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Inverse Crack Problem in Elasticity

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Summary

Exact solution is given to the problem of a penny-shaped crack embedded in a transversely isotropic elastic half-space when arbitrary normal displacements are prescribed at its faces. A new integral representation of the kernel of the governing integral equation allowed to obtain closed form expressions for all the quantities of interest like, stresses inside and outside the crack, stress intensity factor, work done to open the crack, directly through the given displacements. Several illustrative examples are considered.

Introduction

The elastic crack problems considered in the literature generally assume the stress distribution at the crack faces known, and the displacements are to be determined. Investigation of the materials with rigid inclusions led to another formulation of the crack problem, namely, the displacements are considered prescribed at the crack faces, with the other quantities to be determined. The problem so formulated is called here 'inverse crack problem'.

The exact solution to the axisymmetric inverse crack problem was given by Olesiak and Sneddon [1] for isotropic body using the coupled integral equations method. Here, the non-axially-symmetric problem is solved in a closed form for the case of transversely isotropic medium. The problem is reduced to a new type of integral equation which can be solved by a straightforward procedure. One can compute the total force, the moments, the stress intensity factor, etc., directly through the given displacements without calculation of the stresses. Several examples are considered in order to illustrate the advantages of the new method.

Formulation of the Problem and Its Solution

Consider a transversely isotropic space weakened in the plane $z = 0$ by a penny-shaped crack $\varrho \leqq a$. Both sides of the crack are deformed by a smooth rigid inclusion. Stresses in the plane $z = 0$ are to be determined. The normal displacements of the crack faces are prescribed as

$$w = w(\varrho, \psi), \qquad z = 0^+, \qquad 0 \leqq \varrho \leqq a$$
$$w = -w(\varrho, \phi), \qquad z = 0^-, \qquad 0 \leqq \varrho \leqq a.$$

The tangential stresses are zero all over the plane $z = 0$. Due to the symmetry of the problem, it can be reduced to the problem in a half-space $z \geqq 0$ with the boundary conditions at $z = 0$

$$\begin{aligned} w &= w(\varrho, \phi), \qquad 0 \leqq \varrho \leqq a; \\ w &= 0, \qquad \varrho > a. \end{aligned} \tag{1}$$

This problem may be called in a certain sense an inverse crack problem, because usually the stresses are given, and the displacements of the crack faces are to be determined. The direct crack problem was solved in [2] and the relationship between the displacements of the crack faces and the stresses was established as

$$w(\varrho, \phi) = 4H \int_\varrho^a \frac{dx}{\sqrt{x^2 - \varrho^2}} \int_0^x \frac{\varrho_0 \, d\varrho_0}{\sqrt{x^2 - \varrho_0{}^2}} \, L\left(\frac{\varrho\varrho_0}{x^2}\right) \sigma(\varrho_0, \phi). \tag{2}$$

Here the L-operator is defined as

$$L(k) f(y, \psi) = \sum_{n=-\infty}^{\infty} k^{|n|} f_n(y) \, e^{in\psi} \tag{3}$$

and f_n is the Fourier coefficient of f, namely,

$$f_n(y) = \frac{1}{2\pi} \int_0^{2\pi} f(y, \psi) \, e^{-in\psi} \, d\psi. \tag{4}$$

The following properties of the L-operators were stated in [2].

$$L(k) \, L(k_1) = L(kk_1); \qquad L(1) \, f = f. \tag{5}$$

The quantity H in (2) stands for

$$H = \frac{(\gamma_1 + \gamma_2) \, c_{11}}{2\pi(c_{11}c_{33} - c_{13}^2)}$$

$$\gamma_{1,2}^2 = N \pm \sqrt{N^2 - c_{33}/c_{11}}, \qquad N = (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})/2c_{11}c_{44} \tag{6}$$

and c_{ij} are elastic constants of the material of the elastic space.

Now (2) can be considered as an integral equation with respect to the yet unknown stress σ. An exact solution in closed form can be obtained by using the methods similar to those of [2]. Application to both sides of (2) of the operator

$$L(r) \frac{d}{dr} \int_r^a \frac{\varrho \, d\varrho}{\sqrt{\varrho^2 - r^2}} \, L\left(\frac{1}{\varrho}\right)$$

yields

$$2\pi H \int_0^r \frac{\varrho_0\, d\varrho_0}{\sqrt{r^2-\varrho_0^2}}\, L\left(\frac{\varrho_0}{r}\right) \sigma(\varrho_0, \phi) = -L(r)\,\frac{d}{dr}\int_r^a \frac{\varrho\, d\varrho}{\sqrt{\varrho^2-r^2}}\, L\left(\frac{1}{\varrho}\right) w(\varrho, \phi). \quad (7)$$

Here the properties (5) of the L-operators were used along with the integral

$$\int_r^x \frac{\varrho\, d\varrho}{\sqrt{\varrho^2-r^2}\,\sqrt{x^2-\varrho^2}} = \frac{\pi}{2}. \quad (8)$$

The next operator to be applied is

$$L\left(\frac{1}{y}\right)\frac{d}{dy}\int_y^0 \frac{r\, dr}{\sqrt{y^2-r^2}}\, L(r).$$

The result is

$$\pi^2 H y \sigma(y, \phi) = -L\left(\frac{1}{y}\right)\frac{d}{dy}\int_0^y \frac{r\, dr}{\sqrt{y^2-r^2}}\, L(r^2)\,\frac{d}{dr}\int_r^a \frac{\varrho\, d\varrho}{\sqrt{\varrho^2-r^2}}\, L\left(\frac{1}{\varrho}\right) w(\varrho, \phi)$$

and one finally gets the solution

$$\sigma(y, \phi) = -\frac{1}{\pi^2 H y}\, L\left(\frac{1}{y}\right)\frac{d}{dy}\int_0^y \frac{r\, dr}{\sqrt{y^2-r^2}}\, L(r^2)\left(\frac{d}{dr}\right)\int_r^a \frac{\varrho\, d\varrho}{\sqrt{\varrho^2-r^2}}\, L\left(\frac{1}{\varrho}\right) w(\varrho, \phi). \quad (9)$$

Formula (9) is valid inside the crack only. The stresses outside can also be expressed directly through the given normal displacements w. One can use for this purpose the relationship between the normal stresses inside and outside the crack [2], namely

$$\sigma(\varrho, \phi) = -\frac{2}{\pi\sqrt{\varrho^2-a^2}}\int_0^a \frac{\sqrt{a^2-y^2}}{\varrho^2-y^2}\, L\left(\frac{y}{\varrho}\right) \sigma(y, \phi)\, y\, dy, \quad \text{for} \quad \varrho > a. \quad (10)$$

Substitution of (9) into (10) yields after integration with respect to y

$$\sigma(\varrho, \phi) = \frac{L(\varrho^{-1})}{\pi^2 H \varrho^2 \sqrt{\varrho^2-a^2}}\int_0^a \left(a - r\sqrt{\frac{\varrho^2-a^2}{\varrho^2-r^2}}\right) dr\, \frac{d}{dr}\, r L(r^2)$$
$$\cdot \frac{d}{dr}\int_r^a \frac{\varrho_0\, d\varrho_0}{\sqrt{\varrho_0^2-r^2}}\, L\left(\frac{1}{\varrho_0}\right) w(\varrho_0, \phi). \quad (11)$$

Here the following rule of differentiation under the integral sign

$$\frac{d}{dx}\int_0^x \frac{F(t)\, dt}{\sqrt{x^2-t^2}} = \frac{1}{x}\int_0^x \frac{t\, dF(t)}{\sqrt{x^2-t^2}} \quad (12)$$

and the integral

$$\int_r^a \frac{\sqrt{a^2 - y^2}\, dy}{y\sqrt{y^2 - r^2}\,(\varrho^2 - y^2)} = \frac{\pi}{2\varrho^2}\left[\frac{a}{r} - \sqrt{\frac{\varrho^2 - a^2}{\varrho^2 - r^2}}\right]$$

were employed.

Introducing the stress intensity factor N as

$$N = \lim_{\varrho \to a^+} \left[\sigma(\varrho, \phi) \sqrt{\varrho - a}\right]$$

one immediately gets from (11)

$$N(\phi) = \lim_{x \to a} N(x, \phi)$$

where

$$N(x, \phi) = \frac{L(x)}{\pi^2 H \sqrt{2x}} \frac{d}{dx} \int_x^a \frac{\varrho\, d\varrho}{\sqrt{\varrho^2 - x^2}} L\left(\frac{1}{\varrho}\right) w(\varrho, \phi) \tag{13}$$

which gives the possibility to evaluate the stress intensity factor directly through the given displacements and avoiding the necessity to evaluate the stress distribution.

The work done in opening the crack is given by

$$W = \int_0^{2\pi} \int_0^a \sigma(\varrho, \phi)\, w(\varrho, \phi)\, \varrho\, d\varrho\, d\phi .$$

Substituting (9) in the last expression and integrating by parts gives

$$W = 2\pi^2 H \int_0^{2\pi} \int_0^a N^2(x, \phi)\, x\, dx\, d\phi ,$$

where $N(x, \phi)$ is given by (13), which presents a non-axisymmetric generalization to the axisymmetric case considered by Sneddon [5].

Integration by parts in (11) leads to

$$\sigma(\varrho, \phi) = \frac{L(\varrho^{-1})}{\pi^2 H} \int_0^a \frac{r\, dr}{(\varrho^2 - r^2)^{3/2}} L(r^2) \frac{d}{dr} \int_r^a \frac{\varrho_0\, d\varrho_0}{\sqrt{\varrho_0^2 - r^2}} L\left(\frac{1}{\varrho_0}\right) w(\varrho_0, \phi) . \tag{14}$$

Using the rule of differentiation,

$$\frac{d}{dr} \int_r^a \frac{F(\varrho)\, d\varrho}{\sqrt{\varrho^2 - r^2}} = -\frac{F(a)\, a}{r\sqrt{a^2 - r^2}} + \frac{1}{r} \int_r^a \frac{\varrho\, dF(\varrho)}{\sqrt{\varrho^2 - r^2}}$$

and the condition $w(a, \phi) = 0$, expression (14) can be rewritten as

$$\sigma(\varrho, \phi) = \frac{1}{\pi^2 H} \int_0^a \frac{dr}{(\varrho^2 - r^2)^{3/2}} \int_r^a \frac{\varrho_0 \, d\varrho_0}{\sqrt{\varrho_0{}^2 - r^2}} \frac{\partial}{\partial \varrho_0} \left[\varrho_0 L\left(\frac{r^2}{\varrho \varrho_0}\right) w(\varrho_0, \phi) \right]. \quad (15)$$

Change of the order of integration and integration by parts leads to another form

$$\sigma(\varrho, \phi) = -\frac{1}{4\pi^2 H} \int_0^{2\pi} \int_0^a \frac{w(\varrho_0, \phi_0) \, \varrho_0 \, d\varrho_0 \, d\phi_0}{\left(\varrho^2 + \varrho_0{}^2 - 2\varrho\varrho_0 \cos(\phi - \phi_0)\right)^{3/2}}. \quad (16)$$

Expressions (14)—(16) are equivalent, and it is a question of convenience which one to use in each particular case.
The resultant force P can be obtained by integration of (9), namely

$$P = \int_0^a \int_0^{2\pi} \sigma(y, \phi) \, y \, dy \, d\phi. \quad (17)$$

Substitution of (9) into (17) and integration leads to

$$P = -\frac{1}{\pi^2 H} \int_0^{2\pi} d\phi \int_0^a \frac{r \, dr}{\sqrt{a^2 - r^2}} \frac{d}{dr} \int_r^a \frac{w(\varrho, \phi) \, \varrho \, d\varrho}{\sqrt{\varrho^2 - r^2}}. \quad (18)$$

Using (12) and integrating by parts, one can get another equivalent representation, namely

$$P = \frac{a^2}{\pi^2 H} \int_0^{2\pi} d\phi \int_0^a \frac{dr}{(a^2 - r^2)^{3/2}} \int_r^a \frac{w(\varrho, \phi) \, \varrho \, d\varrho}{\sqrt{\varrho^2 - r^2}}. \quad (19)$$

Several additional forms can be obtained by integration in (19) with respect to r and consequent use of various available integral representations for the elliptic integral of the second kind. One of such alternatives gives

$$P = \frac{1}{\pi^2 H} \int_0^{2\pi} d\phi \int_0^a \sqrt{a^2 - r^2} \, dr \int_r^a \frac{w(\varrho, \phi) \, \varrho \, d\varrho}{(a^2 - \varrho^2) \sqrt{\varrho^2 - r^2}}. \quad (20)$$

One can also evaluate the resultant moment of the stresses exerted by the rigid inclusion. Introducing the complex moment $M = M_x + iM_y$, one can deduce

$$M = -i \int_0^{2\pi} \int_0^a \sigma(\varrho, \phi) \, e^{i\varphi} \varrho^2 \, d\varrho \, d\phi. \quad (21)$$

Substitution of (9) into (21) gives

$$M = \frac{i}{\pi^2 H} \int_0^{2\pi} e^{i\phi}\, d\phi \int_0^a \frac{r^3\, dr}{\sqrt{a^2 - r^2}} \frac{d}{dr} \int_r^a \frac{w(\varrho, \phi)\, d\varrho}{\sqrt{\varrho^2 - r^2}}. \tag{22}$$

Integration by parts yields another form, namely

$$M = -\frac{i}{\pi^2 H} \int_0^{2\pi} e^{i\phi}\, d\phi \int_0^a \frac{r^2(3a^2 - 2r^2)}{(a^2 - r^2)^{3/2}}\, dr \int_r^a \frac{w(\varrho, \phi)\, d\varrho}{\sqrt{\varrho^2 - r^2}}. \tag{23}$$

As one can notice

$$\frac{r^2(3a^2 - 2r^2)}{(a^2 - r^2)^{3/2}} = \frac{d}{dr} \frac{r^3}{\sqrt{a^2 - r^2}}.$$

Examples

It is of interest to consider the case when the displacements can be given by an expansion

$$w(\varrho, \phi) = \sum_{n=-\infty}^{\infty} \sqrt{a^2 - \varrho^2}\, C_n \varrho^{|n|} e^{in\psi}. \tag{24}$$

Substitution of (24) into (9) gives the stresses as

$$\sigma(\varrho, \phi) = \frac{1}{2\sqrt{\pi}\, H} \sum_{n=-\infty}^{\infty} C_n \frac{\Gamma(|n| + 3/2)}{\Gamma(|n| + 1)} \varrho^{|n|} e^{in\psi} \quad \text{for} \quad \varrho \leqq a. \tag{25}$$

Using (19) the resultant force is

$$P = \frac{\pi a^2}{4H} C_0. \tag{26}$$

The resultant moment is given by (22) as

$$M = -i \frac{3\pi a^4}{16H} C_{-1}. \tag{27}$$

Substitution of (24) into (14) or (15) yields the expression for the normal stresses outside the crack

$$\sigma(\varrho, \phi) = -\frac{1}{2\pi H} \sum_{n=-\infty}^{\infty} C_n \frac{a^{2|n|+3} e^{in\phi}}{(2\,|n| + 3)\, \varrho^{|n|+3}} F\left(\frac{3}{2}, \frac{3}{2} + |n|; \frac{5}{2} + |n|; \frac{a^2}{\varrho^2}\right) \quad \text{for} \quad \varrho < a. \tag{28}$$

The Gauss hypergeometric function in (28) can be expressed in elementary functions [4]

$$F\left(\frac{3}{2}, \frac{3}{2} + n; \frac{5}{2} + n; z\right) = \frac{(-1)^n\, (3 + 2n)}{n!\, \sqrt{1 - z}} \frac{d^n}{dz^n} \cdot \left\{\frac{(1 - z)^n}{z} \left[1 - \sqrt{\frac{1 - z}{z}} \sin^{-1} \sqrt{z}\right]\right\}.$$

The stresses in (28) become singular when $\varrho \to a$. Using integration by parts in (14) the singular and the nonsingular parts can be written explicitly

$$\sigma(\varrho, \phi) = -\frac{1}{2\pi H} \sum_{n=-\infty}^{\infty} C_n \frac{a^{2|n|+1}}{\varrho^{|n|}} \cdot \left[\frac{1}{\sqrt{\varrho^2 - a^2}} - \frac{1}{\varrho} F\left(\frac{1}{2}, |n| + \frac{1}{2}; |n| + \frac{3}{2}; \frac{a^2}{\varrho^2}\right)\right] e^{in\phi} \tag{29}$$

which yields the stress intensity factor as

$$N(a, \phi) = \frac{\sqrt{a}}{2\sqrt{2}\,\pi H} \sum_{n=-\infty}^{\infty} C_n a^{|n|} e^{in\phi}.$$

The hypergeometric function in (29) can also be expressed in elementary functions [4], namely

$$F\left(\frac{1}{2}, \frac{1}{2} + n, \frac{3}{2} + n; z\right) = (-1)^n \frac{2n+1}{n!} \sqrt{1-z} \frac{d^n}{dz^n} \left[\frac{(1-z)^{n-(1/2)}}{\sqrt{z}} \sin^{-1} \sqrt{z}\right]. \tag{30}$$

For the case of axial symmetry formulae (25) and (29) give

$$\sigma(\varrho, \phi) = \begin{cases} \dfrac{C_0}{4H} & \text{for} \quad \varrho \leqq a \\ -\dfrac{C_0}{2\pi H}\left[\dfrac{a}{\sqrt{\varrho^2 - a^2}} - \sin^{-1}\dfrac{a}{\varrho}\right] & \text{for} \quad \varrho > a. \end{cases} \tag{31}$$

Formula (31) corresponds to the results reported in [1].

Discussion

It is of interest to mention some properties of the solutions obtained. Expression (14) can be rewritten as

$$\sigma(\varrho, \phi) = -\frac{1}{\pi^2 H \varrho} L\left(\frac{1}{\varrho}\right) \frac{d}{d\varrho} \int_0^a \frac{r\,dr}{\sqrt{\varrho^2 - r^2}} L(r^2) \frac{d}{dr} \int_r^a \frac{\varrho_0\,d\varrho_0}{\sqrt{\varrho_0^2 - r^2}} L\left(\frac{1}{\varrho_0}\right) w(\varrho_0, \phi). \tag{32}$$

Comparison of (32) with (9) shows that the stresses inside and outside the crack can be expressed in a similar way with the only difference in the upper limit of the first integral.

Multiplication of both sides of (10) by $\varrho^{|n|} e^{in\phi}$ and integration over the area outside the crack results in

$$\int_0^{2\pi} \int_a^{\infty} \sigma(\varrho, \phi)\, \varrho^{|n|} e^{in\phi}\, d\varrho\, d\phi = -\int_0^{2\pi} \int_0^{a} \sigma(\varrho, \phi)\, \varrho^{|n|+1} e^{in\phi}\, d\varrho\, d\phi. \tag{33}$$

For $n = 0$ and $n = 1$ the integrals in (33) are proportional to the resultant force and the resultant moment respectively. Expression (33) shows that the normal stresses inside and outside the crack are in equilibrium.

All the results of this paper are valid for the case of isotropy provided that $H = (1 - \nu^2)/\pi E$, where ν is the Poisson ratio and E stands for the elasticity modulus.

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