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Quasistatic Problem of a Non-Homogeneous Elastic Layer Containing a Crack

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With 7 Figures

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Summary — Zusammenfassung

Quasistatic Problem of a Non-Homogeneous Elastic Layer Containing a Crack. A non-homogeneous elastic layer is weakened by an infinite, rectilinear crack separating two layers of different elastic materials. The boundary surfaces of the layer are rigidly clamped and the crack surfaces loaded by arbitrary forces satisfying the conditions of antiplane state of strain. Considered are two cases, the crack and its load propagating at a constant velocity along the horizontal axis, and the load being a harmonic function of time, respectively. In the both cases exact values of the stress intensity factor for arbitrary loading (arbitrary load amplitude) of the crack are given. In the limiting cases, solutions of static problems are obtained. The results are illustrated by particular solutions concerning the cases when the crack edge load (or its amplitude) is constant on its entire length.

Ein quasistatisches Problem einer inhomogenen Schicht mit einem Riß. Eine aus zwei verschieden elastischen Schichten zusammengesetzte Schicht wird durch einen streifenförmigen Riß zwischen den Schichten geschwächt. Die Oberflächen der Schicht sind eingespannt und die Rißoberfläche durch beliebige, die Bedingungen des antiebenen Verzerrungszustandes genügenden Kräften belastet. Betrachtet werden die zwei Fälle, daß sich der Riß und seine Belastung mit konstanter Geschwindigkeit horizontal bewegen und daß die Last eine harmonische Funktion der Zeit ist. In beiden Fällen werden die exakten Werte des Spannungserhöhungsfaktors für beliebige Lasten (beliebige Lastamplituden) angegeben. In den Grenzfällen werden die Werte des statischen Problems erhalten. Die Resultate werden an Hand des Spezialfalles einer über die gesamte Länge konstanten Last (oder Amplitude) erläutert.

1. General Formulation

The antiplane state of strain is known to be characterized by the particular form of elastic displacement vector which, in a rectangular coordinate system (x, y, z) may be represented as

$$\boldsymbol{u} = [0, 0, w(x, y, t)].$$

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The only components of the state of strain and stress which do not identically vanish are

$$\varepsilon_{xz} = \frac{1}{2}\frac{\partial w}{\partial x}, \qquad \varepsilon_{yz} = \frac{1}{2}\frac{\partial w}{\partial y}. \qquad \sigma_{xz} = \mu\frac{\partial w}{\partial x}, \qquad \sigma_{yz} = \mu\frac{\partial w}{\partial y}, \tag{1.1}$$

μ being the shear modulus.

In such a case, in absence of body forces, the equations of motion reduce to the single equation

$$\nabla^2 w = \frac{1}{c_T^2}\frac{\partial^2 w}{\partial t^2} \tag{1.2}$$

where $c_T^2 = \mu/\varrho$ is the square of velocity of propagation of transversal elastic waves.

In this paper we shall make extensive use of the two-sided integral Fourier transform defined by the following formulae:

$$F(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x, y)\, e^{i\alpha x}\, dx, \qquad f(x, y) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty+ic}^{+\infty+ic} F(\alpha, y)\, e^{-i\alpha x}\, d\alpha. \tag{1.3}$$

Here the transform parameter is a complex variable, and the path of integration in Eq. $(1.3)_2$ is located within the strip $c_1 < \operatorname{Im}\alpha < c_2$ which represents the region of regularity of $F(\alpha, y)$. We shall also use the following representation [1] of $F(\alpha, y)$:

$$F(\alpha, y) = F^-(\alpha, y) + F^+(\alpha, y) \tag{1.4}$$

in which the one-sided Fourier transforms

$$F^-(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{0} f(x, y)\, e^{i\alpha x}\, dx, \qquad F^+(\alpha, y) = \frac{1}{\sqrt{2\pi}}\int\limits_{0}^{\infty} f(x, y)\, e^{i\alpha x}\, dx, \tag{1.5}$$

are analytic functions in the respective halfplanes $\operatorname{Im}\alpha < c_2$ and $\operatorname{Im}\alpha > c_1$.

In the case of a quasistatic problem in which the fixed rectangular coordinate system (x, y, z) can be replaced by a convectional reference frame (x', y', z')

$$x = x' + ct, \qquad y = y', \qquad z = z'. \tag{1.6}$$

c being the constant velocity of motion of the system (x', y', z'), the equation of motion (1.2) takes the form

$$\beta^2\frac{\partial^2 w}{\partial x'^2} + \frac{\partial^2 w}{\partial y'^2} = 0.$$

Here $\beta^2 = 1 - c^2/c_T^2$. By applying the Fourier transform (1.3) to this equation and to Eq. (1.1) we obtain

$$\begin{aligned}
\Sigma_{xz}(\alpha, y') &= -i\mu\alpha W(\alpha, y'), \\
\Sigma_{yz}(\alpha, y') &= \mu \frac{dW(\alpha, y')}{dy'}, \\
\frac{d^2W(\alpha, y')}{dy'^2} &- \alpha^2\beta^2 W(\alpha, y') = 0.
\end{aligned} \tag{1.7}$$

Equation $(1.7)_3$ is now solved to yield the Fourier transforms of the displacement w and stresses σ_{xz}, σ_{yz},

$$\begin{aligned}
W(\alpha, y') &= A(\alpha) \operatorname{sh} \alpha\beta y' + B(\alpha) \operatorname{ch} \alpha\beta y', \\
\Sigma_{xz}(\alpha, y') &= -i\mu\alpha[A(\alpha) \operatorname{sh} \alpha\beta y' + B(\alpha) \operatorname{ch} \alpha\beta y'], \\
\Sigma_{yz}(\alpha, y') &= \mu\alpha\beta\, [A(\alpha) \operatorname{ch} \alpha\beta y' + B(\alpha) \operatorname{sh} \alpha\beta y'].
\end{aligned} \tag{1.8}$$

The unknown functions $A(\alpha)$ and $B(\alpha)$ are to be determined from the boundary conditions of the problem considered.

In the other type of quasistatic problem in which the displacement and stresses are harmonic functions of time, the transformation

$$g(x, y, t) = g^*(x, y) \exp\,(i\omega t) \tag{1.9}$$

(ω being the harmonic vibration frequency) reduces the equation of motion (1.2) to the form

$$\nabla^2 w^* + \sigma^2 w^* = 0.$$

Here $\sigma = \omega/c_T$. Let us now apply the Fourier transform (1.3) to Eq. (1.1) and to the latter equation; using (1.9) we obtain

$$\begin{aligned}
\Sigma^*_{xz}(\alpha, y) &= -i\mu\alpha W^*(\alpha, y), \\
\Sigma^*_{yz}(\alpha, y) &= \mu \frac{dW^*(\alpha, y)}{dy}, \\
\frac{d^2W^*(\alpha, y)}{dy^2} &- (\alpha^2 - \sigma^2)\, W^*(\alpha, y) = 0.
\end{aligned} \tag{1.10}$$

Equation $(1.10)_3$ is now solved to yield the Fourier transforms of w, σ_{xz} and σ_{yz},

$$\begin{aligned}
W^*(\alpha, y) &= A(\alpha) \operatorname{sh} y\sqrt{\alpha^2 - \sigma^2} + B(\alpha) \operatorname{ch} y\sqrt{\alpha^2 - \sigma^2}, \\
\Sigma^*_{xz}(\alpha, y) &= -i\mu\alpha\left[A(\alpha) \operatorname{sh} y\sqrt{\alpha^2 - \sigma^2} + B(\alpha) \operatorname{ch} y\sqrt{\alpha^2 - \sigma^2}\right], \\
\Sigma^*_{yz}(\alpha, y) &= \mu\sqrt{\alpha^2 - \sigma^2}\left[A(\alpha) \operatorname{ch} y\sqrt{\alpha^2 - \sigma^2} + B(\alpha) \operatorname{sh} y\sqrt{\alpha^2 - \sigma^2}\right].
\end{aligned} \tag{1.11}$$

The unknown functions $A(\alpha)$ and $B(\alpha)$ are to be determined from the corresponding boundary conditions.

2. Quasistatic Motion of the Crack

2.1. General solution. Let us consider an infinite, non-homogeneous and rigidly clamped at the surfaces ($y = h_1$, $y = -h_2$) layer consisting of two layers having different thicknesses and different elastic properties (Fig. 1). Let in the plane separating these two layers be located a flat, semi-infinite crack ($x < 0$, $y = 0$) loaded on its surfaces by forces $\sigma_{yz} = p(x)$ satisfying the conditions of

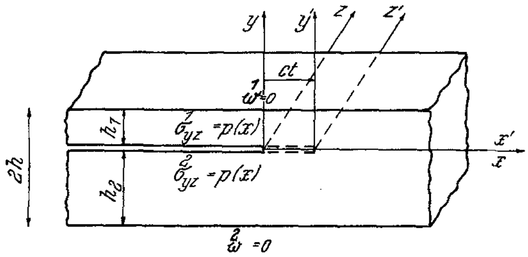


Abb. 1

antiplane state of strain. Both the crack and its loading are assumed to propagate at a constant velocity c along the x-axis of a fixed, rectangular coordinate system (x, y, z). By introducing the moving coordinate system (x', y', z') (1.6) the boundary conditions assume the form

$$
\begin{aligned}
&\overset{1}{w}(x', y') = 0 && \text{for} \quad |x'| < \infty, \quad && y' = h_1,\\
&\overset{2}{w}(x', y') = 0 && \text{for} \quad |x'| < \infty, \quad && y' = -h_2,\\
&\overset{1}{\sigma}_{yz}(x', y') = \overset{2}{\sigma}_{yz}(x', y') = p(x') && \text{for} \quad x' < 0, \quad && y' = 0,\\
&\overset{1}{w}(x', y') = \overset{2}{w}(x', y') && \text{for} \quad x' > 0, \quad && y' = 0,\\
&\overset{1}{\sigma}_{yz}(x', y') = \overset{2}{\sigma}_{yz}(x', y') && \text{for} \quad x' > 0, \quad && y' = 0,
\end{aligned}
\tag{2.1}
$$

where the upper indices $i = 1, 2$ refer to the displacements and stresses of the upper and lower layers, respectively. Applying the Fourier transform (1.3) to the first three relations of (2.1) and using Eqs. (1.8), the problem of determination of the stress intensity factor at the crack tip is reduced to the solution of the following two Wiener-Hopf equations:

$$
\begin{aligned}
\overset{1}{W}(\alpha, 0) &= -\frac{\operatorname{th} \alpha\beta_1 h_1}{\mu_1\beta_1\alpha}\, \overset{1}{\Sigma}_{yz}(\alpha, 0),\\
\overset{2}{W}(\alpha, 0) &= \frac{\operatorname{th} \alpha\beta_2 h_2}{\mu_2\beta_2\alpha}\, \overset{2}{\Sigma}_{yz}(\alpha, 0),
\end{aligned}
\tag{2.2}
$$

where $\beta_{1,2} = 1 - c^2/c^2_{T_{1,2}}$, and μ_1, μ_2, $c_{T_{1,2}}$ denote the shear moduli and transversal wave velocities of the respective upper and lower layers.

Introducing the notations due to Eq. (1.5)

$$\overset{1}{W}{}^{-}(\alpha, 0) - \overset{2}{W}{}^{-}(\alpha, 0) = W^{-}(\alpha),$$

$$\overset{1}{\Sigma}{}^{-}_{yz}(\alpha, 0) + \overset{1}{\Sigma}{}^{+}_{yz}(\alpha, 0) = \Sigma^{+}_{yz}(\alpha) + P(\alpha), \tag{2.3}$$

$$P(\alpha) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{0} p(x)\, e^{i\alpha x'}\, dx',$$

and using the last two relations of Eqs. (2.1), the system (2.2) is reduced to the single Wiener-Hopf equation

$$W^{-}(\alpha) = -H(\alpha)\,[\Sigma^{+}_{yz}(\alpha) + P(\alpha)] \tag{2.4}$$

in which

$$H(\alpha) = \frac{1}{\alpha}\left(\frac{\operatorname{th}\alpha\beta_1 h_1}{\mu_1\beta_1} + \frac{\operatorname{th}\alpha\beta_2 h_2}{\mu_2\beta_2}\right). \tag{2.5}$$

The region of existence of Eq. (2.4) is the strip

$$-\min\left(\frac{\pi}{2\beta_1 h_1}, \frac{\pi}{2\beta_2 h_2}\right) < -\varepsilon < \operatorname{Im}\alpha < 0.$$

In order to solve Eq. (2.4) by the method of factorization [2] it is necessary to factorize, first of all, the function (2.5). Applying here the procedure described in [3] let us write (2.5) in the form

$$H(\alpha) = \overline{H}(\alpha)\, H_1(\alpha). \tag{2.6}$$

The function $\overline{H}$ is required to behave at infinity ($|\alpha| \to \infty$) and at zero ($|\alpha| \to 0$) exactly in the same manner as $H(\alpha)$; $H_1(\alpha)$ should possess no zeros and no singularities within the strip $|\operatorname{Im}\alpha| < \varepsilon_1$, where $0 < \varepsilon \leqq \varepsilon_1 < \min(\pi/2\beta_1 h_1, \pi/2\beta_2 h_2)$. According to the assumptions concerning $\overline{H}$ it may be assumed that

$$\overline{H}(\alpha) = \frac{1+\gamma}{\mu_1\beta_1} R^{-}(\alpha)\, R^{+}(\alpha), \tag{2.7}$$

where

$$R^{\pm}(\alpha) = \frac{1}{\sqrt{\alpha \pm iA}}, \qquad A = \frac{1+\gamma}{\beta_1 h_2\,(k+\lambda)}, \tag{2.8}$$

and $k = h_1/h_2$, $\lambda = \mu_1/\mu_2$, $\gamma = \lambda\beta_1/\beta_2$.

The assumptions concerning $H_1(\alpha)$ are satisfied, and $H_1(\alpha) \to 1$ in the strip $|\operatorname{Im}\alpha| < \varepsilon_1$ for $|\alpha| \to \infty$, and hence the function may be represented in the form [2]

$$H_1(\alpha) = \frac{H_1^{+}(\alpha)}{H_1^{-}(\alpha)}, \tag{2.9}$$

where

$$\ln H_1^+(\alpha) = \frac{1}{2\pi i} \int\limits_{-\infty+i\gamma_2}^{\infty+i\gamma_2} \frac{\ln H(\zeta)}{\zeta - \alpha}\, d\zeta,$$
$$\ln H_1^-(\alpha) = \frac{1}{2\pi i} \int\limits_{-\infty+i\gamma_1}^{\infty+i\gamma_1} \frac{\ln H(\zeta)}{\zeta - \alpha}\, d\zeta, \tag{2.10}$$

here $-\varepsilon_1 < \gamma_2 < \gamma_1 < \varepsilon_1$.

Functions $H^{\pm}(\alpha)$ defined in this manner do not posses zeros and singular points within the respective halfplanes $\operatorname{Im}\alpha > \gamma_2$ and $\operatorname{Im}\alpha < \gamma_1$; in view of the fact that $H_1(0) = H_1(\infty) = 1$ they satisfy the additional condition $H_1^{\pm}(0) = H_1^{\pm}(\infty) = 1$.

Applying now the procedure described in [4], Eq. (2.4) is transformed with the aid of Eqs. (2.6), (2.7), (2.9) to yield

$$-\frac{\mu_1\beta_1}{1+\gamma} \frac{H_1^-(\alpha)\, W^-(\alpha)}{R^-(\alpha)} = R^+(\alpha)\, H_1^+(\alpha)\, \Sigma_{yz}^+(\alpha) + E(\alpha). \tag{2.11}$$

Here

$$E(\alpha) = R^+(\alpha)\, H_1^+(\alpha)\, P(\alpha). \tag{2.12}$$

If $E(\alpha)$ is assumed to be regular at least within the region of existence of Eq. (2.4), it may be represented in the form [2]

$$E(\alpha) = E^+(\alpha) - E^-(\alpha) \tag{2.13}$$

in which

$$E^+(\alpha) = \frac{1}{2\pi i} \int\limits_{-\infty-i\delta_2}^{\infty-i\delta_2} \frac{E(\zeta)}{\zeta - \alpha}\, d\zeta,$$
$$E^-(\alpha) = \frac{1}{2\pi i} \int\limits_{-\infty-i\delta_1}^{\infty-i\delta_1} \frac{E(\zeta)}{\zeta - \alpha}\, d\zeta. \tag{2.14}$$

Here $0 < \delta_1 < \delta_2 < \varepsilon$, and the functions $E^{\pm}(\alpha)$ are regular in the respective half-planes $\operatorname{Im}\alpha > -\varepsilon$ and $\operatorname{Im}\alpha < 0$. Equations (2.11), (2.13) and the generalized Liouville theorem enable us to write the final solution of Eq. (2.4),

$$W^-(\alpha) = \frac{1+\gamma}{\mu_1\beta_1} \frac{E^-(\alpha)\, R^-(\alpha)}{H_1^-(\alpha)} \qquad \text{reg. for } \operatorname{Im}\alpha < 0,$$
$$\Sigma_{yz}^+(\alpha) = -\frac{E^+(\alpha)}{R^+(\alpha)\, H_1^+(\alpha)} \qquad \text{reg. for } \operatorname{Im}\alpha > -\varepsilon. \tag{2.15}$$

These results make it possible to determine the exact value of the stress intensity factor — i.e. the magnitude which is special interest from the point of view of the crack stability problem [5]. This factor and the difference of displacements of the crack edges in the vicinity of the tip is now determined by applying the Abel theorem concerning Fourier transforms [6]; it enables us to determine the behaviour of inverse Fourier transforms at $|x| \to 0$ once the behaviour of Fourier transforms (2.15) at $|\alpha| \to \infty$ is known.

To this end let us use Eqs. (2.8), (2.14) and the properties of functions $E^{\pm}(\alpha)$ and $H_1^{\pm}(\alpha)$; it may be shown that the functions $W^-(\alpha)$ and $\Sigma_{yz}^+(\alpha)$ (2.15) assume for $|\alpha| \to \infty$ the form

$$W^-(\alpha) = -\frac{1+\gamma}{\mu_1\beta_1}\frac{B}{\alpha\sqrt{\alpha}},$$

$$\Sigma_{yz}^+(\alpha) = \frac{B}{\sqrt{\alpha}},$$

where

$$B = \frac{1}{\alpha\pi i}\int\limits_{-\infty-i\delta}^{\infty-i\delta} E(\zeta)\,d\zeta, \qquad 0 < \delta < \varepsilon. \tag{2.16}$$

On the basis of the Abel theorem cited above we conclude that with $|x'| \to 0$ the difference of displacements of the crack edges and the stress σ_{yz} along the positive x-axis are equal to

$$\begin{aligned} \overset{1}{\omega}(x') - \overset{2}{\omega}(x') &= \frac{2N(1+\gamma)}{\mu_1\beta_1}\sqrt{-x'} \qquad \text{for} \qquad x' \to (-0), \\ \overset{1}{\sigma}_{yz}(x') = \overset{2}{\sigma}_{yz}(x') &= \frac{N}{\sqrt{x'}} \qquad \text{for} \qquad x' \to (+0) \end{aligned} \tag{2.17}$$

where

$$N = -\sqrt{-2i}\,B. \tag{2.18}$$

Using (2.18) we can establish the exact value of the stress intensity factor in the case of arbitrary loading of the edges of the crack.

2.2. Constant loading of the edges of the crack. To illustrate the solution derived let us consider the case when the crack edges are loaded on their entire length by a constant load $\sigma_{yz}(x, 0) = p$. In view of Eqs. $(2.3)_3$, (2.8), (2.16), and (2.18) we may write

$$N = -p\sqrt{\frac{-i}{\pi}}\,\frac{1}{2\pi i}\int\limits_{-\infty-i\delta}^{\infty-i\delta}\frac{H_1^+(\zeta)}{\zeta\sqrt{\zeta+iA}}\,d\zeta.$$

Performing the integration and making use of Eq. (2.8) and of the fact that $H_1^+(0) = 1$, the stress intensity factor N is written in the form

$$N = -p\sqrt{\frac{\beta_1 h_2(k+\lambda)}{\pi(1+\gamma)}}. \tag{2.19}$$

From this formula it follows that with increasing velocities of crack propagation and with decreasing thickness of the layer, the stress intensity factor decreases. The factor (2.19) as a function of the crack propagation velocity is shown in Fig. 2 for the cases $\mu_1 = 3\mu$ and $\varrho_1 = 3\varrho_2$, and for $\mu_1 = \mu_2$, $\varrho_1 = \varrho_2$.

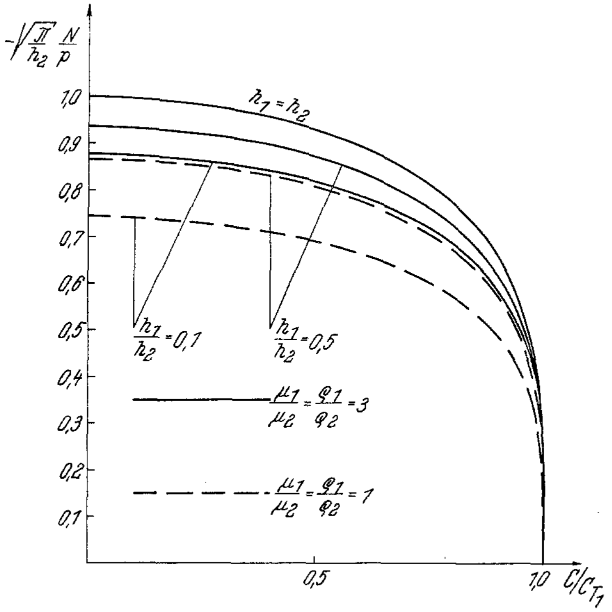


Abb. 2

Moreover, from (2.19) it follows that in the case of a homogeneous layer ($\mu_1 = \mu_2 = \mu$) and non-centrally located crack ($h_1 \neq h_2$) the stress intensity factor is independent of the thicknesses of individual layers. Its value is then the same as in the case of a homogeneous layer with a central crack [7]

$$N = -p \sqrt{\frac{\beta h}{\pi}}.$$

Here $2h = h_1 + h_2$, $\beta = \sqrt{1 - c^2/c_T^2}$.

Passing to the limit in (2.19) with $c \to 0$ we obtain the solution of a corresponding static problem, and then

$$N_s = -p \sqrt{\frac{h_2(k + \lambda)}{\pi(1 + \lambda)}}. \tag{2.20}$$

The variability of N_s as a function of $k = h_1/h_2$ for the cases $\mu_1 = 3\mu_2$ and $\mu_1 = \mu_2 = \mu$ is shown in Fig. 3.

From Eq. (2.20) it follows that in the cases when the layer is homogeneous ($\mu_1 = \mu_2 = \mu$) and the crack is not centrally located ($h_1 \neq h_2$), or when the layer is non-homogeneous ($\mu_1 \neq \mu_2$) and the crack lies in its middle plane ($h_1 = h_2 = h$), the stress intensity factor in the static case is independent of material constants and of the thicknesses of individual layers. Its value is the same as in the case of a homogeneous layer with a centrally located crack [7],

$$N = -p \sqrt{\frac{h}{\pi}}.$$

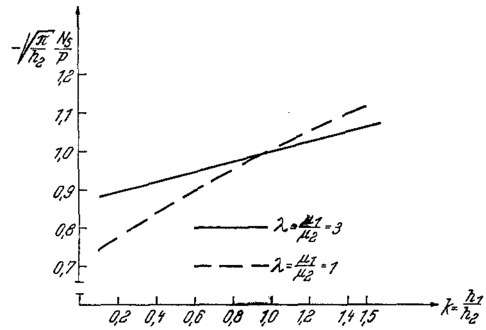


Abb. 3

2.3. Crack with stress-free edges. The solution (2.19) derived in the preceding section may be used to determine the stress intensity factor in the problem in which constant displacements $w = \mp w_0$ are prescribed over the surfaces $y = h_1$ and $y = -h_2$ of the strip, the crack edges remaining stress-free (Fig. 4a).

Applying the method of superposition the solution is represented in the form of a sum of solutions of a continuous layer and prescribed displacements $w = \mp w_0$ on the surfaces $y = h_1$, $y = -h_2$ (Fig. 4b), and of a layer with rigidly clamped surfaces containing a crack subject to constant loads $\bar{\bar{\sigma}}_{yz}(x, 0) = -\bar{\sigma}_{yz}(x, 0) = p_0$ (Fig. 4c). The displacement $\bar{w}$ and stresses $\bar{\sigma}_{yz}$, $\bar{\sigma}_{xz}$ in the problem shown in Fig. 4b are defined by the following relations:

$$
\begin{aligned}
\bar{w}(x, y) &= w_0 \begin{cases} -1 + \dfrac{2(h_1 - y)}{h_2(k + \lambda)} & \text{for } \ |x| < \infty, \quad 0 \leqq y \leqq h_1, \\[2ex] 1 - \dfrac{2\lambda(h_2 + y)}{h_2(k + \lambda)} & \text{for } \ |x| < \infty, \ -h_2 \leqq y \leqq 0, \end{cases} \\
\bar{\sigma}_{xz}(x, y) &= 0 \qquad\qquad\qquad\qquad\qquad \text{for } \ |x| < \infty, \ -h_2 \leqq y \leqq h_1, \\
\sigma_{yz}(x, y) &= -\frac{2\mu_1 w_0}{h_2(k + \lambda)} \qquad\qquad \text{for } \ |x| < \infty, \ -h_2 \leqq y \leqq h_1.
\end{aligned}
\tag{2.21}
$$

It follows that $\bar{\bar{\sigma}}_{yz}(x, 0) = -\bar{\sigma}_{yz}(x, 0) = p_0 = 2\mu_1 w_0/h_2(k + \lambda)$, and with the aid of Eq. (2.19) the stress intensity factor in the problem (2.19) takes the form

$$
N = -2\mu_1 w_0 \sqrt{\frac{\beta_1}{\pi h_2(k + \lambda)(1 + \gamma)}}. \tag{2.22}
$$

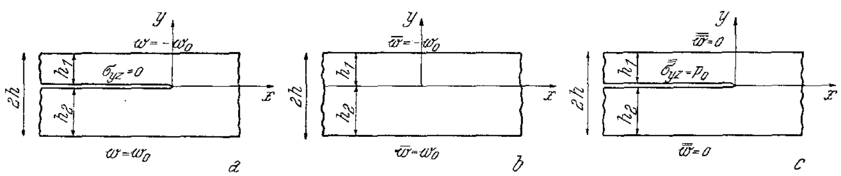


Abb. 4

This relation implies, similarly to the example discussed before, that with increasing crack propagation velocities the stress intensity factor decreases. In contrast to the earlier example, however, decreasing the thicknesses of individual layers is accompanied by increasing values of the factor. Fig. 5.

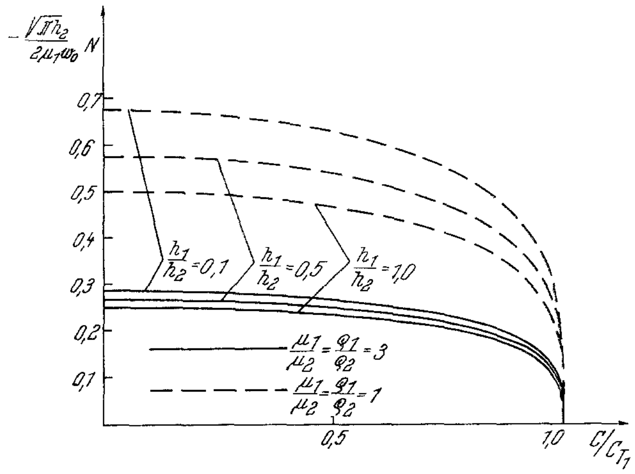


Abb. 5

From Eq. (2.22) it also follows that, similarly as before, in a homogeneous layer ($\mu_1 = \mu_2 = \mu$) with a non-centrally located crack the stress intensity factor remains independent of the ratio h_1/h_2 and is the same as in the case of a layer weakened by a crack lying in its middle plane [7],

$$N = -\mu w_0 \sqrt{\frac{\beta}{\pi h}}$$

where $\beta = \sqrt{1 - c^2/c_T^2}$.

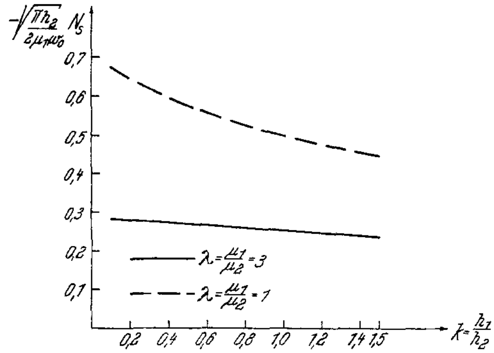
Passing to the limit in (2.22) with $c \to 0$ we obtain the stress intensity factor in a static case,

$$N_s = -\frac{2\mu_1 w_0}{\sqrt{\pi h_2 (k + \lambda)(1 + \lambda)}}. \tag{2.23}$$

The values of N_s as a function of the parameter $k = h_1/h_2$ for $\mu_1 = 3\mu_2$ and $\mu_1 = \mu_2$ is illustrated by the graph in Fig. 6.

Comparison of Eqs. (2.20) and (2.23) yields the conclusion that in a static case, in contrast to the result derived before, only when the layer is homogeneous ($\mu_1 = \mu_2 = \mu$) the stress intensity factor remains independent of the ratio h_1/h_2 and equals [7]

$$N_s = -\frac{\mu w_0}{\sqrt{\pi h}}.$$



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3. Harmonic Oscillation of Loads

3.1. General solution. In this part of the paper we shall tackle the problem of determination of the stress intensity factor at the crack tip under the assumption that the non-homogeneous layer (Fig. 1) contains a crack loaded by harmonic loads which fulfil the conditions of antiplane state of strain. The load applied to the edges of the crack may be written in the form

$$\sigma_{yz} = p_0(x) + p(x) \exp{(i\omega t)}.$$

By the method of superposition the static and quasistatic problems may be separated, and the boundary conditions for the latter problem assume, after transformation (1.9), the form

$$\begin{aligned}
&\overset{1}{w}{}^*(x, y) = 0 && \text{for} \quad |x| < \infty, \ y = h_1, \\
&\overset{2}{w}(x, y) = 0 && \text{for} \quad |x| < \infty, \ y = -h_2, \\
&\overset{1}{\sigma}{}^*_{yz}(x, y) = \overset{2}{\sigma}{}^*_{yz}(x, y) = p(x) && \text{for} \quad x < 0, \quad y = 0, \\
&\overset{1}{w}{}^*(x, y) = \overset{2}{w}{}^*(x, y) && \text{for} \quad x > 0, \quad y = 0, \\
&\overset{1}{\sigma}{}^*_{yz}(x, y) = \overset{2}{\sigma}{}^*_{yz}(x, y) && \text{for} \quad x > 0, \quad y = 0;
\end{aligned} \tag{3.1}$$

the superscripts $i = 1, 2$ referring, as before, to the upper and lower layers, respectively. Application of the Fourier transform (1.3) to Eqs. (3.1) reduces, with the help of Eq. (1.11), the problem to the solution of the following set of two Wiener-Hopf integral equations:

$$\begin{aligned}
\overset{1}{W}{}^*(\alpha, 0) &= -\frac{\operatorname{th} h_1 \sqrt{\alpha^2 - \sigma_1{}^2}}{\mu_1 \sqrt{\alpha^2 - \sigma_1{}^2}} \overset{1}{\Sigma}{}^*_{yz}(\alpha, 0), \\
\overset{1}{W}{}^*(\alpha, 0) &= \frac{\operatorname{th} h_2 \sqrt{\alpha^2 - \sigma_2{}^2}}{\mu_2 \sqrt{\alpha^2 - \sigma_2{}^2}} \overset{2}{\Sigma}{}^*_{yz}(\alpha, 0),
\end{aligned} \tag{3.2}$$

where $\sigma_{1,2} = \omega/c_{T1,2}$. Using the last two of Eqs. (3.1) and introducing the notations according to Eqs. (1.5),

$$\begin{aligned} &\overset{1}{W}{}^*(\alpha, 0) - \overset{2}{W}{}^*(\alpha, 0) = W^-(\alpha), \\ &\overset{1}{\Sigma}{}^*_{yz}(\alpha, 0) = \Sigma^+_{yz}(\alpha) + P(\alpha), \\ &P(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} p(x)\, e^{i\alpha x}\, dx, \end{aligned} \tag{3.3}$$

the system (3.2) is reduced to a single Wiener-Hopf equation

$$W^-(\alpha) = -H(\alpha)\,[\Sigma^+_{yz}(\alpha) + P(\alpha)] \tag{3.4}$$

where

$$H(\alpha) = \frac{\operatorname{th} h_1 \sqrt{\alpha^2 - \sigma_1^2}}{\mu_1 \sqrt{\alpha^2 - \sigma_1^2}} + \frac{\operatorname{th} h_2 \sqrt{\alpha^2 - \sigma_2^2}}{\mu_2 \sqrt{\alpha^2 - \sigma_2^2}}. \tag{3.5}$$

The region of existence of Eq. (4.5) coincides with the strip

$$-\min\left(\sqrt{\pi^2/4h_1^2 - \sigma_1^2},\ \sqrt{\pi^2/4h_2^2 - \sigma_2^2}\right) < -\varepsilon < \operatorname{Im}\alpha < 0. \tag{3.6}$$

Application of the procedure outlined by W. T. KOITER requires the function $H(\alpha)$ to be represented in the form

$$H(\alpha) = \overline{H}(\alpha)\, H_1(\alpha) \tag{3.7}$$

where

$$\begin{aligned} &\overline{H}(\alpha) = \frac{1+\lambda}{\mu_1} R^-(\alpha)\, R^+(\alpha), \\ &R^\pm(\alpha) = \frac{1}{\sqrt{\alpha \pm iA}}, \qquad A = \frac{\sigma_1(1+\lambda)}{\operatorname{tg} h_1\sigma_1 + \varkappa \operatorname{tg} h_2\sigma_2}, \\ &\varkappa = \lambda\sigma_1/\sigma_2 = \sqrt{\lambda\varrho_1/\varrho_2}, \end{aligned} \tag{3.8}$$

$H_1(\alpha)$ being regular and non-zero within the strip $|\operatorname{Im}\alpha| < \varepsilon_1$, $0 < \varepsilon \leqq \varepsilon_1 < \min\left(\sqrt{\pi^2/4h_1^2 - \sigma_1^2},\ \sqrt{\pi^2/4h_2^2 - \sigma_2^2}\right)$.

Repeating now the reasoning used in Sec. 2 we conclude that in the quasistatic case the difference of displacements of the crack edges and the stress σ_{yz} along the positive x-axis for $|x| \to 0$ are expressed by the formulae

$$\begin{aligned} &\overset{1}{w}(x) - \overset{2}{w}(x) = \frac{2(1+\lambda)\,N_q}{\mu_1}\sqrt{-x} \qquad \text{for} \quad x \to (-0), \\ &\overset{1}{\sigma}_{yz}(x) = \overset{2}{\sigma}_{yz}(x) = \frac{N_q}{\sqrt{x}} \qquad \text{for} \quad x \to (+0), \end{aligned} \tag{3.9}$$

and the stress intensity factor — by the formula

$$N_q = N^*(p, \omega) \exp(i\omega t) \tag{3.10}$$

where

$$N^*(p,\omega) = -\frac{\sqrt{-2i}}{2\pi i}\int\limits_{-\infty-i\delta}^{\infty-i\delta} R^+(\zeta)\,H_1^+(\zeta)\,P(\zeta)\,d\zeta. \tag{3.11}$$

By using the Eqs. (3.10), (3.11) we can determine the exact value of the stress intensity factor in the case of an arbitrary harmonic loading of the crack edges under the only assumption that the vibration frequency

$$\omega < \min\left(\pi c_{T_1}/2h_1,\ \pi c_{T_2}/2h_2\right).$$

The static solution is easily obtained by assuming $\omega = 0$ in Eqs. (3.10), (3.11). Superposing the both results we arrive at the final expression for the stress intensity factor in the case when the crack is loaded by $\sigma_{yz} = p_0(x) + p(x)\exp(i\omega t)$, namely

$$N = N^*(p_0, 0) + N^*(p,\omega)\exp(i\omega t). \tag{3.12}$$

3.2. Constant amplitude of vibration. To illustrate the foregoing considerations let us discuss the example of the crack loaded at its edges (Fig. 1) by $\sigma_{yz} = p_0 + p_1\cos\omega t$; $p_0, p_1 = \text{const}$. In view of Eqs. (3.3) and (3.11) we obtain in the quasistatic case

$$N^*(p_1,\omega) = -p_1\sqrt{\frac{i}{\pi}}\,\frac{1}{2\pi i}\int\limits_{-\infty-i\delta}^{\infty-i\delta}\frac{H_1(\xi)}{\zeta\sqrt{\zeta+iA}}\,d\zeta. \tag{3.13}$$

After integration and using Eq. $(3.8)_2$ and the fact that $H_1^+(0) = 1$ Eq. (3.13) assumes the form

$$N^*(p_1,\omega) = -p_1\sqrt{\frac{h_2}{\pi}}\sqrt{\frac{k(\operatorname{tg}\sigma_1' + \varkappa\operatorname{tg}\delta\sigma_1')}{\sigma_1'(1+\lambda)}} \tag{3.14}$$

where $\sigma_1' = \sigma_1 h_1$, $\delta = \lambda/k\varkappa$. The static solution is obtained by putting in Eq. (3.14) $\omega = 0$; the result coincides with Eq. (2.20). Combining the corresponding results (3.12), (2.20), and (3.14) we obtain the final form of the stress intensity factor

$$N = -p_0\sqrt{\frac{h_2}{\pi(1+\lambda)}}\left[\sqrt{k+\lambda} + \frac{p_1}{p_0}\sqrt{\frac{k(\operatorname{tg}\sigma_1' + \varkappa\operatorname{tg}\delta\sigma_1'}{\sigma_1'}}\cos\omega t\right]. \tag{3.15}$$

From this formula it follows that in the limiting case when $\omega \to \min(\pi c_{T_1}/2h_1, \pi c_{T_2}/2h_2)$, a resonance-type phenomenon occurs: arbitrarily small load components p_1 provoke infinitely large values of $N \to \infty$. It is also evident that at a constant frequency ω thinner layers lead to smaller stress intensity factors (Fig. 7). The maximum value of N at a constant frequency ω is given by the formula

$$N_m = -p_0\sqrt{\frac{h_2}{\pi(1+\lambda)}}\left[\sqrt{k+\lambda} + \frac{p_1}{p_0}\sqrt{\frac{k(\operatorname{tg}\sigma_1' + \varkappa\operatorname{tg}\delta\sigma_1')}{\sigma_1'}}\right].$$

Fig. 7 demonstrates the variability of N_m with changing ω for $\mu_1 = 3\mu_2$, $\varrho_1 = 3\varrho_2$, and $\mu_1 = \mu_2$, $\varrho_1 = \varrho_2$.

Assuming in Eq. (3.15) that $k = \lambda = \varkappa = 1$ we obtain the stress intensity factor as given in [8] and referring to the quasistatic case of a homogeneous

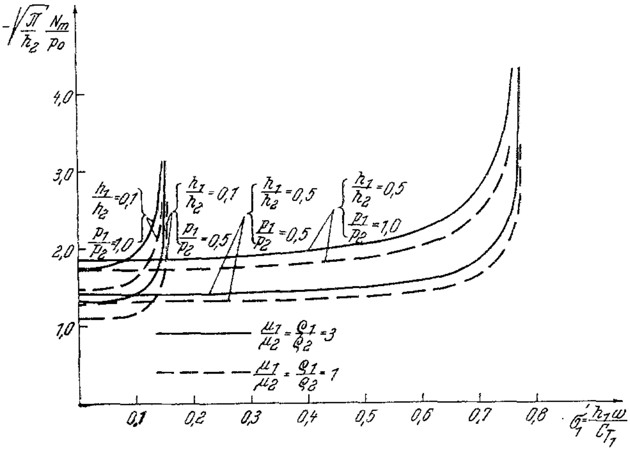

Abb. 7

layer with a central crack,

$$N = -p_0 \sqrt{\frac{h}{\pi}} \left[1 + \frac{p_1}{p_0} \sqrt{\frac{\operatorname{tg} \sigma'}{\sigma'}} \cos \omega t\right]$$

here $\sigma' = h\omega/c_T$.

3.3. Crack with stress-free edges. The solution (3.15) derived in the preceding section may be used to determine the stress intensity factor in the case when the crack is free of stresses whereas the boundary surfaces of the strip $y = h\mathrm{h}_1$, $y = -h_2$ are subject to prescribed displacements $w = \mp(w_0 + w_1 \cos \omega t)$, with

$$\omega < \min\left(\pi c_{T1}/2h_1, \pi c_{T2}/2h_2\right), \quad \text{and} \quad w_0, w_1 = \text{const}.$$

Considering the static and quasistatic problems separately and repeating the way of reasoning used in Sec. 2.3, the determination of the stress intensity factor in the problem formulated above may be reduced to the determination of the parameters p_0, p_1 appearing in Eq. (3.15); this is reduced, in turn, to the determination of stresses $\sigma_{yz}(x, 0)$ in a continuous layer whose boundary surfaces are subjected to prescribed displacements: $w = \mp w_0$ in the static case, and $w = \mp w_1 \cos \omega t$ in the quasistatic case.

The displacement w and stresses σ_{xz}, σ_{yz} in the static case of a continuous layer (Fig. 4b) are given by Eqs. (2.21) and in the corresponding quasistatic case — by the following formulae:

$$w(x, y, t) = -w_1 \cos \omega t \begin{cases} f(\sigma) \sin \sigma_1 y - g(\sigma) \cos \sigma_1 y & \text{for} \quad 0 \leqq y \leqq h_1, \\ \varkappa f(\sigma) \sin \sigma_2 y - g(\sigma) \cos \sigma_2 y & \text{for} \quad -h_2 \leqq y \leqq 0, \end{cases}$$

$$\sigma_{xz}(x, y, t) = 0 \qquad \text{for} \quad -h_2 \leqq y \leqq h_1, \tag{3.16}$$

$$\sigma_{yz}(x, y, t) = -w_1 \cos \omega t \begin{cases} \mu_1 \sigma_1 [g(\sigma) \sin \sigma_1 y + f(\sigma) \cos \sigma_1 y] & \text{for} \quad 0 \leqq y \leqq h_1, \\ \mu_2 \sigma_2 [g(\sigma) \sin \sigma_2 y + \varkappa f(\sigma) \cos \sigma_2 y] & \text{for} \quad -h_2 \leqq y \leqq 0, \end{cases}$$

here

$$f(\sigma) = \frac{\cos \sigma_1' + \cos \delta\sigma_1'}{\cos \sigma_1' \cos \delta\sigma_1' (\operatorname{tg} \sigma_1' + \varkappa \operatorname{tg} \delta\sigma_1')},$$
$$g(\sigma) = \frac{\sin \sigma_1' - \varkappa \sin \delta\sigma_1'}{\cos \sigma_1' \cos \delta\sigma_1' (\operatorname{tg} \sigma_1' + \varkappa \operatorname{tg} \delta\sigma_1')}. \tag{3.17}$$

By applying the Eqs. (2.21), $(3.16)_3$, and $(3.17)_1$ the coefficients p_0 and p_1 sought for are calculated

$$p_0 = \frac{2\mu_1 w_0}{h_2(k + \lambda)},$$
$$p_1 = \frac{\mu_1 w_1 \sigma_1 (\cos \sigma_1' + \cos \delta\sigma_1')}{\cos \sigma_1' \cos \delta\sigma_1' (\operatorname{tg} \sigma_1' + \varkappa \operatorname{tg} \delta\sigma_1')}.$$

The results are now substituted in Eq. (3.15) to yield the final form of the stress intensity parameter

$$N = -\frac{2\mu_1 w_0}{\sqrt{\pi h_2}\,(1 + \lambda)} \left[\frac{1}{\sqrt{k + \lambda}} + \frac{w_1 m(\sigma)}{w_0} \cos \omega t\right]. \tag{3.18}$$

Here

$$m(\sigma) = \frac{\cos \sigma_1' + \cos \delta\sigma_1'}{2 \cos \sigma_1' \cos \delta\sigma_1'} \sqrt{\frac{\sigma_1'}{k(\operatorname{tg} \sigma_1' + \varkappa \operatorname{tg} \delta\sigma_1')}}. \tag{3.19}$$

Passing to the limit with $c \to 0$ in Eqs. (3.18) and (3.19) we obtain the stress intensity factor in the static case of a strip with prescribed displacements $w = \mp(w_0 + w_1)$ at the boundary surfaces; the result coincides with Eq. (2.23).

Equation (3.18) also yields the resonance-type phenomenon when the vibration frequency tends to $\min(\pi c_{T_1}/2h_1, \pi c_{T_2}/2h_2)$. In contrast to the former result, however, thinner layers lead to larger stress intensity factors (at a constant ω).

The maximum of N in Eq. (3.18) at a constant value of ω is given by the formula

$$N_m = -\frac{2\mu_1 w_0}{\sqrt{\pi h_2}\,(1 + \lambda)} \left[\frac{1}{\sqrt{k + \lambda}} + \frac{w_1 m(\sigma)}{w_0}\right].$$

Assuming in Eqs. (3.18) and (3.19) $k = \lambda = \varkappa = 1$ we obtain the stress intensity factor as derived in [8] and referring to a homogeneous layer with a centrally located crack,

$$N = -\frac{\mu w_0}{\sqrt{\pi h}} \left[1 + \frac{w_1}{w_0} \sqrt{\frac{2\sigma'}{\sin 2\sigma'}} \cos \omega t\right]$$

with the notation $\sigma' = \omega h / c_T$.

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