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Bifurcation Analysis of the Triaxial Test on Sand Samples

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With 9 Figures

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Summary — Zusammenfassung

Bifurcation Analysis of the Triaxial Test on Sand Samples. A bifurcation analysis of a strain hardening dilatant sand sample in the triaxial test is carried out. The analysis shows that the triaxial test yields only then the limiting soil properties if 1) the sample is compact enough and 2) if the confining pressure does not exceed a critical value depending on the soil anisotropy and the slenderness of the sample.

Verzweigungsanalyse des Dreiachsialversuchs von Sandproben. Eine Verzweigungsanalyse einer verfestigenden dilatanten Sandprobe im Dreiachsialversuch wurde durchgeführt. Die Analyse zeigt, daß der Dreiachsialversuch nur dann die Grenzeigenschaften des Erdstoffs liefert, wenn 1) die Probe gedrungen genug ist und 2) wenn der Seitendruck nicht einen kritischen Wert überschreitet, der von der Anisotropie des Materials und der Schlankheit der Probe abhängt.

Introduction

For selecting a yield criterion for soils, K. H. Roscoe et al. [1] have presented the results of so-called 'special' triaxial compression and extension tests, where very precise records of the failure patterns have been taken. This series has shown that it is difficult to interpret the experimental data, due to the appreciable bulging or necking of the samples. As an improvement of the standard triaxial test, lubrication at the end plattens has been used. The experiment shows that it is not possible to prevent inhomogeneous strain fields by refinements on the boundaries [2], [3]. These results are forcing us to suppose that a spontaneous homogeneity loss is possible. This possibility can be investigated by asking for bifurcation modes under ideal boundary conditions. If solutions of this type actually exist, than it is reasonable to assume that imperfections can only intensify this tendency.

The bifurcation analysis developed here is based mostly on the techniques of similar investigations on metal plasticity problems [4], [5], [6]. The main differences to these works arise from the special constitutive properties valid for sand, namely friction and dilatancy. The sand behaviour considered here is similar to that of the 'psammic material' introduced by Th. Dietrich [7], i.e. for sand at low pressures. According to this model dilatancy is assumed as an internal constraint, and the elasticity of the grains is very high. This model led to successful results for

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the shear band bifurcation in the biaxial test [8], [9], where the most unsafe solution for the shear band corresponds to the limiting case of 'psammic' behaviour.

With the truly triaxial apparatus of Karlsruhe, M. Goldscheider [10] has confirmed the validity of the Mohr-Coulomb limiting condition, which implies that the angle of friction is not dependent on the mean principal stress (i.e. on σ_2/σ_1 and σ_2/σ_3 for $\sigma_3 < \sigma_2 < \sigma_1 < 0$). We here assume the validity of the Mohr-Coulomb yield condition for simplicity, and a unique hardening rule is taken for both the triaxial compression and extension. The corresponding stress-strain curve is taken from biaxial experiments which undergo homogeneous deformation up to the peak [8], [9].

For determining the volumetric flow rule a simplified shear strength — dilatancy relationship is used, which is a modification of Taylor's formula [11], [12]. Concerning the deviatoric flow rule it can be shown that in the neighbourhood of a triaxial compression or extension and for a smooth plastic potential no new informations can be obtained.

Experimental Results

The failure modes observed in the triaxial test on dry sand samples with lubricated end plattens are almost axisymmetric [2], [3]. Fig. 1a shows the sand sample before the test. The force piston was clamped into the top cap to prevent tilting of the specimens during a test. The end caps carry polished stainless steel plates and a silicone grease to prevent boundary friction. They carry also a small porous stone to keep the sample centred. To reduce the influence of weight the height of the sample has been chosen unusually small ($R/H = 0.6$). Fig. 1b demonstrates the failure pattern of a dense sample at an axial strain of -41%. This mode is the most possible one at the triaxial compression test. At a strain controlled test, bulging begins slowly to develop. Fig. 1c shows the alternative pattern, which demonstrates the non-influence of weight into the failure mode.

Concerning the triaxial extension test with lubricated end plattens, H. Meissner et al. [3] refer that necking is observed (cf. [1]).

Fig. 2 shows a typical stress-strain plot of a triaxial compression test of a dense sample with medium grained sand from Karlsruhe. To detect the failing of homogeneity, the shape of the sample was observed by a theodolite, and the sample diameters in various horizontal planes were measured (cf. [1]). The maximum angle of friction $\phi_{zp} = 41°$ was observed at a state where the maximal difference between two diameters of the sample was about 1% of the mean diameter. For the same sand and the same porosity the maximum angle of friction at a biaxial test is $\phi_{rp} = 47°$ (Fig. 2, [8], [9]). The difference between ϕ_{rp} and ϕ_{zp} can be explained by the special stress distribution corresponding to the bifurcation solution. Due to $\phi_{zp} < \phi_{rp}$ it is reasonable to concentrate the bifurcation analysis in the hardening regime. The material softening is mostly overestimated in conventional tests due to various localizations [8], [9]. Fig. 1d shows that in the compression test not only diffuse modes but also shear bands do occur. It should be noted that first bulging takes place, and that the shear bands develop at a later stage in the softening regime, as predicted by J. W. Rudnicki and J. R. Rice [13].

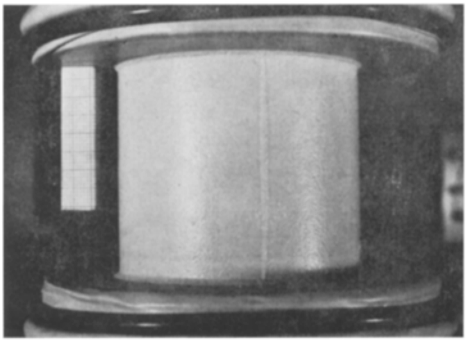


Fig. 1a. Failure patterns — sample before the test

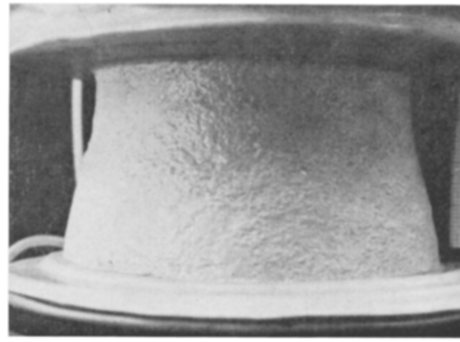


Fig. 1b. Failure pattern at an axial strain of 41%

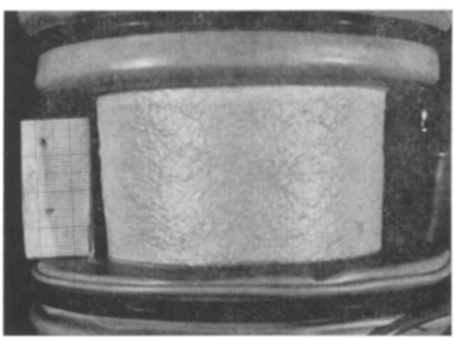


Fig. 1c. Failure patterns — alternative mode

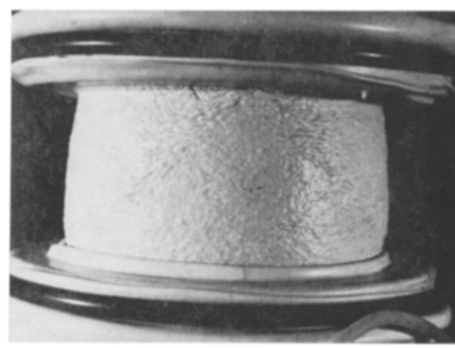


Fig. 1d. Failure patterns — bulging with post-failure shear bands

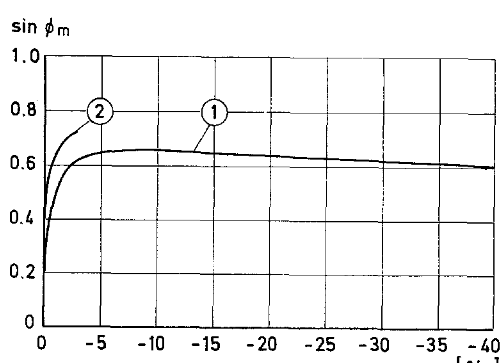


Fig. 2. Stress-strain curves for *1* triaxial compression ($n_0 = 36.4\%$, $\sigma_c = 20$ N/cm^2), *2* biaxial test ($n_0 = 36.2\%$, $\sigma_c = 12$ N/cm^2)

Formulation

Let a homogeneous cylindrical dry sand sample in an undistorted configuration C_0 be subjected to a smooth, quasi-static axisymmetric motion of extension [14]. Call the resultant configuration C. Let C be the reference configuration, R the radius and H the height of the sample in C (Fig. 3). Let σ_c be the confining pressure and w_t the vertical displacement of the top of the sample.

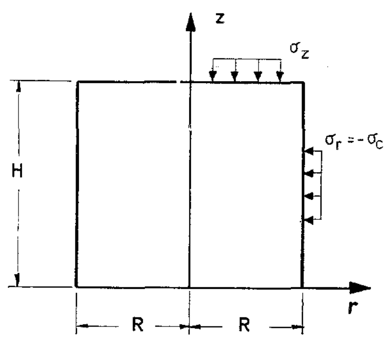


Fig. 3. Current configuration C

Due to lubrication no shear stress can develop at the end caps. For any change of the boundary conditions there is a new configuration C'. Let us employ a single fixed cylindrical co-ordinate system with its z-axis along the axis of the sample and put

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ \theta \\ z \end{bmatrix}; \quad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} r' \\ \theta' \\ z' \end{bmatrix} \tag{1}$$

for the cylindrical co-ordinates of a particle X in C and C', respectively. Here only axisymmetric deformation modes will be considered, i.e. the tangential displacement and the tangential derivative vanishes identically. Let u be the radial and w the axial displacement, the displacement gradient reads:

$$[u_{i|j}] = \begin{bmatrix} \frac{\partial u}{\partial r} & 0 & \frac{\partial u}{\partial z} \\ 0 & \frac{u}{r} & 0 \\ \frac{\partial w}{\partial r} & 0 & \frac{\partial w}{\partial z} \end{bmatrix}, \tag{2}$$

where $(\cdot)_{|j}$ denotes the covariant derivative with respect to the co-ordinate x_j. It is assumed that $u_{i|j}$ is infinitesimal everywhere in C, so that all terms of an order higher than one in $u_{i|j}$ can be neglected. Let

$$\varepsilon_{ij} := \frac{1}{2}(u_{i|j} + u_{j|i}); \qquad \omega_{ij} := \frac{1}{2}(u_{i|j} - u_{j|i}) \tag{3}$$

be the infinitesimal strain and rotation tensor.

The Cauchy stress tensor in C denoted by σ_{ij}:

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_\theta & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}. \tag{4}$$

For the considered axisymmetric motions of extension it is

$$\begin{aligned} &\sigma_z < \sigma_r = \sigma_\theta = -\sigma_c < 0 \text{ for compression or} \\ &\sigma_r = \sigma_\theta = -\sigma_c < \sigma_z < 0 \text{ for extension.} \end{aligned} \tag{5}$$

The stress in C' is suitably described in terms of the 1. Piola-Kirchhoff stress tensor Σ_{ij}. Σ_{ij} is defined by referring the nominal traction in C' on the oriented surface element in the reference configuration C. Within a linear theory the increment:

$$\Delta\Sigma_{ij} := \Sigma_{ij} - \sigma_{ij} \tag{6}$$

reads [15]:

$$\Delta\Sigma_{ij} = \Delta\sigma_{ij} + \sigma_{ij}u_{k|k} - \sigma_{ik}u_{k|j}, \tag{7}$$

where $\Delta\sigma_{ij}$ is the corresponding Cauchy stress increment. $\Delta\sigma_{ij}$ can be decomposed into a constitutive part Δs_{ij} and into a rotational part $\bar{\Delta}\sigma_{ij}$ (so-called Jauman part):

$$\Delta\sigma_{ij} = \Delta s_{ij} + \bar{\Delta}\sigma_{ij}, \tag{8}$$

$$\bar{\Delta}\sigma_{ij} = \omega_{ik}\sigma_{kj} - \sigma_{ik}\omega_{kj}. \tag{9}$$

Strictly speaking is Δs_{ij} the increment of the corrotated Cauchy stress tensor, which can be used for expressing incremental constitutive laws [16]. Substitution from Eq. (8) and (9) into Eq. (7) yields a decomposition of $\Delta\Sigma_{ij}$ in a constitutive and in a geometrical part:

$$\Delta\Sigma_{ij} = \Delta s_{ij} + \bar{\Delta}\Sigma_{ij} \tag{10}$$

$$\bar{\Delta}\Sigma_{ij} = \omega_{ik}\sigma_{kj} - \sigma_{ik}\varepsilon_{kj} + \sigma_{ij}\varepsilon_{kk}. \tag{11}$$

The field equations for continued equilibrium in C' can be expressed in terms of $\Delta\Sigma_{ij}$ [15], [16]:

$$\Delta\Sigma_{ij|j} = 0. \tag{12}$$

According to Eq. (10) and (11) the quantities

$$I := \bar{\Delta}\Sigma_{1j|j}; \qquad III := \bar{\Delta}\Sigma_{3j|j} \tag{13}$$

yield

$$I = 2t\,\frac{\partial\omega}{\partial z}; \qquad III = 2t\left(\frac{\partial\omega}{\partial r} + \frac{\omega}{r}\right) \tag{14}$$

where

$$\omega := \frac{1}{2}\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right) \tag{15}$$

$$t := \frac{1}{2}(\sigma_r - \sigma_z). \tag{16}$$

Introducing the above equations into Eq. (12) yields:

$$\frac{\partial s_{rr}}{\partial r} + \frac{\partial s_{rz}}{\partial z} + \frac{1}{r}(\Delta s_{rr} - \Delta s_{\theta\theta}) + 2t\frac{\partial \omega}{\partial z} = 0$$
$$\frac{\partial \Delta s_{rz}}{\partial r} + \frac{\partial \Delta s_{zz}}{\partial z} + \frac{1}{r}\Delta s_{rz} + 2t\left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r}\right) = 0 \quad (17)$$

(cf. [18]).

Boundary Conditions

For the considered infinitesimal transition $C \to C'$ the confining pressure σ_c is assumed to remain constant. The boundary conditions for the cylindrical edge of the sample should express the fact, that a follower traction of constant intensity acts always normal on it. Mathematically, this condition reads [17]:

for $r = R$: $$\Delta\Sigma_{ij}n_j = \sigma_c(n_k\delta_{il} - n_i\delta_{kl})\, u_{k|l}\,, \quad (18)$$

where $\{n_i\}^T = \{1, 0, 0\}$ is the boundary normal and δ_{ij} the Kronecker delta. Substitution from Eq. (11) and (12) into (18) yields:

for $r = R$: $$\Delta s_{rz} = -2t\varepsilon_{rz}; \qquad \Delta s_{rr} = 0. \quad (19)$$

On the other hand, the ends of the sample are subject to frictionless constraints, the axial displacement being there prescribed, i.e.:

for $z = 0$: $$\Delta s_{rz} = 0; \qquad w = 0$$
and (20)
for $z = H$: $$\Delta s_{rz} = 0; \qquad w = w_t.$$

Deformation Modes

As already mentioned, only axisymmetric deformation modes will be considered. First we define the so-called trivial mode, which is an axisymmetric motion of extension:

$$\begin{bmatrix} \mathring{u} \\ \mathring{v} \\ \mathring{w} \end{bmatrix} = \begin{bmatrix} q_1 r \\ 0 \\ q_3 z \end{bmatrix}. \quad (21)$$

This mode obeys to the boundary conditions:

$$\mathring{w}(r, 0) = 0; \qquad \mathring{w}(r, H) = w_t. \quad (22)$$

For the non-trivial mode a non-linear displacement field is assumed:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \hat{u}(\varrho)\cos\zeta \\ 0 \\ \hat{w}(\varrho)\sin\zeta \end{bmatrix}, \quad (23)$$

with

$$\varrho := \frac{r}{R}; \qquad \zeta := m\pi\frac{z}{H}\ (m = 1, 2, \ldots). \quad (24)$$

This field satisfies the homogeneous boundary conditions:

$$\tilde{w}(r, 0) = \tilde{w}(r, H) = 0. \tag{25}$$

The deformation mode considered here is a linear combination of the trivial and the non-trivial solution:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \mathring{u} \\ \mathring{v} \\ \mathring{w} \end{bmatrix} + \eta \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix}, \tag{26}$$

where $\eta > 0$ is a suitably *small* real number. From Eq. (26) the strain field can be derived; we write formally:

$$[\varepsilon_{ij}] = \begin{bmatrix} \mathring{\varepsilon}_r & 0 & 0 \\ 0 & \mathring{\varepsilon}_\theta & 0 \\ 0 & 0 & \mathring{\varepsilon}_z \end{bmatrix} + \eta \begin{bmatrix} \tilde{\varepsilon}_{rr} & 0 & \tilde{\varepsilon}_{rz} \\ 0 & \tilde{\varepsilon}_{\theta\theta} & 0 \\ \tilde{\varepsilon}_{rz} & 0 & \tilde{\varepsilon}_{zz} \end{bmatrix}. \tag{27}$$

Denote that

$$\mathring{\varepsilon}_r = \mathring{\varepsilon}_\theta = q_1 \quad \text{and} \quad \mathring{\varepsilon}_z = q_3 \tag{28}$$

are constant in C.

Invariants

For formulating the subsequent constitutive relationships, the following stress measures will be used:

$$\begin{aligned} p &:= \frac{1}{3}\,\sigma_{kk} = \frac{1}{3}\,(2\sigma_r + \sigma_z) \\ \tau &:= \sqrt{\frac{3}{2}\,\overset{*}{\sigma}_{ij}\overset{*}{\sigma}_{ij}} = |\sigma_r - \sigma_z| \end{aligned} \tag{29}$$

where (*) denotes the deviator. Corresponding to the above definitions the corresponding stress measures in C' are defined in terms of the rotated Cauchy stress tensor s_{ij}:

$$p' := \frac{1}{3}\,s_{kk}; \qquad \tau' := \sqrt{\frac{3}{2}\,\overset{*}{s}_{ij}\overset{*}{s}_{ij}}, \tag{30}$$

where

$$s_{ij} = \sigma_{ij} + \Delta s_{ij}. \tag{31}$$

For Δs_{ij} being infinitesimal everywhere in C and

$$p' = p + \Delta p; \qquad \tau' = \tau + \Delta\tau \tag{32}$$

from the above equation follows:

$$\begin{aligned} \Delta p &= \frac{1}{3}\,(\Delta s_{rr} + \Delta s_{\theta\theta} + \Delta s_{zz}) \\ \Delta\tau &= \frac{1}{2}\,|\Delta s_{rr} + \Delta s_{\theta\theta} - 2\Delta s_{zz}|. \end{aligned} \tag{33}$$

For calculating the strain measures we introduce the following notations:

$$I_{1\varepsilon} := \varepsilon_{kk}; \qquad J_{2\varepsilon} := \overset{*}{\varepsilon}_{ij}\overset{*}{\varepsilon}_{ij}; \qquad J_{3\varepsilon} := \overset{*}{\varepsilon}_{ij}\overset{*}{\varepsilon}_{jk}\overset{*}{\varepsilon}_{ki}. \tag{34}$$

Let $\mathring{\varepsilon}_1 > \mathring{\varepsilon}_2 > \mathring{\varepsilon}_3$ be the principal values of a strain deviator. It is convenient to write [10]:

$$\mathring{\varepsilon}_1 = D \cos \alpha_\varepsilon; \qquad \mathring{\varepsilon}_2 = -D \cos (\pi/3 + \alpha_\varepsilon); \qquad \mathring{\varepsilon}_3 = -D \cos (\pi/3 - \alpha_\varepsilon), \quad (35)$$

where

$$\cos 3\alpha_\varepsilon := \sqrt{6}\ J_{3\varepsilon}/J_{2\varepsilon}^{3/2}; \qquad D := \left(\frac{2}{3} J_{2\varepsilon}\right)^{1/2}. \quad (36)$$

With

$$\begin{aligned} &\varepsilon_3 < \varepsilon_2 = \varepsilon_1 \text{ for compression or} \\ &\varepsilon_1 = \varepsilon_2 < \varepsilon_3 \text{ for extension} \end{aligned} \quad (37)$$

from Eq. (35) follows

$$\alpha_\varepsilon = \pi/3 \quad \text{or } 0, \quad (38)$$

correspondingly. In addition to the above definitions we introduce the angle of orientation β of the principal axes of strain in the (r, z)-plane:

$$\tan 2\beta := \frac{2\varepsilon_{rz}}{\varepsilon_{rr} - \varepsilon_{zz}}. \quad (39)$$

By using these definitions it can be shown that the following representation holds:

$$\sqrt{J_{2\varepsilon}} = \frac{\sqrt{2}}{2} \left| \frac{\varepsilon_{rr} - \varepsilon_{zz}}{\cos 2\beta \sin (\pi/3 + \alpha_\varepsilon)} \right|. \quad (40)$$

By introducing the so-called Lode factor

$$L_\varepsilon := 2 \frac{\varepsilon_2 - \varepsilon_3}{\varepsilon_1 - \varepsilon_3} - 1 = -\sqrt{3} \cot (\pi/3 + \alpha_\varepsilon) \quad (41)$$

and taking into account that, according to Eq. (27), as well $(\varepsilon_{rr} - \varepsilon_{\theta\theta})$ as ε_{rz} are proportional to the small parameter η, follows that for

$$\eta \to 0 : \alpha_\varepsilon \to (\pi/3 \text{ or } 0) \quad \text{and} \quad \beta \to 0. \quad (42)$$

This argument allows to introduce

$$\gamma := \sqrt{\frac{3}{2} J_{2\varepsilon}} \simeq |\varepsilon_{rr} - \varepsilon_{zz}| \quad (43)$$

as a measure for the infinitesimal shearing strain intensity.

At least it should be noted that near to the triaxial compression or extension the increment of the third stress invariant is proportional to that of the second one. This means that introducing the third stress invariant no new information can be expected.

Constitutive Equations

We assume that the Mohr-Coulomb friction law is valid. Let ϕ_m be the mobilized angle of friction

$$\sin \phi_m := \left| \frac{\sigma_r - \sigma_z}{\sigma_r + \sigma_z} \right|. \quad (44)$$

$\sin \phi_m$ obeys to a strain hardening rule (Fig. 4):

$$\sin \phi_m = T(g), \tag{45}$$

where $T(\cdot)$ is a hardening function and g a finite Eulerian measure of the shearing strain intensity of the deformation measured from the undistorted configuration C_0. Introducing [1]

$$\sin \psi_\sigma := -\frac{\tau}{3p} \tag{46}$$

from Eq. (5) and (44) follows:

$$\sin \psi_\sigma = \frac{2 \sin \phi_m}{3 \mp \sin \phi_m}, \tag{47}$$

where throughout in this paper, if two signs, the upper holds for compression and the lower for extension.

For the increments Eq. (46) yields:

$$\Delta\tau = -3\Delta p \sin \psi_\sigma - 3p\Delta \sin \psi_\sigma. \tag{48}$$

Let h be the hardening rate (tangent modulus), defined as follows:

$$\Delta \sin \psi_\sigma = h\gamma, \tag{49}$$

where $\Delta g = \gamma$ is used. For writing Eq. (48) explicitly in terms of the stress and strain increments, we have to distinguish between a continuation of a triaxial compression or extension, i.e. we assume that for $C \to C'$

$$\begin{gathered} \varepsilon_{rr} > \varepsilon_{zz} \quad \text{or} \quad \varepsilon_{rr} < \varepsilon_{zz} \\ \text{and} \\ \Delta \mathring{s}_{zz} < 0 \quad \text{or} \quad \Delta \mathring{s}_{zz} > 0 \end{gathered} \tag{50}$$

if $C_0 \to C$ is a triaxial compression or extension. By using these assumptions, Eq. (48) yields:

$$\Delta s_{rr} + \Delta s_{\theta\theta} - 2\Delta s_{zz} = \mp 6\Delta p \sin \psi_\sigma \pm 4t \frac{h}{\sin \psi_\sigma} (\varepsilon_{rr} - \varepsilon_{zz}). \tag{51}$$

Note that $t := \frac{1}{2} (\sigma_r - \sigma_z)$ is positive or negative if respectively the current configuration corresponds to a triaxial compression or extension.

The increment of mean pressure Δp is regarded as a statically indeterminate quantity. This is consistent with the assumption that the strain field obeys to the rule of dilatancy, i.e. we assume that

$$I_{1\varepsilon} = \sin \psi_\varepsilon \, \gamma, \tag{52}$$

where ψ_ε is an angle of dilatancy. Between ψ_σ and ψ_ε according to D. Taylor's [11] proposition, some dependency can be assumed. A simple modification of Taylor's formula, proposed by I. Vardoulakis [12], reads:

$$\sin \psi_\varepsilon = \text{const} \, (\sin \phi_m - \sin \phi_c), \tag{53}$$

where ϕ_c corresponds to the critical state.

For the remaining shear stress increments, following Biot's proposition [18] we write:

$$\Delta s_{rr} - \Delta s_{\vartheta\theta} = 2\mathring{\mu}(\varepsilon_{rr} - \varepsilon_{\theta\theta}) \tag{54}$$

$$\Delta s_{rz} = 2\mu\varepsilon_{rz}. \tag{55}$$

$\mathring{\mu}$ and μ are two generally different shear moduli, which are at this moment no more restricted. It is clear that

$$\varepsilon_{rr} - \varepsilon_{\theta\theta} = \tilde{\varepsilon}_{rr} - \tilde{\varepsilon}_{\theta\theta}; \qquad \varepsilon_{rz} = \tilde{\varepsilon}_{rz}, \tag{56}$$

so that $\mathring{\mu}$ and μ correspond to unloading. Substitution from Eq. (54) into Eq. (51) yields:

$$\begin{aligned} \Delta s_{rr} &= (1 \mp \sin\psi_\sigma)\,\Delta p + \mathring{\mu}(\varepsilon_{rr} - \varepsilon_{\theta\theta}) \pm \frac{2}{3}\,t\,\frac{h}{\sin\psi_\sigma}\,(\varepsilon_{rr} - \varepsilon_{zz}) \\ \Delta s_{zz} &= (1 \pm 2\sin\psi_\sigma)\,\Delta p \mp \frac{4}{3}\,t\,\frac{h}{\sin\psi_\sigma}\,(\varepsilon_{rr} - \varepsilon_{zz}). \end{aligned} \tag{57}$$

It can be easily seen that Eq. (57) is consistent to Eq. (50), if for the considered deformation mode Eq. (27) the parameter η is chosen suitably small and the analysis is concentrated in the hardening regime ($h > 0$).

Internal Constraint

We discuss now the rule of dilatancy Eq. (52). Substitution from Eq. (34) and (43) into Eq. (52) yields:

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \pm\sin\psi_\varepsilon(\varepsilon_{rr} - \varepsilon_{zz}), \tag{58}$$

where use of Eq. (50) has been made. Assuming that the trivial mode obeys to the internal constraint, Eq. (58) also holds for the non-trivial mode, then

$$\tilde{\varepsilon}_{zz} = -\left(\delta^2\tilde{\varepsilon}_{rr} + \frac{1}{2}\,(1 + \delta^2)\,\tilde{\varepsilon}_{\theta\theta}\right) \tag{59}$$

where

$$\delta^2 := \tan^2\,(\pi/4 \mp \psi_\varepsilon/2). \tag{60}$$

On the other hand, Eq. (23) yields:

$$\tilde{\varepsilon}_{rr} = \frac{1}{R}\,\hat{u}'\cos\zeta; \qquad \tilde{\varepsilon}_{\theta\theta} = \frac{1}{R}\,\frac{\hat{u}}{\varrho}\cos\zeta; \quad \tilde{\varepsilon}_{zz} = \frac{m\pi}{H}\,\hat{w}\cos\zeta \tag{61}$$

where $(\cdot)' \equiv \frac{d}{d\varrho}$. From Eq. (61) and (59) follows at least a restriction for the non-trivial mode:

$$\hat{w} = -K^{-1}\left(\delta^2\hat{u}' + \frac{1}{2}\,(1 + \delta^2)\,\frac{\hat{u}}{\varrho}\right) \tag{62}$$

$$K := m\pi\,\frac{R}{H}\ (m = 1, 2, \ldots). \tag{63}$$

With Eq. (62) $\hat{w}$ can be eliminated from Eq. (61), there is

$$\begin{aligned}
\tilde{\varepsilon}_{rr} - \tilde{\varepsilon}_{\theta\theta} &= \frac{1}{R}\left(\hat{u}' - \frac{\hat{u}}{\varrho}\right)\cos\zeta \\
\tilde{\varepsilon}_{rr} - \tilde{\varepsilon}_{zz} &= \frac{1}{R}(1+\delta^2)\left(\hat{u}' + \frac{1}{2}\frac{\hat{u}}{\varrho}\right)\cos\zeta \\
2\varepsilon_{rz} &= -\frac{1}{R}K^{-1}\left(\delta^2\hat{u}'' + \frac{1}{2}(1+\delta^2)\frac{\hat{u}'}{\varrho} - \left(\frac{1}{2}(1+\delta^2) - K^2\varrho^2\right)\frac{\hat{u}}{\varrho^2}\right)\sin\zeta \\
2\omega &= -\frac{1}{R}K^{-1}\left(\delta^2\hat{u}'' + \frac{1}{2}(1+\delta^2)\frac{\hat{u}'}{\varrho} - \left(\frac{1}{2}(1+\delta^2) + K^2\varrho^2\right)\frac{\hat{u}}{\varrho^2}\right)\sin\zeta .
\end{aligned} \tag{64}$$

We denote here that the chosen kinematic field for the non-trivial mode satisfies the boundary conditions of the ends of the sample identically:

$$\text{for } z = 0, H: \qquad \tilde{\varepsilon}_{rz} = 0; \qquad \tilde{w} = 0. \tag{65}$$

Field Equations

The solution of the considered bifurcation problem consists of determining the non-trivial mode Eq. (23) in such a manner that the field equations for continued equilibrium, the constitutive equations and the boundary conditions are satisfied. Substitution from Eq. (54), (55) and (57) into Eq. (17) and use of the representation Eq. (64) for the non-trivial mode yields:

$$\begin{aligned}
-(1 \mp \sin\psi_\sigma)\frac{\partial \Delta p}{\partial r} = \frac{\mu+t}{R^2}\Bigg((\hat{\mu} - \delta^2 + 2\hat{h})\,\hat{u}'' + \left(\hat{\mu} - \frac{1}{2}(1+\delta^2) + \hat{h}\right)\frac{\hat{u}'}{\varrho} \\
+ \left(-\hat{\mu} + \frac{1}{2}(1+\delta^2) - \hat{h} + \xi K^2\varrho^2\right)\frac{\hat{u}}{\varrho}\Bigg)\cos\zeta
\end{aligned} \tag{66}$$

$$\begin{aligned}
(1 \pm 2\sin\psi_\sigma)\frac{\partial \Delta p}{\partial z} = K^{-1}\frac{\mu+t}{R^2}\Bigg(\delta^2\hat{u}''' + \frac{1}{2}(1+3\delta^2)\frac{\hat{u}''}{\varrho} \\
+ \left(-\frac{1}{2}(1+\delta^2) - (4\hat{h}+\xi)K^2\varrho^2\right)\frac{\hat{u}'}{\varrho^2} \\
+ \left(\frac{1}{2}(1+\delta^2) - (2\hat{h}+\xi)K^2\varrho^2\right)\frac{\hat{u}}{\varrho^3}\Bigg)\sin\zeta
\end{aligned} \tag{67}$$

where

$$\begin{aligned}
\hat{\mu} &:= \frac{\mathring{\mu}/\mu}{1+t/\mu} \\
\hat{h} &:= \frac{t/\mu}{1+t/\mu}\tilde{h}; \qquad \tilde{h} := \pm\frac{1}{3}(1+\delta^2)\frac{h}{\sin\psi_\sigma} \\
\xi &:= \frac{t/\mu - 1}{t/\mu + 1}.
\end{aligned} \tag{68}$$

As already mentioned, *full* friction mobilization and the dilatancy flow rule are assumed. This means, that the mean pressure increment cannot be determined by the field equations for continued equilibrium and the constitutive equations. For existing such a pressure field, Δp, the integrability conditions

$$\frac{\partial^2 \Delta p}{\partial r\, \partial z} = \frac{\partial^2 \Delta p}{\partial z\, \partial r} \tag{69}$$

must hold. Eq. (69), (66) and (67) yield a single differential equation for the admissable displacement field $\hat{u}(\varrho)$, namely:

$$\begin{aligned} & A_4 \hat{u}^{\mathrm{IV}} + A_3 \frac{\hat{u}'''}{\varrho} + (A_{20} + A_{21}\varrho^2) \frac{\hat{u}''}{\varrho^2} + (A_{10} + A_{11}\varrho^2) \frac{\hat{u}'}{\varrho^3} \\ & \quad + (A_{00} + A_{01}\varrho^2 + A_{02}\varrho^4) \frac{\hat{u}}{\varrho^4} = 0 \end{aligned} \tag{70}$$

where

$$\begin{aligned} A_4 &= \delta^2 \\ A_3 &= \frac{1}{2}(1 + 3\delta^2) \\ A_{20} &= -(1 + 2\delta^2); \qquad A_{21} = -K^2\big(\xi + \lambda^2(\hat{\mu} - \delta^2) + 2(2 + \lambda^2)\,\hat{h}\big) \\ A_{10} &= \frac{3}{2}(1 + \delta^2); \qquad A_{11} = -K^2\left(\xi + \lambda^2\left(\hat{\mu} - \frac{1}{2}(1 + \delta^2)\right) + (2 + \lambda^2)\,\hat{h}\right) \\ A_{00} &= -A_{10}; \qquad A_{01} = -A_{11}; \qquad A_{02} = -K^2\xi \end{aligned} \tag{71}$$

and

$$\lambda^2 := \frac{1 \pm 2 \sin \psi_\sigma}{1 \mp \sin \psi_\sigma} = \tan^2(\pi/4 \pm \phi_m/2). \tag{72}$$

The differential Eq. (70) is of the Fuchs type [19] and satisfies the coefficient criterion for $\varrho = 0$; $\varrho = 0$ being a point of definiteness. This means that for finding a solution of Eq. (70) the Frobenius method can be used, i.e. we try for solutions of the form:

$$\Psi(\varrho) = \sum_{n=0}^{\infty} c_n \varrho^{\alpha+n}; \qquad c_0 \neq 0, \tag{73}$$

where α is to be determined.

Analytical Solution

Inserting Eq. (73) into Eq. (70) yields:

$$\begin{aligned} & \Phi_0(\alpha)\, c_0 \varrho^{\alpha-4} + \Phi_0(\alpha + 1)\, c_1 \varrho^{\alpha-3} + \big(\Phi_0(\alpha + 2)\, c_2 + \Phi_2(\alpha)\, c_0\big)\, \varrho^{\alpha-2} \\ & + \big(\Phi_0(\alpha + 3)\, c_3 + \Phi_2(\alpha + 1)\, c_1\big)\, \varrho^{\alpha-1} \\ & + \sum_{n=0}^{\infty} \big(\Phi_0(\alpha + n + 4)\, c_{n+4} + \Phi_2(\alpha + n + 2)\, c_{n+2} + A_{02} c_n\big)\, \varrho^{\alpha+n} = 0, \end{aligned} \tag{74}$$

where

$$\Phi_0(\alpha) := (\alpha - 3)(\alpha - 1)^2 \left(\delta^2 \alpha + \frac{1}{2}(1 + \delta^2)\right) \tag{75}$$

$$\Phi_2(\alpha) := (\alpha - 1)(A_{21}\alpha + A_{11}). \tag{76}$$

For $c_0 = 1$ Eq. (74) yields the equation for determining α:

$$\Phi_0(\alpha) = 0 \leftrightarrow \alpha_1 = 3; \qquad \alpha_2 = \alpha_3 = 1; \qquad \alpha_4 = -\frac{1}{2}\frac{1+\delta^2}{\delta^2}. \tag{77}$$

It can be shown that $\alpha_1 = 3$ and $\alpha_2 = 1$ belong to different undergroups. The corresponding solutions are also the only solutions satisfying the boundness condition at $\varrho = 0$. These solutions have the following form:

$$\Psi_i(\varrho) = \sum_{n=0}^{\infty} c_{2n}^i \varrho^{\alpha_i+2n} \qquad (i = 1, 2) \tag{78}$$

with

$$c_0^1 = 1; \qquad c_2^1 = -\Phi_2(3)/\Phi_0(5)$$
$$c_0^2 = 1; \qquad c_2^2 = 0 \tag{79}$$
$$c_{2n+4}^i = -\left(\Phi_2(\alpha_i + 2n + 2)\, c_{2n+2}^i + A_{02} c_{2n}^i\right)/\Phi_0(\alpha_i + 2n + 4).$$

The general solution for the field $\hat{u}(\varrho)$ reads:

$$\hat{u}(\varrho) = C_1 \Psi_1(\varrho) + C_2 \Psi_2(\varrho). \tag{80}$$

The constants C_1 and C_2 have to be chosen in such a manner that the remaining boundary conditions at the cylindrical edge of the sample are satisfied.

Bifurcation Condition

Substitution from Eq. (55) and $(64)_3$ into the first boundary condition Eq. $(19)_1$ yields:

$$\delta^2 \hat{u}''(1) + \frac{1}{2}(1+\delta^2)\,\hat{u}'(1) - \left(\frac{1}{2}(1+\delta^2) - K^2\right)\hat{u}(1) = 0. \tag{81}$$

For computing Δs_{rr} at $\varrho = 1$ (Eq. $(19)_2$), we first have to evaluate Δp. There is

$$\Delta p = \int \frac{\partial \Delta p}{\partial \varrho}\, d\varrho + \int \frac{\partial \Delta p}{\partial \zeta}\, d\zeta \tag{82}$$

because the integration constant is obviously zero. The expressions $\partial\Delta p/\partial\varrho$ and $\partial\Delta p/\partial\zeta$ are given by Eqs. (66) and (67).

By setting

$$-(1 \mp \sin\psi_\sigma)\frac{\partial \Delta p}{\partial \varrho} = \frac{\mu + t}{R} = f_1(\varrho)\cos\zeta \tag{66a}$$

$$(1 \pm 2\sin\psi_\sigma)\frac{\partial \Delta p}{\partial \zeta} = K^{-2}\frac{\mu + t}{R} f_2(\varrho)\sin\zeta \tag{67a}$$

follows

$$(1 \mp \sin\psi_\sigma)\,\Delta p = -\frac{\mu + t}{R}\left(\int f_1(\varrho)\, d\varrho + \frac{f_2(\varrho)}{K^2\lambda^2}\right)\cos\zeta. \tag{83}$$

With the notations

$$F_1(\varrho) := \int f_1(\varrho)\, d\varrho; \qquad F_2(\varrho) := f_2(\varrho)/K^2\lambda^2 \tag{84}$$

and

$$F_3(\varrho) := (\hat{\mu} + 2\hat{h})\,\hat{u}' - (\hat{\mu} - \hat{h})\frac{\hat{u}}{\varrho} \tag{85}$$

the remaining boundary condition Eq. $(19)_2$ reads:

$$F_1(1) + F_2(1) - F_3(1) = 0. \tag{86}$$

By using the representation Eq. (80) the above boundary conditions Eq. (81) and (84) yield a homogeneous linear system of equations for determining the constants C_i:

$$d_{ik}C_k = 0 \qquad (i, k = 1, 2). \tag{87}$$

For having non-trivial solutions it must be

$$\det(d_{ik}) = 0. \tag{88}$$

Eq. (88) is called the bifurcation condition and represents a condition for determining the bifurcation stress $(t/\mu)_{cr}$.

For calculating the bifurcation stress $(t/\mu)_{cr}$ we first consider the solutions Eq. (78):

$$\begin{aligned} \Psi_1(\varrho) &= \varrho^3 + c_2{}^1\varrho^5 + c_4{}^1\varrho^7 + \cdots \\ \Psi_2(\varrho) &= \varrho \;\; + c_4{}^2\varrho^5 + c_6{}^2\varrho^7 + \cdots \end{aligned} \tag{78}$$

Let

$$\Psi_1{}^0 = \varrho^3; \qquad \Psi_2{}^0 = \varrho; \qquad \overline{\Psi}^{2n} = \varrho^{3+2n}. \tag{89}$$

According to Eq. (81), (86) and (89), let d_{ik}^0 be the value of d_{ik} for $\Psi_k{}^0$ and $\bar{d}_i{}^{2n}$ for $\overline{\Psi}^{2n}$, i.e.:

$$\begin{aligned} d_{11} &= d_{11}^0 + c_2{}^1\bar{d}_1{}^2 + c_4{}^1\bar{d}_1{}^4 + \cdots \\ d_{12} &= d_{12}^0 + c_4{}^2\bar{d}_1{}^2 + c_6{}^2\bar{d}_1{}^4 + \cdots \\ d_{21} &= d_{21}^0 + c_2{}^1\bar{d}_2{}^2 + c_4{}^1\bar{d}_2{}^4 + \cdots \\ d_{22} &= d_{22}^0 + c_4{}^2\bar{d}_2{}^2 + c_6{}^2\bar{d}_2{}^4 + \cdots \end{aligned} \tag{90}$$

where

$$\begin{aligned} d_{11}^0 &= 1 + 7\delta^2 + K^2, \\ d_{12}^0 &= K^2, \\ d_{21}^0 &= (2/K^2\lambda^2 - 1/2)\,(1 + 7\delta^2) + 2\hat{\mu} - 14\hat{h}/\lambda^2 + (K^2/4 - 4/\lambda^2)\,\xi, \\ d_{22}^0 &= 3(1 - 2/\lambda^2)\,\hat{h} + (K^2/2 - 2/\lambda^2)\,\xi \end{aligned} \tag{91}$$

and

$$\begin{aligned} \bar{d}_1{}^{2n} &= (1 + n)\left(1 + (7 + 4n)\,\delta^2\right) + K^2, \\ \bar{d}_2{}^{2n} &= a_0 + a_1\hat{\mu} + a_2\hat{h} + a_3\xi, \end{aligned} \tag{92}$$

$$\begin{aligned} a_0 &= 2(1 + n)\left((1 + n) + (7 + 11n + 4n^2)\,\delta^2\right)/K^2\lambda^2 - \left(1 + (7 + 4n)\,\delta^2\right)/2, \\ a_1 &= 2, \\ a_2 &= -2(7 + 4n)/\lambda^2, \\ a_3 &= K^2/(4 + 2n) - (4 + 2n)/\lambda^2. \end{aligned} \tag{93}$$

The bifurcation condition Eq. (88) has been calculated for one, two or three terms approximation of the series expansion Eq. (78). Correspondingly, the bifurcation condition yields an algebraic equation for determining $(t/\mu)_{cr}$ of the first, second or third degree. Higher approximations have to be solved iteratively, which sometimes deteriorates the precision arising from the consideration of a higher term.

Computational Results

For calculating the bifurcation stresses of dry sand samples the stress-strain curve has been taken from biaxial tests, which undergo homogeneous deformation up to the limiting state [8], [9]. Fig. 4 shows the chosen stress-strain curve, which is given by the function:

$$T(g) = 0.0976 \ln (1 + 30000g). \tag{94}$$

In the dilatancy rule Eq. (53) the constant is chosen equal to unity and $\phi_c = 34°$. The bifurcation stresses have been computed for

$$0 \leqq \mathring{\mu}/\mu \leqq 1 \quad \text{and} \quad 0.1 \leqq R/H \leqq 2. \tag{95}$$

Only the first bifurcation mode is considered, i.e. $m = 1$ in Eq. (24), because tests with lubricated end plattens are considered.

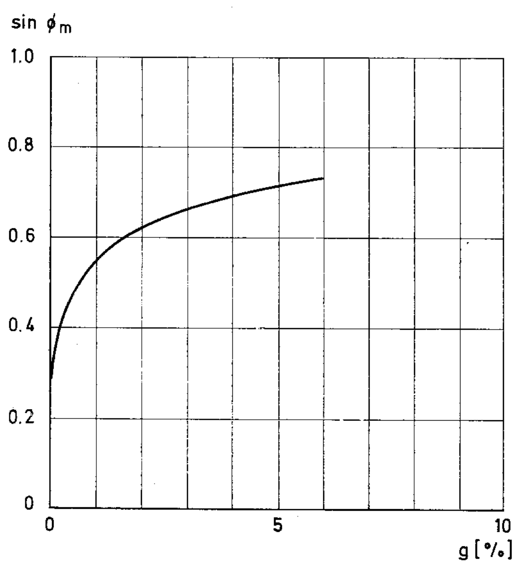


Fig. 4. Hardening rule

In Figs. 5 and 6 the bifurcation stress t/μ is plotted over R/H for various states of strain and for $\mathring{\mu}/\mu = 0.5$. These curves are approximately proportional to the corresponding shearing strain intensity g. This result is typical for the computational range of Eq. (95). Let now h_t be the tangent modulus of the considered stress-strain curve:

$$h_t := \frac{dT}{dg} = \frac{2928}{1 + 30000g}. \tag{96}$$

The above property of Figs. 5 and 6 allows then to assume, according to Eq. (96), that μ is proportional to h_t for consistency.

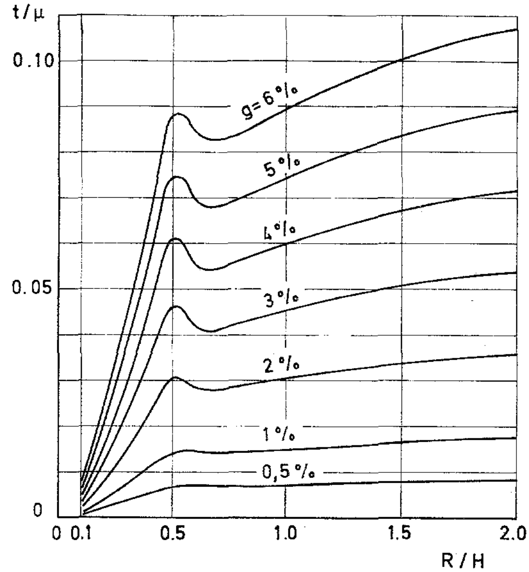


Fig. 5. Lowest bifurcation load for compression, $\overset{*}{\mu}/\mu = 0.5$

Figs. 7 and 8 show the results for max $\phi_m = 47°$ ($g = 6\%$) and for various $\overset{*}{\mu}/\mu$-ratios.

For having an estimate of the actual bifurcation stress for sand, we have chosen the parameters h_t/μ and $\overset{*}{\mu}/\mu$ in such a manner, that at an adequate low pressure level the limit condition can be reached. For this we have assumed that for $R/H = 0.5$ the critical confining pressure is for compression: $\sigma_{c,cr} = 3 \div 5$ N/cm^2. For fiting this $h_t/\mu = 0.1$ N^{-1}cm^2 has been taken. In Fig. 9 the critical stress ratio

$$\left(\frac{t}{s}\right)_{cr} = \left(\frac{\sigma_r - \sigma_z}{\sigma_r + \sigma_z}\right)_{cr} \tag{97}$$

is plotted for various states of strain.

Conclusions

The central result of the above analysis is that in the triaxial test bifurcation is always possible in the hardening regime. This means that if the confining pressure exceeds a critical value, then it is not possible to carry on a homogeneous deformation. The bifurcation stress generally increases by increasing the R/H-

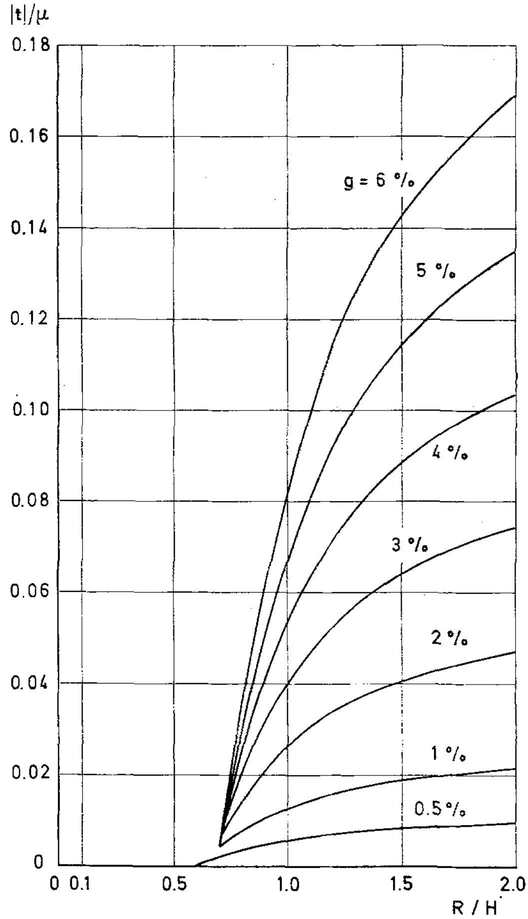


Fig. 6. Lowest bifurcation load for extension, $\mathring{\mu}/\mu = 0.5$

ratio. The anisotropy of the sample in a deformed state, given in terms of $\mathring{\mu}/\mu$ affects the result in compression more than in extension. The bifurcation stress for extension is very small for $R/H < 0.7$. This means that extension tests should not be carried out with slender samples. For having comparable results in compression and extension samples with $R/H > 1$ should be used (Fig. 9).

Fig. 9 demonstrates the fact, that at very low pressure levels the triaxial test yields higher angles of friction. This property has often been misunderstood and has led to the assumption that the angle of friction depends on the pressure level. As already mentioned, the global shear strength in triaxial tests appears to be lower than the limiting one due to the inhomogeneous post-bifurcation stress field.

Consequently the bifurcation analysis shows that the triaxial test yields only then the limiting soil properties if 1) the samples ar not slender ($R/H > 1$) and 2) if the confining pressure is less than the corresponding critical value.

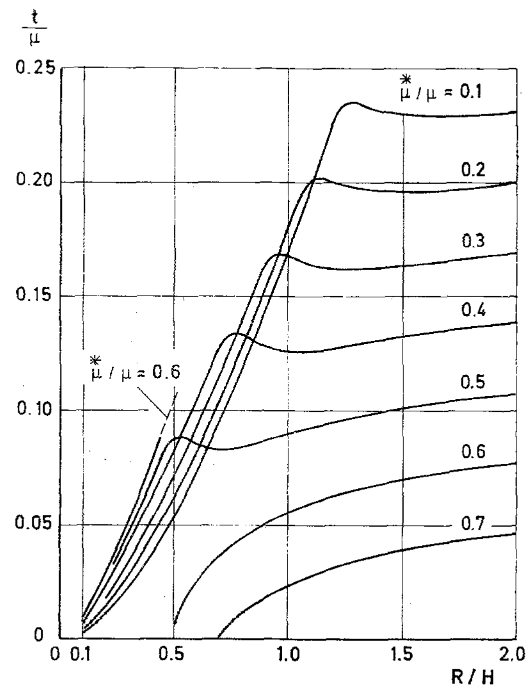


Fig. 7. Lowest bifurcation load for compression, $g = 6\%$ ($\phi_m = 47°$)

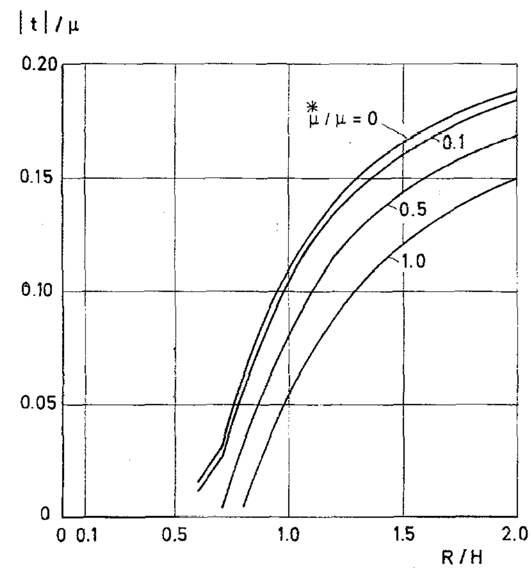
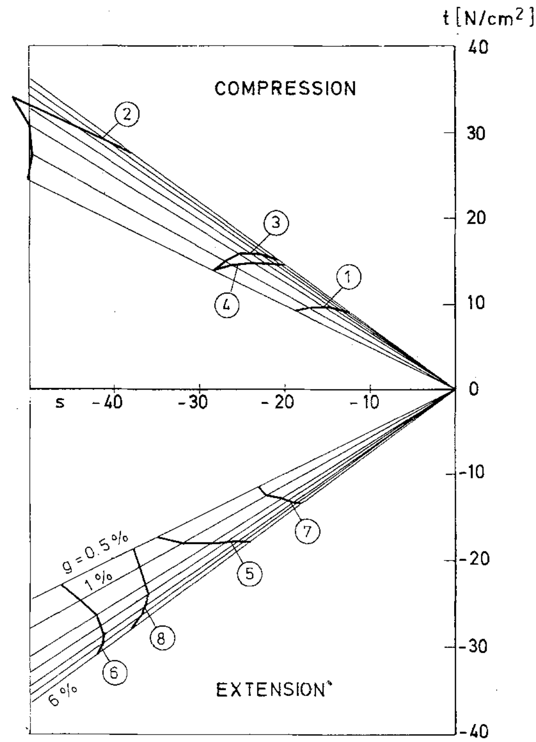


Fig. 8. Lowest bifurcation load for extension, $g = 6\%$ ($\phi_m = 47°$)



No.	h_t/μ	$\mathring{\mu}/\mu$	R/H	σ_c^{cr}	No.	h_t/μ	$\mathring{\mu}/\mu$	R/H	σ_c^{cr}
①	0.1	0.1	0.5	3	⑤	0.1	0.0	1.0	42
②	0.1	0.1	1.0	10	⑥	0.1	0.0	2.0	73
③	0.1	0.5	0.5	6	⑦	0.1	0.5	1.0	32
④	0.1	0.5	1.0	5	⑧	0.1	0.5	2.0	65

Fig. 9. Critical stress ratios and corresponding confining pressures

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