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Magnetohydrodynamic flow in a rectangular duct with suction and injection

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Summary. The effects of suction or injection on an incompressible laminar flow in a rectangular duct with nonconducting walls in the presence of an imposed transverse magnetic field are examined. Analytical solutions are obtained for the velocity and magnetic field, which are useful for obtaining the current density and electric field strength.

1 Introduction

The increasing number of technical applications using magnetohydrodynamic (MHD) effects has made it desirable to extend many of the available hydrodynamic solutions to include the effects of magnetic fields for those cases when the viscous fluid is electrically conducting. For example, liquid-metal MHDs take their roots in conventional hydrodynamics of incompressible media, which become important in the metallurgical industry, nuclear reactor sodium cooling system, energy storage and electrical power generation [1], [3]. The greatest advantage of induction-type pumps over other types of MHD devices is the absence of electrodes [4]. Induction pumps have been used to pump coolents in nuclear reactors. These designs are also being considered in MHD power generation [5]. The basic equations of incompressible MHD are nonlinear. But there are many interesting cases where the equations become linear in terms of the unknown quantities and may be solved without difficulty. These cases constitute a somewhat restricted version of MHD from which nonlinear phenomena are excluded. They do involve the mutual interaction of the magnetic and velocity fields, but only in a degenerate way. Linear MHD permits exact solutions and adopts the approximations that the density and transport properties be constant. The boundary conditions are linear in terms of the unknowns. No fluid is incompressible, but all may be treated as such whenever the pressure changes are small in comparison with the bulk modulus. Sparrow et al. [6] linearized the inertia terms by introducing a stretched coordinate in the flow direction and obtained a closed-form solution for hydrodynamic flows. Using this method, Snyder [7] has analyzed MHD flows in the entrance region of a rectangular channel and provided a good bibliography of the earlier work. Chen and Chen [8] and Hwang [9] have considered the entry flow with arbitrary inlet velocity profile. These studies of entry flow in channels are needed for operational MHD devices like power generators and MHD accelerators. Shercliff [10] has examined the steady motion of an electrically conducting, viscous fluid along channels in the presence of an imposed transverse magnetic field when the walls do not conduct currents. It is a case of considerable practical interest because of the utility of induction flow-meters, which rely on the

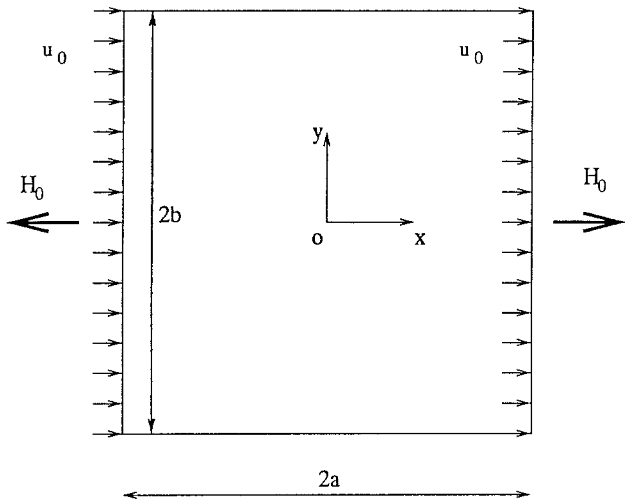

Fig. 1. A cross-section of rectangular duct normal to the flow direction (z-axis)

generation of a measurable potential difference in the fluid in a direction perpendicular to the motion and to the magnetic field. Moreover laminar flow occurs more readily under a magnetic field since turbulence tends to be damped by eddy currents. The equations which determine the velocity profile, induced currents and field are derived and solved exactly in the case of a rectangular channel. Gold [11] has obtained an exact solution for incompressible laminar flows in circular pipes with nonconducting walls and transverse magnetic fields.

In this paper, the effects of suction or injection on the flow rate are examined when an electrically conducting liquid flows in a rectangular duct in the presence of an imposed transverse magnetic field. The porous walls are assumed to be nonconducting and at right angles to the magnetic fiels, and suction is applied at one wall and injection at the other wall (see Fig. 1). Analytical solutions have been obtained for the velocity and the magnetic field, which are useful for obtaining the current density and the electric field strength. These analytical solutions can provide not only a check against the finite difference/finite element model, but also a means by which the effect of a parameter change can be readily gauged, which is useful in understanding the flow phenomena.

2 Basic equations

The motion of an electrically conducting fluid in the presence of a magnetic field obeys the well known equations of MHD. The fluid is treated as a continuum, and the classical results of fluid dynamics and electrodynamics are combined to express the phenomenon. For the steady flow of a viscous, incompressible fluid with constant properties, the full MHD system can be reduced to just two equations involving the velocity, pressure, and magnetic field, i.e., the modified Navier-Stokes equation and the induction equation, along with the solenoidal conditions on the two vector quantities:

$$\varrho(\vec{V}.\nabla)\,\vec{V} = -\nabla p + (\nabla \times \vec{H}) \times \mu_e\vec{H} + \mu\nabla^2\vec{V}\,, \tag{1}$$

$$(\vec{V}.\nabla)\,\vec{H} - (\vec{H}.\nabla)\,\vec{V} = \frac{1}{\sigma_e\mu_e}\,\nabla^2\vec{H}\,, \tag{2}$$

$$\nabla . \vec{V} = 0 \,, \tag{3}$$

$$\nabla . \vec{H} = 0 \,, \tag{4}$$

where $\varrho, p, \mu, \mu_e, \vec{V}$, and σ_e are the fluid density, pressure, viscosity, permeability, velocity, and electrical conductivity, respectively, and $\vec{H}$ is the magnetic field strength.

In the present problem conditions are invariant in the flow direction (z-direction), apart from a pressure gradient. Differentiating Eq. (1) with respect to z shows that $\nabla(\partial p/\partial z)$ vanishes. Hence $\partial p/\partial z$ is a constant. The main characteristics of such flows that need to be known are the volumetric flow rate through the duct for a given pressure gradient and magnetic field and the stability of the flow. We consider the steady laminar flow of an incompressible electrically conducting viscous fluid in a channel with rectangular cross-section as shown in Fig. 1. The walls are located at $x = \pm a$ and $y = \pm b$. A constant magnetic field, H_0 is applied parallel to the x-axis. Fluids are injected with a constant velocity, u_0, at $x = -a$ and sucked with the same velocity at $x = a$. The components of velocity and magnetic field are taken in the form:

$$\nu_x = u_0 \,, \qquad \nu_y = 0 \,, \qquad \nu_z = \nu_z(x, y) \tag{5}$$

$$H_x = H_0 \,, \qquad H_y = 0 \,, \qquad H_z = H_z(x, y) \,. \tag{6}$$

Substituting these expressions into Eq. (1), we obtain

$$-\frac{\partial p}{\partial x} - \mu_e H_z \frac{\partial H_z}{\partial x} = 0 \,, \tag{7}$$

$$-\frac{\partial p}{\partial y} - \mu_e H_z \frac{\partial H_z}{\partial y} = 0 \,, \tag{8}$$

$$\varrho u_0 \frac{\partial \nu_z}{\partial x} = -\frac{\partial p}{\partial z} + \mu_e H_0 \frac{\partial H_z}{\partial x} + \mu \left(\frac{\partial^2 \nu_z}{\partial x^2} + \frac{\partial^2 \nu_z}{\partial y^2} \right) . \tag{9}$$

From Eqs. (7)–(9) we can write

$$p(x, y, z) = -\frac{1}{2} \mu_e H_z^2 + k_1 z + k_2 \,, \tag{10}$$

and

$$\frac{\partial p}{\partial z} = \text{constant} = k_1 = -k\mu \,. \tag{11}$$

The components of velocity and the magnetic field given in Eqs. (5) and (6) are identically satisfied by the continuity equation (3) and Maxwell's equation (4), while Eq. (2) becomes

$$u_0 \frac{\partial H_z}{\partial x} = H_0 \frac{\partial \nu_z}{\partial x} + \frac{1}{\sigma_e \mu_e} \left(\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} \right) . \tag{12}$$

Equations (7)–(12) define the variables $p(x, y, z)$, $\nu_2(x, y)$ and $H_z(x, y)$ subject to the following homogeneous boundary conditions: There is no fluid slip at the walls, hence

$$\nu_z = 0 \quad \text{at} \quad x = \pm a \quad \text{and} \quad y = \pm b \,, \tag{13}$$

while the assumption of non-conducting walls implies that

$$H_z = 0 \quad \text{at} \quad x = \pm a \quad \text{and} \quad y = \pm b \,. \tag{14}$$

When Eqs. (7)–(14) have been solved for p, ν_z, and H_z, we obtain the current density $\vec{J}$ and the electric field strength, $\vec{E}$ from Ampéres law and Ohm's law, respectively:

$$E_x = \frac{1}{\sigma_e}\frac{\partial H_z}{\partial y}, \qquad E_y = -\frac{1}{\sigma_e}\frac{\partial H_z}{\partial x} + \nu_z H_0 - u_0 H_z, \qquad E_z = 0 \tag{15}$$

$$j_x = \frac{\partial H_z}{\partial y}, \qquad j_y = -\frac{\partial H_z}{\partial x}, \qquad j_z = 0. \tag{16}$$

Defining $\nu = \nu_z / < \nu_z >$ and $h = H_z/(R_e H_0)$ in Eqs. (9)–(14), we can write the following equations in nondimensional form:

$$R_{es}\frac{\partial \nu}{\partial \xi} = K + \frac{Ha^2}{P_{rm}}\frac{\partial h}{\partial \xi} + \frac{\partial^2 \nu}{\partial \xi^2} + \frac{\partial^2 \nu}{\partial \eta^2}, \tag{17}$$

$$P_{rm}R_{es}\frac{\partial h}{\partial \xi} = P_{rm}\frac{\partial \nu}{\partial \xi} + \frac{\partial^2 h}{\partial \xi^2} + \frac{\partial^2 h}{\partial \eta^2}. \tag{18}$$

Here $\xi = \frac{x}{a}$; $\eta = \frac{y}{a}$; $\lambda = \frac{b}{a}$ is the aspect ratio, $\upsilon = \frac{\mu}{\varrho}$ is the kinematic viscosity, $R_e = \frac{a < \nu_z >}{\upsilon}$ is the Reynolds number, $R_{es} = \frac{a u_0}{\upsilon}$ is the suction Reynolds number, $Ha = \mu_e H_0 a \sqrt{\frac{\sigma_e}{\mu}}$ is the Hartmann number, $P_{rm} = \sigma_e \mu_e \upsilon$ is the magnetic Prandtl number, and $K = -\frac{k_1 a^2}{\mu < \nu_z >}$ is the interaction parameter.

The average velocity is

$$< \nu_z > = \frac{1}{4ab}\int_{-a}^{a}\int_{-b}^{b} \nu_z \, dx \, dy. \tag{19}$$

3 Analytical solution

We assume the solution of the problem in the form

$$\nu = \sum_{n=0}^{\infty} \nu_n(\xi)\cos(\alpha_n \eta), \tag{20}$$

$$h = \sum_{n=0}^{\infty} h_n(\xi)\cos(\alpha_n \eta), \tag{21}$$

where $\alpha_n = (2n+1)\pi/(2\lambda)$.

It should be noted that the nondimensional components of velocity (ν) and magnetic field (h) in Eqs. (20) and (21) are expressed as Fourier series in η, with unknown coefficients functions of ξ, for obtaining the solution of the coupled linear partial differential equations (17) and (18). Equations (20) and (21) satisfy the boundary conditions at $\eta = \pm\lambda$. It will be more convenient for determining the unknown coefficients $\nu_n(\xi)$ and $h_n(\xi)$ from Eqs. (17) and (18), by expressing the constant term, K, in Eq. (17) as Fourier series in η. Without loss of generality, we can replace the constant term, K, in Eq. (17) by multiplying it with unity and expressing the unity as Fourier series in η. The constant term, K, in Eq. (17) can be written in the form

$$K = K \times \text{Unity} = K \times \sum_{n=0}^{\infty} k_n \cos(\alpha_n \eta), \tag{22}$$

where $k_n = 4(-1)^n/[\pi(2n+1)]$.

Substituting the relations (20)–(22) in Eqs. (17)–(19), and equating the coefficients of $\cos(\alpha n \eta)$, we obtain the following set of ordinary differential equations ($n = 0$ to ∞) with boundary conditions:

$$(D^2 - R_{es}D - \alpha_n{}^2)\,\nu_n + \frac{Ha^2}{P_{rm}} Dh_n + Kk_n = 0, \tag{23}$$

$$P_{rm}D\nu_n + (D^2 - P_{rm}R_{es}D - \alpha_n{}^2)\,h_n = 0, \tag{24}$$

$$\nu_n = 0, \quad h_n = 0 \quad \text{at} \quad \xi = \pm 1, \tag{25}$$

where $D = d/d\xi$.

The solution of Eqs. (23)–(25) can be written in the form

$$\nu_n(\xi) = K\frac{k_n}{\alpha_n{}^2}\left[1 + \sum_{i=1}^{4} \nu_{ni} \exp(m_{ni}\xi)\right], \tag{26}$$

$$h_n(\xi) = -K\frac{P_{rm}}{Ha^2}\frac{k_n}{\alpha_n{}^2}\sum_{i=1}^{4} h_{ni}\nu_{ni} \exp(m_{ni}\xi), \tag{27}$$

where

$$h_{ni} = (m_{ni}^2 - R_{es}m_{ni} - \alpha_n{}^2)/m_{ni}. \tag{28}$$

$m_{ni}(i = 1$ to $4)$ are the real roots of the quartic equation

$$m_n{}^4 - (1 + P_{rm})\,R_{es}m_n{}^3 - (2\alpha_n{}^2 - P_{rm}R_{es}^2 + Ha^2)\,m_n{}^2 + (1 + P_{rm})\,R_{es}\alpha_n{}^2 m_n + \alpha_n{}^4 = 0, \tag{29}$$

and $\nu_{ni}(i = 1$ to $4)$ are obtained by solving Eqs. (30)–(33):

$$\sum_{i=1}^{4} \cosh(m_{ni})\,\nu_{ni} + 1 = 0, \tag{30}$$

$$\sum_{i=1}^{4} \sinh(m_{ni})\,\nu_{ni} = 0, \tag{31}$$

$$\sum_{i=1}^{4} h_{ni} \cosh(m_{ni})\,\nu_{ni} = 0, \tag{32}$$

$$\sum_{i=1}^{4} h_{ni} \sinh(m_{ni})\,\nu_{ni} = 0. \tag{33}$$

The flow rate, Q, is defined by

$$Q = \int_{-a}^{a}\int_{-b}^{b} \nu_z\,dx\,dy = 4ab < \nu_z > = -\frac{4a^3bk_1}{\mu K} = -\frac{4a^3b}{\mu K}\frac{\partial p}{\partial z}, \tag{34}$$

which is linearly varying with the pressure gradient.

From the definition, the average value of the nondimensional velocity, ν, becomes unity, i.e.,

$$< \nu > = 1\,, \tag{35}$$

which implies that

$$K = \lambda^{-2}\left[\frac{1}{3} + \frac{32}{\pi^4}\sum_{n=0}^{\infty}\sum_{i=1}^{4}\frac{\nu_{ni}\sinh\left(m_{ni}\right)}{\left(2n+1\right)^4 m_{ni}}\right]^{-1}. \tag{36}$$

4 Results and discussion

The effects of suction or injection in the steady motion of an electrically conducting, viscous fluid in a rectangular duct in the presence of an imposed transverse magnetic field are examined. The porous walls are assumed to be nonconducting and at right angles to the magnetic field, and suction is applied at one wall and injection at the other wall. The main characteristics of such flows that need to be known are: the volumetric flow rate, Q, through the duct for given pressure gradient and magnetic field and the stability of the flow. When all walls are non-conducting and the Hartmann number $Ha \gg 1$, the velocity profile in the boundary layer on the walls has no point of inflexion, and the flow in such a duct is probably stabilized by the magnetic field [12], [13]. Analytical solutions for the velocity and magnetic field have been obtained by using Eqs. (26) and (27) in Eqs. (20) and (21). The flow rate, Q, defined in Eq. (34) is found to be linearly varying with the constant pressure gradient along the flow direction (z-direction). The interaction parameter of the pressure gradient and mean velocity component, K, in Eq. (36) is a function of the Hartmann number (Ha), the magnetic Prandtl number (P_{rm}), the suction Reynolds number (R_{es}), and the aspect ratio (λ). The current density ($\vec{J}$) and the electric field strength ($\vec{E}$) can be obtained directly by using the analytical solutions for the velocity and magnetic fields in Eqs. (15) and (16). A convergence study has been made on the number of terms in the present series solutions, and the results have been com-

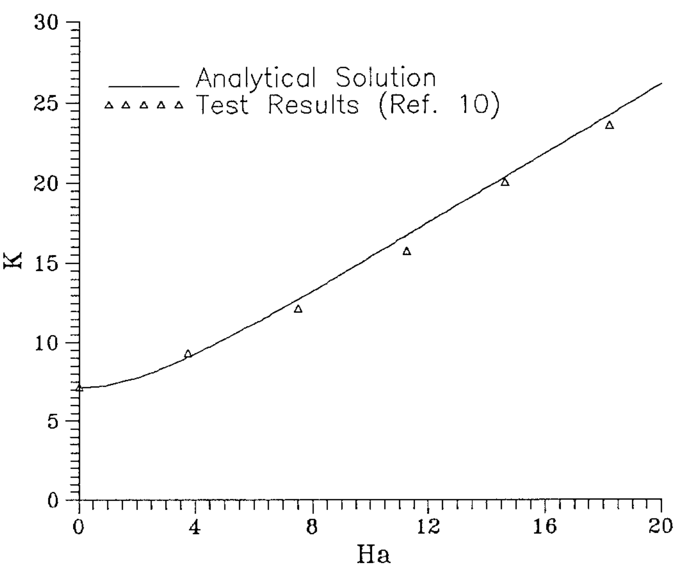

Fig. 2. Variation of K with $Ha\,(R_{es} = 0)$

Table 1. Variation of the interaction parameter (K) of the pressure gradient and average velocity with the Hartmann number (Ha) for different values of R_{es} and λ. ($P_{rm} = 0.1$)

Ha	$\lambda = 0.5$		$\lambda = 1.0$		$\lambda = 1.5$	
	$R_{es} = 0$	$R_{es} = 10$	$R_{es} = 0$	$R_{es} = 10$	$R_{es} = 0$	$R_{es} = 10$
0	17.49	23.75	7.114	15.34	5.108	13.61
1	17.58	23.76	7.278	15.25	5.299	13.47
2	17.84	23.78	7.740	15.01	5.824	13.09
3	18.25	23.82	8.425	14.71	6.581	12.61
4	18.79	23.92	9.259	14.46	7.473	12.21
5	19.45	24.08	10.18	14.37	8.439	12.05
10	23.75	26.25	15.34	16.85	13.61	14.79
20	34.23	35.13	26.10	26.61	24.17	24.59
30	45.24	45.71	36.82	37.11	34.67	34.91
40	56.31	56.61	47.47	47.66	45.11	45.28
50	67.34	67.56	58.06	58.20	55.51	55.64
100	121.8	121.9	110.4	110.5	107.1	107.2

puted by considering the first 75 terms in the series solution. The accuracy of the present analysis is examined with the experimental data of Hartmann and Lazarus for a square channel available in Ref. [10]. The analytical results of K with Ha for $R_{es} = 0$, shown in Fig. 2, compare well with the existing test results. When the suction Reynolds number $R_{es} = 0$, the values of K are found to be independent of the magnetic Prandtl number (P_{rm}), which is evident from the real roots of the quatric equation (29). Table 1 gives the variation of K with Ha for the specified values of $R_{es} = 10$, $P_{rm} = 0.1$, and $\lambda = 0.5, 1$ and 1.5. It is noted that the interaction parameter (K) increases with the Hartmann number (Ha) and decreases with the aspect ratio (λ) for the specified values of $P_{rm} = 0.1$ and $R_{es} = 10$. The closed-form solution obtained for the problem can be used to generate contour plots for the velocity, magnetic field, electric field and current density profiles, for the specified values of Ha, R_{es}, P_{rm} and λ, without difficulty.

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