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On One Variational Principle for Irreversible Phenomena*

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With 2 Figures

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Summary — Zusammenfassung

On One Variational Principle for Irreversible Phenomena. The present paper exhibits a variational principle in which variation and differentation with respect to time are not permutable. The formulation is equally valid for systems with finite and infinite number of degrees of freedom. The merit and efficiency of the method is demonstrated by means of several examples. The numerical aspects of the variational principle are also studied. Special attention is paid to the linear and nonlinear heat conduction problem. Heat conduction with cylindrical symmetry and temperature dependent material properties is discussed in details.

Zu einem Variationsprinzip irreversibler Erscheinungen. In dieser Arbeit wird ein Variationsprinzip angegeben, bei dem Variation und Ableitung nach der Zeit nicht vertauschbar sind. Die Formulierung gilt für Systeme mit endlich vielen Freiheitsgraden wie auch für solche mit unendlich vielen. Vorzüge der Methode werden an Hand einiger Beispiele demonstriert. Untersucht werden auch numerische Aspekte des Variationsprinzips. Besondere Betonung liegt dabei auf lineare wie nichtlineare Wärmeleitungsprobleme. Die Wärmeleitung zylindersymmetrischer Probleme und temperaturabhängiger Materialeigenschaften sind im Detail erörtert.

1. Introduction

The present paper exhibits a variational principle for purely irreversible phenomena. The development is based on the following well known facts.

a) For almost all important processes of irreversible physics the exact Lagrangian of the problem in the sense of classical mechanics does not exist. For example, the parabolic differential equation of heat conduction in solids, even in the linear case, has not any Lagrangian density function.

b) In order to describe the corresponding phenomena by the variational technique, some artificial restrictions must be made, concerning the basic rules of variational calculus. The variational principle of ROSEN [1] and CHAMBERS [2] may serve as a good example of a restricted variational principle. In this variational formulation the functional of a problem contains one physical quantity (temperature, for example) but this quantity is represented by two different symbols; one is subject to variation and the other is not varied at all. By setting

* Dedicated to the memory of Prof. Dr. RASTKO STOJANOVIC, teacher and friend.

the two symbols the same after the variation process has been performed, the exact differential equation of the process in consideration is obtained.

c) The merit and efficiency of the corresponding variational formulation should be testified by the possibility of obtaining approximate solutions using some of the direct methods of variational calculus.

In the present paper *we will assume that the variation and differentiation with respect to time of a function are not permutable processes if the physical system is non-conservative.* In other words, the commutative properties of the variations and a differential with respect to time is a privilege of conservative physics.

At this point it is worthwhile to enumerate some of the assumptions on which the theoretical treatment is based. The main ones are

i) There is a one to one time correspondence between the paths actually describing the natural motion and infinitely near (varied) motion. As HÖLDER pointed out [3], this supposition assures that the variation and integration processes are permutable.

ii) Time is not varied during the process of motion, i.e., natural motion and varied motion have the same terminal configurations and the time of transit is the same in the actual and varied path.

The fact that we are imposing special rules for the variations of first derivatives with respect to time is very important because we can take this rules as a measure of nonconservativity of particular system. This technique has led the author to the present study because one can develop a variational principle for any differential equation, ordinary or partial, describing a nonconservative process. This technique can be used for obtaining approximate solutions. Special attention will be paid to the problem of linear and nonlinear heat conduction in solids.

2. The Variational Principle

In Lagrangian formulation of field theory, the basic dynamical equations are derived from an action integral by introducing suitable Lagrangian densities. However, it is well known that the general equations of dissipative physics in use at the present time can not be derived from Hamilton's principle. Hence, as a consequence of the fact, that the differential equations of a dissipative process are not equivalent with the exact functional (variational) derivative of a Lagrangian density, all basic properties of variational calculus (strong and weak relative extermums, fundamental lemma of variational calculus [10], etc.) are not applicable to nonconservative mechanics.

It seems that the inability to include dissipative forces in the compact from of Lagrangian analysis lies in the fact that the dependence of a dissipative force of velocity affects the process of variation of velocities of a system. The primary purpose of this note is to introduce the variational rules for velocities of a physical system, in accordance with the mechanism of dissipation. In other words, we will abandon the well known rule: "the variation of the velocity is equal to the derivative of the variation" in dealing with nonconservative physics, and introduce some incommutable rules in accordance with the specific dissipative mechanism.

The incommutable rules have been employed by many authors in nonholonomic mechanics. The use of these rules can be traced to the works of T. LEVI-CIVITA and AMALDI [15] and SUSLOV [16]. NEIMARK and FUFAEV [17] and LURE [18] have pointed out that the commutative rules of variational calculus are quite arbitrary as far as conservative mechanics is concerned. They claim that these rules are adapted in accordance with the differential equations of conservative mechanics.

It seems reasonable, accepting this point of view, to adapt the incommutable rules in such a way that the physically admissible values for the dynamical variables of a dissipative system are those for which the first variation of an integral vanish.

For our purposes we will write the general differential equation of a physical system in the form

$$\varphi_L\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, x_i, t\right) + \varphi_D\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, x_i, t\right) = 0 \tag{1}$$

where x_i are the coordinates, t denotes time and u is the field variable. The part[1] φ_L in (1) is the "Lagrange part" which is derivable from a Hamilton's principle of the form:

$$I = \int_{t_0}^{t_1}\int_V L\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, x_i, t\right) dV\,dt. \tag{2}$$

In other words, the variational equation

$$\delta I = 0 \tag{3}$$

together with the boundary conditions

$$\delta u|_V = 0, \quad \text{on the boundaries of } V \text{ for every moment } t \tag{4}$$

and

$$\delta u|_{t_0} = \delta u|_{t_1} = 0, \quad \text{everywhere in } V\text{, including the boundaries of } V, \tag{5}$$

is equivalent with the equation $\varphi_L\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, x_i, t\right) = 0$, in the strict sense of classical variational calculus.

Note, that the time interval $[t_0, t_1]$ is arbitrary and $dV = \prod_{j=1}^{3} dx_j$, is the elementary volume of geometrical space.

The nonconservative part φ_D of (1) can not be derived from a variational principle of Hamilton's type. Usually, the part φ_D is connected with the mechanism of dissipation of the system under consideration. In order to be able to derive the differential Eq. (1) using the variational integral (2) we will introduce the dissipative characteristics of our physical system through incommutative rules of variation and differentiation with respect to time, of the function u.

[1] Henceforth, the explicit dependence of φ_L will be omitted except where it is necessary or desirable, thus $\varphi_L\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, x_i, t\right)$ is simply written φ_L, etc.

Let

$$\delta \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \delta$$
$$\delta \left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t} \delta u + \theta \left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial t}, u, x_i, t\right) \delta u \tag{6}$$

be the incommutative rules of the system considered where θ is a suitably chosen functional, which is equal to zero if the system is conservative.

Let us take the variation of (2) with respect to u using (6). This step gives[2]

$$\delta I = \int_{t_0}^{t_1} \int_V \left\{ \frac{\partial L}{\partial u} \delta u + \frac{\partial L}{\partial \left(\frac{\partial u}{\partial x_i}\right)} \frac{\partial}{\partial x_i} \delta u + \frac{\partial L}{\partial \left(\frac{\partial u}{\partial t}\right)} \left[\frac{\partial}{\partial t} \delta u + \theta \cdot \delta u\right]\right\} dV\, dt. \tag{7}$$

Using integration by parts the final term is of the form:

$$\delta I = \int_{t_0}^{t_1} \int_V \left\{ [L] + \frac{\partial L}{\partial \left(\frac{\partial u}{\partial t}\right)} \theta \right\} \delta u\, dV\, dt$$
$$+ \int_V \frac{\partial L}{\partial \left(\frac{\partial u}{\partial t}\right)} \cdot \delta u \Big|_{t_0}^{t_1} dV + \int_{t_0}^{t_1} \int_S \frac{\partial L}{\partial \left(\frac{\partial u}{\partial x_i}\right)} \delta u \Big|_S dS \tag{8}$$

where S is the boundary of V, and $[L]$ is the variational (functional) derivative defined by $[L] \equiv \frac{\partial L}{\partial u} - \frac{\partial}{\partial x_i} \frac{\partial L}{\partial \left(\frac{\partial u}{\partial x_i}\right)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \left(\frac{\partial u}{\partial t}\right)}$.

We will assume that the functionals L and θ are selected in such a way, that

$$[L] \equiv \varphi_L \tag{9}$$

and

$$\frac{\partial L}{\partial \left(\frac{\partial u}{\partial t}\right)} \theta \equiv \varphi_D. \tag{10}$$

Supposing that δu is arbitrary along the actual trajectory, and boundary conditions (4) and (5) are fulfilled, the variational equation (8) $\delta I = 0$ is equivalent with the equation of motion (1)

$$\varphi_L + \varphi_D = 0.$$

If the boundary conditions (4) are not valid, i.e., the function u is not specified on the boundary S of V, then an appropriate number of boundary conditions will be supplied during the course of the variational analysis. Clearly, the incom-

[2] Repeated indices are summed throughout.

mutative rules (6) do not change the outstanding feature of Hamilton's principle that it implies boundary conditions as well as differential equations.

Let us now consider the case of a dynamical nonconservative system with n-degrees of freedom. x_i $(i = 1, 2, \dots, n)$ are regarded as the generalized coordinates and Q_i are generalized nonconservative forces which, in the general case, are given functions of position, time and velocities. The conservative part of the system can be completely described by a Lagrange's function $L(x_1, x_2, \dots, x_n; \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n; t)$. Hence, the action integral of the conservative part is

$$I = \int_{t_0}^{t_1} L\, dt. \tag{11}$$

Let us define the incommutative rules in the form

$$\delta \dot{x}_i = \frac{d}{dt}\, \delta x_i + \Lambda_k^{\,i}\, (\dot{x}, \dot{x}, t)\, \delta x_k, \tag{12}$$

where the system of functions $\Lambda_k^{\,i}$ is chosen in accordance with the equations

$$\frac{\partial L}{\partial \dot{x}_k}\, \Lambda_i^{\,k} = Q_i. \tag{13}$$

Using (11) and (12) we find

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x_i}\, \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \left(\frac{d}{dt}\, \delta x_i + \Lambda_k^{\,i}\, \delta x_k\right)\right] dt.$$

Integration by parts and use of Eq. (13) gives:

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x_i} - \frac{d}{dt}\, \frac{\partial L}{\partial \dot{x}_i} + Q_i\right] \delta x_i\, dt + \frac{\partial L}{\partial \dot{x}_i}\, \delta x_i \Big|_{t_0}^{t_1}. \tag{14}$$

If we suppose the standard boundary conditions

$$\delta x_i \Big|_{t_0}^{t_1} = 0 \tag{15}$$

then, for arbitrary variations δx_i, the equation $\delta I = 0$ is equivalent with the Euler-Lagrange equations

$$\frac{d}{dt}\, \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = Q_i \quad (i = 1, 2, \dots, n). \tag{16}$$

Finally, it should be noted, that the variational principle presented in this paper may be employed for obtaining approximate solutions. The applications of well known methods of Ritz and partial integration are straightforward and do not require any general clarifications.

3. Applications of the Theory

The theory just developed is now applied to various problems of classical mechanics and heat conduction.

A. Whittaker's Equations [20]

Let us consider a holonomic mechanical system of n degrees of freedom. If x^i are regarded as the generalized coordinates, for a large class of dynamical systems the kinetic energy is given by a quadratic form of generalized velocities $\dot{x}^i$ ($i = 1, 2, \ldots, n$):

$$T = \frac{1}{2} a_{ij}\dot{x}^i\dot{x}^j. \tag{17}$$

a_{ij} is the fundamental metric tensor of second order which is a function of position x^i. Suppose that there exists the generalized potential

$$\Pi = \Pi(x^i, t) \tag{18}$$

and the Lagrangian function is

$$L = \frac{1}{2} a_{ij}\dot{x}^i\dot{x}^j - \Pi(x^i, t). \tag{19}$$

Hence the action integral is of the form

$$I = \int_{t_0}^{t_1} L\, dt. \tag{20}$$

Let us assume that our system is subject to external resisting forces which are directly proportional to the velocities, the dissipative function of which is of the form

$$2\varphi = b_{ij}\dot{x}^i\dot{x}^j, \tag{21}$$

where b_{ij} is a symmetrical tensor which is frequently a function of position.

Let the rule for the velocity variation be of the form

$$\delta\dot{x}^j = \frac{d}{dt}\, dx^j - a^{js}b_{ls}\, \delta x^l, \tag{22}$$

where a^{js} is the contravariant tensor with the property ([4], p. 14)

$$a^{ij}a_{kj} = \delta_k{}^i, \tag{23}$$

where $\delta_k{}^i$ is Kronecker delta. After performing the usual manipulations, we find, using (19), (22) and (23),

$$\delta I = a_{ij}\dot{x}^i\,\delta x^j \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(a_{ij}\ddot{x}^j + [mn, i]\,\dot{x}^m\dot{x}^n + b_{ij}\dot{x}^j - \frac{\partial\Pi}{\partial x^i}\right) \delta x^i\, dt,$$

or, if δI is to vanish for all variations, then

$$a_{ij}\ddot{x}^j + [mn, i]\,\dot{x}^m\dot{x}^n + b_{ij}\dot{x}^j - \frac{\partial\Pi}{\partial x^i} = 0 \tag{24}$$

which is the system of equations of motion in tensorial form in the presence of dissipative forces. The symbol $[mn, i]$ is the first Chirtoffel symbol.

B. A Numerical Example

Using the same variational formulation it is possible to obtain the differential equations of the first order as the consequence of variation of an action integral with the specific variational properties of the first derivative. To be more specific let us consider the almost trivial example

$$\dot{x} = -2xt, \quad x(0) = 1, \quad 0 \leqq t \leqq 1. \tag{25}$$

Consider the action integral

$$I = \int_0^1 (\dot{x} + x^2 t)\, dt - x \Big|_{t=1} \tag{26}$$

with the velocity variation rule

$$d\dot{x} = \frac{d}{dt}\, \delta x + \dot{x}\, \delta x. \tag{27}$$

Note at this point that the term $\int_0^1 \dot{x}\, dt$ in (26) *should not be integrated* before the process of variation is finished. This requirement is the direct consequence of the fact that we are imposing special rules for the variation of velocity. Actually we are faced with the situation typical for many branches of physics. As B. DEWITT pointed out ... "For most of our purposes the *form* of the action functional will have more importance than its actual value." ([5], p. 1). The last term in (26) involving only the function evaluated at the boundary has been added to Eq. (26) because the variation δx is not specified at $t = 1$, hence this term plays the role of a natural boundary condition. It can easily be verified that the variation of (26) with the rule (27) is

$$\delta I = \int_0^1 \left[\frac{d}{dt}\,(\delta x) - (\dot{x} + 2xt)\, \delta x\right] dt - \delta x \Big|_{t=1}$$

hence, $\delta I = 0$ is equivalent to Eq. (25).

It should be noted that all of the problems which are treated in this section are also amenable to numerical treatment.

Let us demonstrate the application of the Rayleigh-Ritz direct method for obtaining an approximate solution of Eq. (25). It is obvious that $\dot{x} = 0$ when $t = 0$. At the same time Eq. (25) is invariant with respect to a time transformation $\bar{t} = -t$. Chosing the form of the trial polynomial we have to take into account these two properties and the given boundary condition. We shall assume:

$$x = 1 + At^2 + Bt^4 \tag{28}$$

and use our technique to find A and B. In order to describe the specific nature of Eq. (27) we will introduce two kinds of velocities

$$\dot{x} = 2At + 4Bt^3$$

and (29)

$$\dot{x} = 2at + 4bt^3$$

where a and b are alias for A and B. Now, let us write (27) in the form

$$\delta \dot{x}\Big|_{a,b} = \frac{d}{dt}\,\delta x + \dot{x}\,\delta x\Big|_{A,B} \tag{30}$$

and the action integral (26)

$$I = \int_0^1 \left(\dot{x}\Big|_{a,b} + x^2 t\right) dt - x\Big|_{t=1}. \tag{31}$$

Introducing (28) and (29) into (30) and integrating this equation with respect to t from 0 to 1 we get

$$\delta a + \delta b = \delta A + \delta B + \frac{1}{2} A\delta A + \frac{1}{3} A\delta B + \frac{2}{3} B\delta A + \frac{1}{2} B\delta B. \tag{32}$$

When we compute I given by (31) we find

$$I = a + b + \frac{1}{6} A^2 + \frac{1}{10} B^2 + \frac{1}{2} A + \frac{1}{3} B + \frac{1}{4} AB - (1 + A + B).$$

Now we have

$$\delta I = \delta a + \delta b + \frac{1}{3} A\delta A + \frac{1}{5} B\delta B + \frac{1}{2}\delta A + \frac{1}{3}\delta B + \frac{1}{4} A\delta B + \frac{1}{4} B\delta A - \delta A - \delta B = 0. \tag{33}$$

Introducing (32) into (33), collecting corresponding terms with δA and δB, the equation $\delta I = 0$ will yield:

$$\frac{5}{6} A + \frac{11}{12} B = -\frac{1}{2}$$

$$\frac{7}{12} A + \frac{7}{10} B = -\frac{1}{3},$$

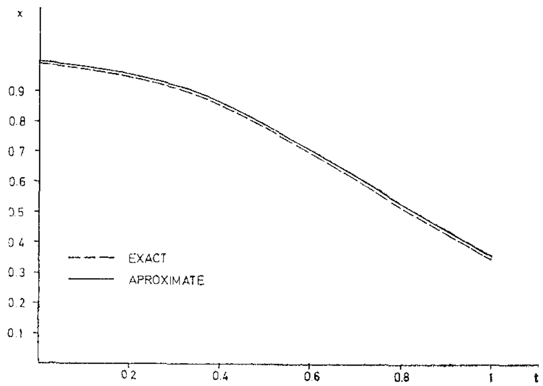

Fig. 1

with the solution $A = -32/35$ and $B = 2/7$. Hence, the approximate solution is of the form

$$x = 1 - \frac{32}{35} t^2 + \frac{2}{7} t^4. \tag{34}$$

The approximate and exact solutions are presented graphically in Fig. 1. It is seen that the agreement with the exact solution is quite satisfactory.

C. Applications to Heat Conduction in Solids

In this section we will apply the previous considerations to problems which are described by partial differential equations. The transient heat conduction in solids is chosen to demonstrate the feasibility and efficacy of the variational principle. Special attention will be paid to nonlinear heat conduction and heat conduction through cylindrical bodies. Various boundary conditions are also studied.

Let us consider the action integral

$$I = \int_{t_0}^{t_1} \int_{v} \left[\varepsilon \frac{\partial T}{\partial t} + \frac{\alpha}{2} \sum_{i=1}^{3} \left(\frac{\partial T}{\partial x_i} \right)^2 \right] dv\, dt, \tag{35}$$

where T is the temperature, α is the thermal diffusivity x_1, x_2, x_3 are rectangular coordinates, $dv = \prod_{j=1}^{3} dx_j$, t is the time and ε is an arbitrary constant parameter dimensionally equal to T. The following commutative rules are introduced

$$\begin{aligned} \delta \left(\frac{\partial T}{\partial x_i} \right) &= \frac{\partial}{\partial x_i} \delta T \\ \delta \left(\frac{\partial T}{\partial t} \right) &= \frac{\partial}{\partial t} \delta T + \frac{1}{\varepsilon} \frac{\partial T}{\partial t} \delta T . \end{aligned} \tag{36}$$

From (35) and (36) it is seen that there is no loss of generality in assuming that the numerical value of ε is $\varepsilon = 1$. We now take the first variation of I i.e.

$$\delta I = \int_{t_0}^{t_1} \int_{v} \left[\varepsilon \delta \left(\frac{\partial T}{\partial t} \right) + \alpha \sum \frac{\partial T}{\partial x_i} \delta \left(\frac{\partial T}{\partial x_i} \right) \right] dv\, dt. \tag{37}$$

Substituting (36) and integrating by parts we have

$$\delta I = \int_{t_0}^{t_1} \int_{v} \left\{ \frac{\partial T}{\partial t} - \alpha \sum \frac{\partial^2 T}{\partial x_i^2} \right\} dv\, dt\, \delta T + \int_{t_0}^{t_1} \alpha \sum \frac{\partial T}{\partial x_i} \delta T \Big|_S + \int_{v} \varepsilon (\delta T) \Big|_{t_0}^{t_1} dv, \tag{38}$$

where S is the boundary of the body. If we suppose that

$$\delta T \Big|_S = \delta T \Big|_{t_0}^{t_1} = 0 \tag{39}$$

the equation $\delta I = 0$ is equivalent with

$$\frac{\partial T}{\partial t} = \alpha \sum_{i=1}^{3} \frac{\partial^2 T}{\partial x_i^2} . \tag{40}$$

This equation is the well known transient heat conduction equation in linear form.

The action integral (35) may be easily written in other systems of orthogonal coordinates. For example in the case of cylindrical coordinates the corresponding action integral is of the form

$$I = \int\limits_t \int\limits_r \int\limits_\theta \int\limits_z \left\{ \varepsilon \frac{\partial T}{\partial t} + \frac{\alpha}{2} \left[r \left(\frac{\partial T}{\partial r} \right)^2 + \frac{1}{r} \left(\frac{\partial T}{\partial \theta} \right)^2 + r \left(\frac{\partial T}{\partial z} \right)^2 \right] \right\} dv\, dt \tag{41}$$

with the commutative rules

$$\begin{gathered} \frac{\partial}{\partial r} \delta = \delta \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial z} \delta = \delta \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \theta} \delta = \delta \frac{\partial}{\partial \theta} \\ \delta \left(\frac{\partial T}{\partial t} \right) = \frac{\partial}{\partial t} \delta T + \frac{1}{\varepsilon} r \frac{\partial T}{\partial t} \delta T . \end{gathered} \tag{42}$$

It is easy to show that $\delta I = 0$ is equivalent with the heat conduction equation in cylindrical form

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right]. \tag{43}$$

In the case of temperature dependent thermal properties i.e. if thermal conductivity k and thermal capacity c are given functions of temperature, the corresponding action integral is

$$I = \int\limits_{t_0}^{t_1} \int\limits_v \left\{ \varepsilon \frac{\partial T}{\partial t} + \frac{k^2(T)}{2} \sum_{i=1}^{3} \left(\frac{\partial T}{\partial x_i} \right)^2 \right\} dv\, dt. \tag{44}$$

At the same time the commutative rules are

$$\begin{gathered} \delta \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \delta \\ \delta \left(\frac{\partial T}{\partial t} \right) = \frac{\partial}{\partial t} \delta T + \frac{1}{\varepsilon} c(T)\, k(T) \frac{\partial T}{\partial t} \delta T . \end{gathered} \tag{45}$$

The variational equation $\delta I = 0$ together with (45) is equivalent with the equation

$$c(T) \frac{\partial T}{\partial t} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[k(T) \frac{\partial T}{\partial x_i} \right]. \tag{46}$$

The dimension of constant parameter ε in (44) and (45) is the same as $c(T)\, k(T)\, \delta T$.

If our system constains a heat source the rate of which per unit volume and unit time is denoted by $A = A(x, y, z, t)$, then the action integral (44) should be modified in the following way

$$I = \int\limits_{t_0}^{t_1} \int\limits_v \left\{ \varepsilon \frac{\partial T}{\partial t} + \frac{k^2(T)}{2} \sum_{i=1}^{3} \left(\frac{\partial T}{\partial x_i} \right)^2 - A \int k(T)\, dT \right\} dv\, dt \tag{47}$$

and the rules (45) remain unchanged.

In the next section, by the help of several examples we will demonstrate the procedure of obtaining approximate solutions using direct methods of given variational formulation.

D. Heating of an Infinite Cylinder

We now turn to an examination of approximate method for the solution of heat conduction problems based on the variational principle stated in this paper. The application of the method to a problem with cylindrical symmetry was selected because it is known that various approximate methods are often inapropriate in dealing with heat conduction in spherical and cylindrical bodies. For example, the monograph devoted to the Biot variational principle [6] does not contain any problem from this area. Lardner and Pohle [7] and Goodman [8] have demonstrated that, for problems involving polar or spherical symetry, the integral method is inappropriate unless a special modification in the assumed temperature profile is made. We shall demonstrate that the variational technique can be applied without any modifications in the temperature profiles.

The problem we wish to solve is transient heat conduction in an infinite circular cylinder with constant heat flux through the surface. The cylinder is at zero initial temperature. Thermal properties will be taken to be constant so the governing equation is linear. If the axis of the cylinder coincides with the z axes, the initial and boundary conditions are independent of the coordinates θ and z, the temperature is a function of r and t only and differential equation and corresponding boundary condition are of the form

$$c\,\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\,\frac{\partial T}{\partial r}\right), \tag{48}$$

$$k\,\frac{\partial T}{\partial r} - p = 0, \qquad r = R, \qquad t > 0, \tag{49}$$

where k, c, p are given constants and R is radius of the cylinder.

Let us consider the action integral

$$I = \int\limits_{t_0}^{t_1}\int\limits_{r=a}^{R}\left\{\varepsilon\,\frac{\partial T}{\partial t} + \frac{k}{2}\,r\left(\frac{\partial T}{\partial r}\right)^2\right\}dr\,dt - \int\limits_{t_0}^{t_1} pRT\,\Big|_{r=R}\,dt - \int\limits_{r=a}^{R}\varepsilon\,T\,\Big|_{t_0}^{t_1}dr, \tag{50}$$

where the time interval $[t_0, t_1]$ is arbitrary and the boundary $r = a$ depends of the character of temperature profile. The commutative rules are (42) i.e.

$$\frac{\partial}{\partial r}\,\delta = \delta\,\frac{\partial}{\partial r}$$
$$\delta\left(\frac{\partial T}{\partial t}\right) = \frac{\partial}{\partial t}\,\delta T + \frac{1}{\varepsilon}\,cr\,\frac{\partial T}{\partial t}\,\delta T. \tag{51}$$

As we mentioned previously, the first term on the right hand side of (50) should not be integrated before the process of variation is performed. It is easy to show that $\delta I = 0$, together with (51) will yield the Eqs. (48) and (49) if we assume that

$\delta T|_{r=a} = 0$. The last term at the right-hand side of (50) plays the role of a natural boundary condition because the temperature is not specified at the boundaries. In accordance with the real conditions of heating we will study two phases of the process. In the first phase the heat front is penetrating into the cylinder and when it reaches the axis of the cylinder the second phase begins. If we choose the parabolic distribution during the first phase the assumed profile will be

$$T_I = \begin{cases} \frac{p}{2k(R-l)}(r-l)^2 & \text{for} \quad l < r < R \\ 0 & \text{for} \quad 0 < r < l \end{cases} \tag{52}$$

where $l = l(t)$ is the location of the heat front measured from the center of the cylinder. The penetration distance $l(t)$ satisfies the initial condition

$$l(0) = R. \tag{53}$$

Note, that the profile (52) was selected in such a way that the boundary condition (49) is satisfied and $T|_{l=r} = 0$. The penetration time τ is the root of the equation $l(\tau) = 0$. It is clear now, that we have to choose $a = l$ in the lower bound of (50). Substitution of (52) into (50) and (51) and integration of these equations with respect to r from l to R yields, for $\varepsilon = 1$,

$$I = \int_{t_0}^{t_1} \left\{ -\frac{1}{3}\frac{p}{k}(R-l)\,\dot{l} + \frac{p^2}{2k}\left[\frac{(R-l)^2}{4} + \frac{l(R-l)}{3} - R(R-l)\right]\right\} dt \\ - \frac{p}{6k}(R-l)^2 \Big|_{t_0}^{t_1}, \tag{54}$$

$$\delta \dot{l} = (\delta l)^{\cdot} - \frac{1}{40}\frac{cp}{k}(11R + 5l)\,\dot{l}\,\delta l. \tag{55}$$

It is interesting to note, that the action integral is of the Bolza type [10].

If we substitute (55) into $\delta I = 0$ we have

$$\int_{t_0}^{t_1} \left\{ \frac{d}{dt}\left[-\frac{p}{3k}(R-l)\,\delta l\right] + \left[0{,}1\,\dot{l}(R-l)(11R+5l) + \frac{k}{c}(5R-l)\right]\delta l \right\} dt \\ + \frac{p}{3k}(R-l)\,\delta l \Big|_{t_0}^{t_1}.$$

Integrating the first term, we get the following differential equation

$$0{,}1\,\dot{l}(R-l)(11R+5l) + \frac{k}{c}(5R-l) = 0. \tag{56}$$

For the second phase we will choose the temperature profile in the form

$$T_{II} = \frac{2pR}{k}\left[\left(\frac{1}{2}\frac{r}{R}\right)^2 + u(t)\right] \tag{57}$$

and the lower bound in (50) is $a = 0$. Substituting (57) into (50) and (51), the variational equation $\delta I = 0$ for the second phase is

$$c \frac{du}{dt} = \frac{k}{R^2}. \tag{58}$$

With condition (53) the solutions of (56) and (58) are easily obtained.

The solution of (56) is

$$5\left[1 - \left(\frac{l}{R}\right)^2\right] + 62\left(1 - \frac{l}{R}\right) + 288 \ln \frac{4}{5 - \frac{l}{R}} = 20 \frac{kt}{cR^2} \tag{59}$$

and the penetration time $l(\tau) = 0$ is

$$\tau = \frac{cR^2}{k} 0{,}137. \tag{60}$$

The solution of (58) is

$$u = \frac{k}{cR^2}(t - \tau) \tag{61}$$

hence

$$T_{II} = \frac{2pR}{k}\left[\frac{kt}{cR^2} + \left(\frac{1}{2}\frac{r}{R}\right)^2 - 0{,}137\right]. \tag{62}$$

Note, that the exact solution for long times are given in ref. [11] in the form:

$$T = \frac{2pR}{k}\left[\frac{kt}{cR^2} + \left(\frac{1}{2}\frac{r}{R}\right)^2 - 0{,}125\right]. \tag{63}$$

It is seen that the agreement with the approximate solution (62) is satisfactory.

The same result was obtained and compared for both phases with the exact solution in ref. [9], using the Galerkin method.

E. Semi-Infinite Body with Temperature Dependent Heat Capacity

As an illustration we shall treat a nonlinear problem. Consider a semi-infinite body that occupies the region $x > 0$. The constant heat flux F is applied to the surface $x = 0$. The initial temperature of the body is zero. We assume the heat capacity to be a linear function of the temperature as follows

$$c(T) = c_0(1 + \alpha T), \tag{64}$$

where c_0 and α are given constants. The thermal conductivity k is assumed constant. Hence the mathematical form of the problem is

$$c(T) \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}. \tag{65}$$

$$k \frac{\partial T}{\partial x} = -F, \quad x = 0, \quad t > 0. \tag{66}$$

Consider the following action integral

$$I = \int_{t_0}^{t_1}\int_0^{x_1}\left[\varepsilon\,\frac{\partial T}{\partial t} + \frac{k}{2}\left(\frac{\partial T}{\partial x}\right)^2\right] dx\,dt - \alpha\int_{t_0}^{t_1} FT\Big|_{x=0} dt - \int_0^{x_1} \varepsilon T\Big|_{t_0}^{t_1} dx \tag{67}$$

with commutative rule

$$\delta\left(\frac{\partial T}{\partial t}\right) = \frac{\partial}{\partial t}\,\delta T + \frac{1}{\varepsilon}\,c(T)\,\frac{\partial T}{\partial t}\,\delta T. \tag{68}$$

The time interval $[t_0, t_1]$ is arbitrary and the bound x_1 depends on the assumed temperature profile. Before proceeding with the calculations, note that $\delta I = 0$ of (67) together with (68) is equivalent with (65) and (66) if $T|_{x_1} = 0$. For relatively small values of parameter α in (64), it is reasonable to suppose that the solution of (65) and (66) is not drastically different that in the case $\alpha = 0$. For the linear variant of the same problem, several authors supposed that the adequate trial solution, which is expressed in terms of the generalized coordinates, should be assumed as a polynomial [7], [8]. Following the natural requirement that the trial solution must be relatively simple, the temperature field is taken as

$$T = \frac{z}{\theta^3}\,(\theta - x)^3, \tag{69}$$

where z is the surface temperature

$$T(0, t) = z \tag{70}$$

and $\theta = \theta(t)$ is the depth of penetration. Note that the coordinate z is not independent of θ. From (69) and (66) we have

$$z - \frac{F\theta}{3k} = 0. \tag{71}$$

Substituting Eq. (69) into (67) and (68), taking $\varepsilon = 1$ and integrating with respect to x from 0 to θ we have, respectively,

$$I = \int_{t_0}^{t_1}\left[\frac{1}{4}\,z\dot{\vartheta} + \frac{1}{4}\,\dot{\theta}z + \frac{9}{10}\,k\,\frac{z^2}{\theta} - Fz\right] - \frac{1}{4}\,(z\theta)\Big|_{t_0}^{t_1} \tag{72}$$

$$\begin{aligned}\frac{1}{4}\,(\theta\delta\dot{z} + z\delta\dot{\theta}) = {} & \frac{1}{4}\,[\theta(\delta z)^{\cdot} + z(\delta\theta)^{\cdot}] \\ & + c_0\left\{\frac{1}{7}\,\theta\dot{z}\delta z + \frac{1}{14}\,z\dot{z}\delta\theta + \frac{1}{14}\,z\dot{\theta}\delta z + \frac{3}{35}\,\frac{z^2\dot{\theta}}{\theta}\,\delta\theta\right. \\ & \left. + \alpha\left[\frac{1}{10}\,\theta z\dot{z}\delta z + \frac{1}{30}\,z^2\dot{z}\delta\theta + \frac{1}{30}\,z^2\dot{\theta}\delta z + \frac{1}{40}\,\frac{z^2\dot{\theta}}{\theta}\,\delta\theta\right]\right\}.\end{aligned} \tag{73}$$

However, the coordinates θ and z are not independent and we have a constrained optimization problem. It is well known that the solution to this problem is the same as that obtained by extremizing

$$I' = \int_{t_0}^{t_1} \left[\frac{1}{4} z\dot{\theta} + \frac{1}{4}\dot{\theta} z + \frac{9}{10} k \frac{z^2}{\theta} - Fz + \lambda\left(z - \frac{F\theta}{3k}\right)\right] dt - \frac{1}{4}(z\theta)\Big|_{t_0}^{t_1} \quad (74)$$

where λ is a constant Lagrange multiplier.

Taking that the first variation of (74) is equal to zero and using (73) we get

$$\begin{aligned}\delta I' = &\int_{t_0}^{t_1} \frac{d}{dt}\left[\frac{1}{4}\delta(\theta z)\right] dt \\ &+ \int_{t_0}^{t_1} \Big\{\delta z \left[\frac{1}{7} c_0 \theta \dot{z} + \frac{1}{14} c_0 z\dot{\theta} + \frac{1}{10} c_0 \alpha \theta z \dot{z} + \frac{1}{30} c_0 \alpha z^2 \dot{\theta} + \frac{9}{5} k \frac{z}{\theta} - F + \lambda\right] \\ &+ \delta\theta \Big[\frac{1}{14} c_0 z\dot{z} + \frac{3}{35}\frac{z^2\dot{\theta}}{\theta} + \frac{1}{30} c_0 \alpha z^2 \dot{z} + \frac{1}{40} c_0 \alpha \frac{z^3\dot{\theta}}{\theta} - \frac{9}{10} k \frac{z^2}{\theta^2} \\ &- \frac{F\lambda}{3k}\Big]\Big\} dt - \frac{1}{4}\delta(z\theta)\Big|_{t_0}^{t_1} = 0.\end{aligned} \quad (75)$$

Integrating the first term and canceling it with the last in (75), the equation $\delta I' = 0$ is equivalent with the following differential equations:

$$\frac{1}{7} c_0 \theta \dot{z} + \frac{1}{14} c_0 z\dot{\theta} + \frac{1}{10} c_0 \alpha \theta z \dot{z} + \frac{1}{30} c_0 \alpha z^2 \dot{\theta} + \frac{9}{5} k \frac{z}{\theta} - F + \lambda = 0, \quad (76)$$

$$\frac{1}{14} c_0 z\dot{z} + \frac{3}{35}\frac{z^2\dot{\theta}}{\theta} + \frac{1}{30} c_0 \alpha z^2 \dot{z} + \frac{1}{40} c_0 \alpha \frac{z^3\dot{\theta}}{\theta} - \frac{9}{10} k \frac{z^2}{\theta^2} - \frac{F\lambda}{3k} = 0. \quad (77)$$

These two equations together with (71) form a complete set for finding θ, z and λ. Eliminating λ from (76) and (77) and using (71) we obtain the following differential equation for the surface temperature

$$\frac{13}{35} c_0 z\dot{z} + \frac{23}{100} c_0 \alpha z^2 \dot{z} - \frac{7}{30}\frac{F^2}{k} = 0. \quad (78)$$

Integrating with respect to the initial condition $z(0) = 0$ we have

$$\frac{13}{7} z^2 + \frac{23}{36}\alpha z^3 = \frac{7}{3}\frac{F^2}{kc_0} t. \quad (79)$$

The relationship between the surface temperature and time is shown in Fig. 2. Unfortunately, the exact solution of this problem is not known and the direct comparison is not possible. For the linear case $\alpha = 0$, we have from (79)

$$z = 1.121\, F \sqrt{\frac{t}{c_0 k_0}}. \quad (80)$$

The exact solution for $\alpha = 0$ is $z = 1.128\, F \sqrt{\frac{t}{c_0 k_0}}$ and the error is about 0.7%.

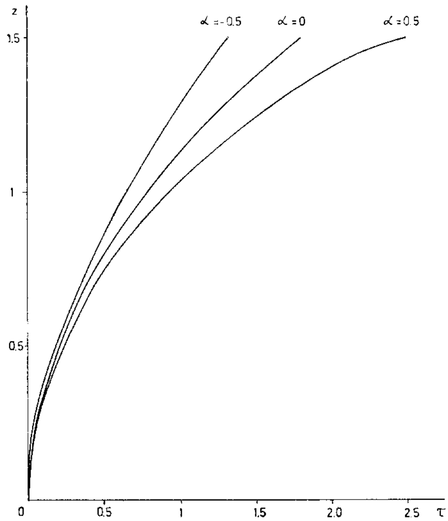

Fig. 2

Using the heat balance method Goodman has obtained for this problem $z = 1.15\, F \sqrt{\frac{t}{c_0 k_0}}$. In a different variational approach [14] the author has obtained $z = 1.128\, F \sqrt{\frac{t}{c_0 k_0}}$.

It should be noted that the procedure used in this example with two dependent coordinates can be used only in the case when a Lagrangian of the problem exists.

Discussion

1. The variational principle set out in this paper has been found to be applicable to a wide range of problems of dissipative mechanics and heat conduction. All considerations are based on the supposition that the process of variation and differentiation with respect to time is not commutative for nonconservative physical systems.

2. The variational principle presented in this paper is structurally different from the variational formulations of Glansdorff and Prigogine [19], Biot [6] and a variational principle given by the author et. al. ([12], [13], [14]). But apparently the numerical results obtained by the help of these formulations are more or less the same. In addition, it is felt that, due to the excellent agreement between the results of Example (D) obtained by the variational method and Galerkin's method, there is a direct relationship between Galerkin's method and the variational method in the case of linear problems. However, from the standpoint of approximate solutions the variational approach offers some advantages such as the use of dependent coordinates, Lagrange's multipliers and the natural boundary conditions.

3. It appears that a study of conservation laws of nonconservative mechanics by the help of variational principle developed herein could have interesting physical implications. An investigation about this problem will be reported on elsewhere.

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