F_{11}F_{23}F_{34} - F_{11}F_{33}F_{24} - F_{21}F_{13}F_{34} \\
&\quad + F_{21}F_{33}F_{14} + F_{31}F_{13}F_{24} - F_{31}F_{23}F_{14} \\
N_3^{(j)} &= i(F_{11}F_{22}F_{34} + F_{11}F_{32}F_{24} + F_{21}F_{12}F_{34} - F_{21}F_{32}F_{14} \\
&\quad - F_{31}F_{12}F_{24} + F_{31}F_{14}F_{22}) \\
N_4^{(j)} &= F_{11}F_{22}F_{33} - F_{11}F_{32}F_{23} - F_{21}F_{12}F_{33} \\
&\quad + F_{21}F_{13}F_{32} + F_{31}F_{12}F_{23} - F_{31}F_{13}F_{22}
\end{aligned}\right\} \tag{A-3}$$

$$\begin{aligned}
p^{(j)} &= i(-F_{12}F_{23}F_{34} - F_{12}F_{33}F_{24} - F_{13}F_{22}F_{34} \\
&\quad + F_{13}F_{32}F_{24} + F_{14}F_{22}F_{33} - F_{32}F_{23}F_{14}).
\end{aligned} \tag{A-4}$$

It should be noticed that any element F_{mn} in the above expressions is a function of λ_j.

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Y. Benveniste and J. Aboudi
Department of Solid Mechanics, Materials,
and Structures
School of Engineering
Tel-Aviv University
Ramat-Aviv, Tel-Aviv
Israel