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The Kelvin-Helmholtz Instability for a Viscous Interface

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Summary

This paper discusses the Kelvin-Helmholtz instability for the interface between a viscous and an inviscid fluid. The incipient boundary layer is modelled as a two dimensional viscous interface.

1. Introduction

Interfacial mechanics and surface tension have been described using varied ideas which we mention. Dussan [1] and Joseph [2] investigated the boundary between two fluids and introduced surface tension directly by assuming a specific form for the stress discontinuity at the fluid interface. Jenkins and Barratt [3], [4], [5] developed a virtual work approach and were necessarily confined to static problems although this was not a severe restriction as far as they were concerned since their interest lay in the applicability of their ideas to the theory of liquid crystals and related director type materials.

Originally Scriven [6] and later Gurtin and Murdoch [7], Moeckel [8] and Lindsay and Straughan [9] have considered an approach in which the interface is regarded as a two dimensional continuum. Recently confusion has arisen as to the relationship between the work of [7], [8], [9] and that of Green, Laws and Naghdi [10] and Green and Naghdi [11] where a fluid sheet theory is derived treating the sheet as a two dimensional continuum to each point of which is attached one or more directors. Much of the confusion has arisen from the mistaken belief that the director theory could not take account of the fluids either side of it. It is clear from Naghdi [12] that this is not the case. Since [7], [8] and [9] are membrane theories, they are embedded in any director theory. In fact, early work of Green, Naghdi and Wainwright [17] on the Cosserat surface contained a section in which they dealt with the membrane theory unambiguously using invariance of the energy

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equation and in the process include terms which describe the effect of the bulk continua either side of the membrane.

In order to explain the effect of viscosity of the bulk fluids on the interface itself we consider the "Kelvin-Helmholtz" instability. This is the phenomenon by which movement of one layer of fluid over another can, according to linear theory, cause instabilities. Of course, non-linear effects control these instabilities. The case of two incompressible inviscid fluids has been considered by Chandrasekhar [13]. For technical simplification we shall investigate the situation in which both fluids are incompressible but only the denser fluids is viscous. However, the analysis makes it clear how to introduce viscosity into the less dense fluid and the interface itself.

Miles [14] discusses a similar problem for the flow of a light inviscid fluid over a dense viscous fluid and comes to the conclusion that, for gratvity waves, the inclusion of dominant viscous terms has little effect. His analysis is based on an asymptotic discussion of the complete Orr-Sommerfeld equation.

In conclusion, we find that the presence of viscosity reduces the destabilising velocity. In fact

$$V_{\substack{\text{critical}\\ \text{(with viscosity)}}} = \left(\frac{\varrho_1}{\varrho_1 + \varrho_2}\right)^{1/2} V_{\substack{\text{critical}\\ \text{(without viscosity)}}}$$

where ϱ_1, ϱ_2 are the densities of the upper and lower fluids respectively. When the inviscid fluid is "light", it is clear that $(\varrho_2/\varrho_1 + \varrho_2) \simeq 1$ and so we find that viscosity has little effect in agreement with Miles.

2. Basic Equations and Notation

Using standard methods of Coninuum Mechanics, Moeckel [8] has derived field equation expressing conservation of mass, momentum and energy in a material surface in the respective forms

$$\begin{gathered}
\dot{\gamma} + \gamma(V^{\alpha}_{,\alpha} - u_n b_{\alpha}{}^{\alpha}) = 0, \\
\gamma \dot{V}^k - S^{k\alpha}_{,\alpha} = [t^{ki} n_i] + \gamma f^k, \\
S^{k\alpha} = S^{\beta\alpha} x^k_{,\beta} \\
\gamma \dot{\varepsilon} + q^{\alpha}_{,\alpha} - S^{k\alpha} V_{k,\alpha} - \gamma r = -[Q^i n_i + t_{ki} n^k (V^i - v^i)],
\end{gathered} \tag{2.1}$$

where γ is the surface density, $1/2\ b_{\alpha}{}^{\alpha}$ is the mean curvature of the surface, $S^{k\alpha}$ is the surface stress, ε is the specific internal surface energy, q^{α} is the surface heat flux vector, f^i is the surface body force, r is the surface heat supply, $V^i = u_n n^i + V^{\alpha} x^i_{,\alpha}$ is the surface velocity and $[\varphi] = \varphi^+ - \varphi^-$ denotes the jump in a quantity across the interface. Further, the field equations describing the bulk media either side of the interface or material surface are in the usual notation

$$\begin{gathered}
\dot{\varrho} + \varrho v^i_{,i} = 0, \\
\varrho \dot{v}^i - t^{ik}_{,k} - \varrho F^i = 0, \\
t^{ik} = t^{ki}, \\
\varrho \dot{E} + Q^i_{,i} - t^{ik} v_{i,k} - \varrho R = 0,
\end{gathered} \tag{2.2}$$

where the use of a capital indicates that we are dealing with a bulk quantity. In Eqs. (2.2) the subscript comma indicates covariant differentiation with respect to spatial coordinates ξ_i whereas in (2.1) the subscript comma indicates covariant differentiation with respect to some set of surface coordinates θ^α where α has values 1 or 2.

3. Kelvin-Helmholtz Instability

The aim of this paper is to investigate the effect of bulk and interfacial viscosity on the propagation of surface gravity waves on the boundary $z = 0$ between a viscous fluid occupying $z < 0$ and an inviscid fluid occupying $z > 0$, both fluids being incompressible. In effect, we propose to model the boundary layer by a two dimensional viscous interface.

Suppose now that the inviscid fluid flows with speed V in the positive x direction whereas the viscous fluid is stationary, then the linearised equations governing disturbances in these fluids are respectively

$$\varrho \left(\frac{\partial v_i}{\partial t} + V \frac{\partial v_i}{\partial x} \right) + (p + \varrho \Omega)_{,i} = 0, \tag{3.1.1}$$

$$\varrho \frac{\partial u_i}{\partial t} + (p + \varrho \Omega)_{,i} - \mu \Delta u_i = 0. \tag{3.1.2}$$

Further, on the interface we require that

$$\begin{gathered}
\gamma \left(\frac{\partial V^i}{\partial t} + V^\alpha V^i_{,\alpha} \right) - S^{\alpha\beta} b_{\alpha\beta} n^i - S^{\alpha\beta}_{,\alpha} x^i_{,\beta} = [n_k t^{ik}] + \gamma f^i, \\
\frac{\partial \gamma}{\partial t} + V^\alpha \gamma_{,\alpha} + \gamma d_\alpha{}^\alpha = 0,
\end{gathered} \tag{3.2}$$

where $d_{\alpha\beta} = V_{(\alpha,\beta)} - u_n b_{\alpha\beta}$. Let us consider the propagation of the interfacial shape $\eta = a e^{ik(x-ct)}$ where $|a|$ is small. We shall assume that the movement of the inviscid fluid is irrotational and so we can find a velocity potential ϕ such that $\boldsymbol{v}_{\text{inviscid}} = -\operatorname{grad} \phi$ where $\boldsymbol{v}_{\text{inviscid}}$ is the perturbation of the inviscid fluid velocity from the steady state $(V, 0, 0)$. From (3.1.1) we may conclude that in

$z > 0$ (region to be denoted by a subscript 1)

$$p_1 + \varrho_1 \Omega - \varrho_1 \dot{\varphi} = \text{constant}, \tag{3.3}$$

where $(\dot{\ })$ is a convected derivative taken with the velocity $(V, 0, 0)$, i.e., $\dot{\varphi} = \frac{\partial \varphi}{\partial t} + V \frac{\partial \varphi}{\partial x}$. Since the region $z < 0$ contains a viscous fluid then we cannot expect the flow in this region to be irrotational. We shall look for a solution of (3.1.2) in the form

$$\boldsymbol{u} = \big(u(z),\ 0,\ w(z)\big)\, e^{ik(x-ct)}. \tag{3.4}$$

We also require $\boldsymbol{u}$ to be divergence free, satisfy the boundary condition $\boldsymbol{u}_{z\,\text{cpt.}}(0, t) = -ikcae^{ik(x-ct)}$ and tend to zero as z tends to negative infinity. In $z < 0$ (region to be denoted by a subscript 2), calculation reveals that

$$\boldsymbol{u} = (i\alpha A e^{k\alpha z} + iBe^{kz},\ 0,\ Ae^{k\alpha z} + Be^{kz})\, e^{ik(x-ct)}, \tag{3.5}$$

where

$$\begin{aligned} &\text{(a)}\quad \alpha^2 = 1 - ic/k\nu, \qquad \nu = \mu/\varrho_2, \qquad \operatorname{Re} \alpha > 0, \\ &\text{(b)}\quad A + B = -ikca. \end{aligned} \tag{3.6}$$

Further computation from (3.1.2) leads to

$$p_2 + \varrho_2 \Omega - icB\varrho_2 e^{ik(x-ct)}\, e^{kz} = \text{constant}. \tag{3.7}$$

Since the velocity potential for region one is $\varphi = -icae^{ik(x-ct)}e^{-kz}$, it follows from (3.3), (3.7) that

$$[p] - [\varrho]\, g\eta - \varrho_1 akc(V - c)\, e^{ik(x-ct)} + icB\varrho_2 e^{ik(x-ct)} = \text{constant}$$

from which we may deduce that

$$\begin{aligned} &\frac{d}{dt}\,[p] + ikcga[\varrho]\, e^{ik(x-ct)} - i\varrho_1 ack^2(V - c)^2\, e^{ik(x-ct)} \\ &\qquad + c^2kB\varrho_2 e^{ik(x-ct)} = 0. \end{aligned} \tag{3.8}$$

The propagation conditions are going to be derived from Eqs. (3.2).

We may deal with these equations in their fully linearised form in which variations in surface density γ may be considered. However, since γ is very small, we shall ignore it in what we are about to do. In particular we shall suppose that surface tension σ is constant. The simplified equations (3.2) become

$$\begin{aligned} S^{\alpha\beta} b_{\alpha\beta} &= [p] + 2\mu n_i d_{ik} n_k, \\ S^{\alpha\beta}_{\ \ ,\alpha}\, a_{\beta\theta} &= 2\mu n_k d_{ik} x_{i,\theta}. \end{aligned} \tag{3.9}$$

To first order

$$\begin{aligned}&\text{(a)}\quad V^1 = i(\alpha A + B)\, e^{ik(x-ct)}, \qquad V^2 = 0,\\ &\text{(b)}\quad d_\alpha{}^\alpha = V^1_{,1} = -k(\alpha A + B)\, e^{ik(x-ct)}.\end{aligned} \tag{3.10}$$

The most general expression for $S^{\alpha\beta}$ linear in $d^{\alpha\beta}$ is given by

$$S^{\alpha\beta} = \nu_0 a^{\alpha\beta} + \nu_1 b^{\alpha\beta} + \nu_2 a^{\alpha\beta} d_\lambda{}^\lambda + \nu_3 b^{\alpha\beta} d_\lambda{}^\lambda + \nu_4 a^{\alpha\beta}\,\mathrm{tr}\,(bd)$$
$$+ \nu_5 b^{\alpha\beta}\,\mathrm{tr}\,(bd) + \nu_6 d^{\alpha\beta} + \nu_7 b^{(\alpha\lambda} d_\lambda{}^{\beta)}$$

and when this is further linearised about $V^i = 0$, $b_{\alpha\beta} = 0$, the result is

$$S^{\alpha\beta} = (\sigma + \pi_2 d_\lambda{}^\lambda)\, a^{\alpha\beta} + \pi_6 d^{\alpha\beta}, \tag{3.11}$$

where π_2, π_6 are viscosities and are to be treated as constants. Substitution into Eqs. (3.9) reveal that

$$[p] = \sigma \frac{\partial^2 \eta}{\partial x^2} - 2\mu k(B + \alpha A)\, e^{ik(x-ct)}, \tag{3.12.1}$$

$$2B + A(1 + \alpha^2) = -k(\alpha A + B)\, \pi/\mu, \tag{3.12.2}$$

where $\pi = \pi_2 + \pi_6$. Returning to (3.8), further calculation indicates that

$$-2\mu k\alpha A + B(ic\varrho_2 - 2uk) + a\left[-\sigma k^2 - g[\varrho] + \varrho_1 k(V - c)^2\right] = 0. \tag{3.13}$$

In view of (3.12.2) and (3.6)

$$\begin{aligned}A &= -k^2 \nu a \frac{2 + k\lambda}{1 - \dfrac{ik^2\lambda\nu(\alpha - 1)}{c}},\\ B &= a\nu k^2[1 + \alpha^2 + k\lambda(2 - \alpha)] \Big/ \left[1 - \frac{ik^2\lambda\nu(\alpha - 1)}{c}\right]\end{aligned} \tag{3.14}$$

in which $\lambda = \pi/\mu$. For consistency of solution with A, B and a not all zero, it is necessary that the wavespeed c satisfy the dispersive relationship

$$\begin{aligned}&4\alpha k^2\nu^2\varrho_2 - \varrho_2(ic - 2\nu k)^2 + \lambda\varrho_2 k^2\nu[4\nu k(\alpha - 1) + ic(2 - \alpha)]\\ &-\left[1 - \frac{ik^2\lambda\nu(\alpha - 1)}{c}\right]\left[\sigma k - \varrho_1(V - c)^2 + \frac{g}{k}[\varrho]\right] = 0.\end{aligned} \tag{3.15}$$

The new feature of (3.15) is the dependence of c on ν and interfacial viscosity. The classical case in which $\nu = \lambda = 0$ has been studied by Chandrasekhar [13]. Here c satisfies the quadratic equation

$$(\varrho_1 + \varrho_2)\, c^2 - 2\varrho_1 V c = \sigma k + \frac{g}{k}[\varrho] - \varrho_1 V^2$$

with solution

$$c = V\varrho_1/(\varrho_1 + \varrho_2) \pm \frac{\left[\left(\sigma k + \frac{g}{k}[\varrho]\right)(\varrho_1 + \varrho_2) - \varrho_1\varrho_2 V^2\right]^{1/2}}{(\varrho_1 + \varrho_2)}. \qquad (3.16)$$

In the situation in which

$$V^2 > \frac{\varrho_1 + \varrho_2}{\varrho_1\varrho_2}(\sigma k + g[\varrho]/k),$$

c is complex valued and thus the Kelvin-Helmholtz instability ensues.

When $\nu \neq 0$, it is mathematically convenient to write $c = i(\alpha^2 - 1)\,k\nu$ so that (3.15) becomes

$$\begin{aligned}&(\alpha + 1)\left[(\varrho_1 + \varrho_2)\,\alpha^4 + 2(\varrho_2 - \varrho_1 + i\varrho_1 V/k\nu)\,\alpha^2 - 4\varrho_2\alpha\right.\\&\left.+ \varrho_1 + \varrho_2 - 2V\varrho_1 i/k\nu + (\sigma k + g[\varrho]/k - \varrho_1 V^2)/k^2\nu^2\right]\\&- \lambda k\left[(\varrho_2 + \varrho_1)\,\alpha^4 - \varrho_2\alpha^3 + (\varrho_2 - 2\varrho_1 + 2V\varrho_1 c/k\nu)\,\alpha^2\right.\\&\left.+ \varrho_2\alpha + \varrho_1 - 2\varrho_2 - 2iV\varrho_1/k\nu + (\sigma k + g[\varrho]/k - \varrho_1 V^2)/k^2\nu^2\right] = 0,\end{aligned} \qquad (3.17)$$

where we are only interested in solutions for which $\operatorname{Re}\alpha > 0$.

If these solutions α are to represent stable solutions then in addition $\operatorname{Im} c < 0$ from which we may conclude that $\operatorname{Re}\alpha^2 < 1$. Let us consider the classical situation in which $\gamma = 0$. In this event α is the solution of the quartic

$$\begin{aligned}&(\varrho_1 + \varrho_2)\,\alpha^4 + 2(\varrho_2 - \varrho_1 + i\varrho_1 V/k\nu)\,\alpha^2 - 4\varrho_2\alpha\\&+ \varrho_1 + \varrho_2 - 2V\varrho_1 i/k\nu + \left(\sigma k + \frac{g[\varrho]}{k} - \varrho_1 V^2\right)\Big/ k^2\nu^2 = 0.\end{aligned} \qquad (3.18)$$

As a general observation, we note that the sum of the four roots of (3.18) is zero and hence the roots are either all purely complex or there is at least one root with a positive real part. It is easily verified that the first hypothesis is void.

We begin by investigating the case in which $V = 0$. Here α satisfies

$$(\varrho_1 + \varrho_2)\,\alpha^4 + 2(\varrho_2 - \varrho_1)\,\alpha^2 - 4\varrho_2\alpha + \varrho_1 + \varrho_2 + (\sigma k + g[\varrho]/k)/k^2\nu^2 = 0. \qquad (3.19)$$

Since all the coefficients in this polynomial are real then either

(a) the solutions are all real,

or

(b) there are two real solutions and two complex conjugate pair solutions,

or

(c) there are no real solutions and four solutions arranged in two complex conjugate pairs.

Let us define a real function f by the rule

$$f(x) = (\varrho_1 + \varrho_2)\, x^4 + 2(\varrho_2 - \varrho_1)\, x^2 - 4\varrho_2 x + \varrho_1 + \varrho_2 + (\sigma k + g[\varrho]/k)\, k^2 \nu^2$$

and observe that if $\pi^* = (\sigma k + g[\varrho]/k)\, k^2 \nu^2$ then

$$f(0) = \varrho_1 + \varrho_2 + \pi^*, \qquad f(1) = \pi^*. \tag{3.20}$$

Further since $f'(0) = -4\varrho_2 < 0$ and $f'(1) = 4\varrho_2 > 0$ and $f''(x) > 0$ then $f(x)$ attains a minimum value within (0, 1). Moreover, provided $k < k_0$, this minimum value is positive. On the other hand if $k \geqq k_0$ then since both $f(0)$ and $f(1)$ are positive, we have two real solutions α_1, α_2 such that $0 < \alpha_1 < \alpha_2 < 1$ with equality when $k = k_0$. The other solutions are represented $p \pm iq$ (p, q real) where $p = -\dfrac{1}{2}(\alpha_1 + \alpha_2)$. In particular $-1 < p < 0$.

If $k < k_0$ then the solutions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ may be represented

$$\alpha_1 = \beta + i\theta, \quad \alpha_2 = \beta - i\theta, \quad \alpha_3 = -\beta + i\varphi, \quad \alpha_4 = -\beta - i\varphi$$

where from the outset β, θ and φ may be taken as positive without any loss in generality. Moreover Eq. (3.19) in this instance is equivalent to

$$(x^2 - 2\beta x + \beta^2 + \theta^2)\,(x^2 + 2\beta x + \beta^2 + \varphi^2) = 0 \tag{3.21}$$

and hence by comparison with (3.19)

$$\theta^2 + \varphi^2 - 2\beta^2 = 2(\varrho_2 - \varrho_1)/(\varrho_1 + \varrho_2), \tag{3.22.1}$$

$$\beta(\varphi^2 - \theta^2) = 2\varrho_2/(\varrho_1 + \varrho_2), \tag{3.22.2}$$

$$(\beta^2 + \theta^2)\,(\beta^2 + \varphi^2) = 1 + \pi^*/(\varrho_1 + \varrho_2). \tag{3.22.3}$$

There are precisely two solutions satisfying $\mathrm{Re}\,\alpha > 0$ whichever value of k we choose. It is obviously that when $k \geqq k_0$ there is stability. When $k < k_0$ then for both solutions with $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\alpha^2 = \beta^2 - \theta^2$. From (3.22.1, 2)

$$\beta^2 - \theta^2 = [\varrho_1 - \varrho_2 + \varrho_2/\beta]/(\varrho_1 + \varrho_2) \tag{3.23}$$

and consequently since $\beta > 0$, it satisfies

$$(\varrho_1 + \varrho_2)\, \beta^3 + (\varrho_2 - \varrho_1)\, \beta - \varrho_2 > 0$$

from which it is clear that $\beta > \dfrac{1}{2}$. Returning to (3.23) we see that

$$\beta^2 - \theta^2 < 1. \tag{3.24}$$

Thus in any event, when $V = 0$ two wavespeeds are possible and both are stable.

When $V \neq 0$, the situation is considerably complicated since the solutions to the quartic polynomial are no longer complex conjugate pairs. Our aim is to show that if V is suitably large then there is instability. We recall that we wish to analyse the solutions of (3.18) for which $\operatorname{Re} \alpha > 0$. Let

$$q = (\varrho_1 + \varrho_2) \operatorname{Re} \alpha^2 + \varrho_2 - \varrho_1, \qquad w = (\varrho_1 + \varrho_2) \operatorname{Im} \alpha^2 + \varrho_1 V/k\nu \tag{3.26}$$

then

$$q^2 - w^2 = (\varrho_2/\varrho_1) R^2 - 4\varrho_1\varrho_2 + 4\varrho_2(\varrho_1 + \varrho_2) x - \frac{(\varrho_1 + \varrho_2)}{k^2\nu^2} \left[\sigma k + g[\varrho]/k\right], \tag{3.27.1}$$

$$qw = 2\varrho_2\big(R + (\varrho_1 + \varrho_2) y\big) \tag{3.27.2}$$

where $R = \varrho_1 V/k\nu$ and $\alpha = x + iy$. We begin by observing that $\alpha = 1$ (i.e., $x = 1$, $y = 0$) trivially satisfies (3.27.2) and also satisfies (3.27.1) provided k is a solution of

$$\varrho_1 V^2 = \alpha k + g[\varrho]/k. \tag{3.28}$$

In this case α satisfies

$$(\varrho_1 + \varrho_2) \alpha^4 + 2(\varrho_2 - \varrho_1 + iR) \alpha^2 - 4\varrho_2\alpha + (\varrho_1 + \varrho_2) - 2Ri = 0$$

which on factorisation yields that $\alpha = 1$ or $g(\alpha) = 0$ where

$$g(\alpha) = \alpha^3 + \alpha^2 + (\beta + i\theta)\alpha + i\theta - 1$$

with

$$\beta = \frac{3\varrho_2 - \varrho_1}{\varrho_1 + \varrho_2}, \qquad \theta = 2R/(\varrho_1 + \varrho_2). \tag{3.29}$$

Since $g(-1) = -(1 + \beta) \neq 0$ then $\alpha = -1$ can never be a zero of g and thus we may deduce that

$$\alpha^2 = -(\beta + i\theta) + \frac{(1 + \beta)}{(1 + \alpha)}.$$

Hence

$$\operatorname{Re}(\alpha^2) = -\beta + \frac{(1 + \beta)(1 + x)}{(1 + x)^2 + y^2} \tag{3.30}$$

where $x = \operatorname{Re}(\alpha)$. If $x > 0$ then $\operatorname{Re}(\alpha^2) < 1$, i.e., any solution of $g(\alpha) = 0$ with $\operatorname{Re}(\alpha) > 0$ is stable and so $\alpha = 1$ is the destabilising solution. Returning to (3.28), we can only find a wavenumber k such that $\alpha = 1$ is a solution of (3.27) provided

$$V^2 > (1/\varrho_1) \min_{(k \in R^+)} \left[\sigma k + g[\varrho]/k\right].$$

Let $k = K$ be the larger solution to (3.28). We aim to determine $\left.\frac{dq}{dk}\right|_{k=K}$. From (3.26) and (3.27) it follows that

$$\begin{aligned}
&\text{(a)}\quad 2q\frac{dq}{dk} = 2w\frac{dw}{dk} + 2(\varrho_2/\varrho_1)\,R\frac{dR}{dk} + 4(\varrho_1+\varrho_2)\,\varrho_2\frac{dx}{dk} \\
&\qquad + \frac{(\varrho_1+\varrho_2)}{k^2\nu^2}\left[\sigma + \frac{3g[\varrho]}{k^2}\right], \\
&\text{(b)}\quad q\frac{dw}{dk} + w\frac{dq}{dk} = 2\varrho_2\left(\frac{dR}{dk} + (\varrho_1+\varrho_2)\frac{dy}{dk}\right), \\
&\text{(c)}\quad \frac{dq}{dk} = 2(\varrho_1+\varrho_2)\left(x\frac{dx}{dk} - y\frac{dy}{dk}\right), \\
&\text{(d)}\quad \frac{dw}{dk} = \frac{dR}{dk} + 2(\varrho_1+\varrho_2)\left(x\frac{dy}{dk} + y\frac{dx}{dk}\right).
\end{aligned} \tag{3.31}$$

In view of the fact that $\frac{dR}{dk} = -R/k$, we may verify after some algebra on Eqs. (3.31) that at $x = 1$, $y = 0$

$$2(R^2 + \varrho_2{}^2)\left.\frac{dq}{dk}\right|_{k=K} = \frac{\varrho_1(\varrho_1+\varrho_2)}{K^2\nu^2}\left[\frac{g[\varrho]}{K^2} - \sigma\right], \tag{3.32}$$

where V^2 has been replaced from expression (3.28). Since the minimum of $\sigma k + g[\varrho]/k$ occurs at $k = k_0$ with k_0 the positive solution of $\sigma - g[\varrho]/k^2 = 0$ then since $K > k_0$, it follows from (3.32) that $\left.\frac{dq}{dk}\right|_{k=K} < 0$ and hence that there is an interval (K^*, K) on which $q > 2\varrho_2$ i.e., $\operatorname{Re}(\alpha^2) > 1$. Thus we have shown that there is a region of instability provided $V > V_{\text{crit.}}$ where

$$V^2_{\text{crit.}} = \underset{(k\in R^+)}{\text{minimum}}\ \frac{1}{\varrho_1}\left[\sigma k + \frac{g[\varrho]}{k}\right]. \tag{3.33}$$

4. Conclusions

When an incompressible inviscid fluid of density ϱ_1 flows with steady shear velocity over a similar but denser fluid, Chandrasekhar [13] has shown that the configuration is unstable for speeds in excess of the Kelvin-Helmholtz velocity V_{KH} where

$$V^2_{KH} = \left(\frac{\varrho_1+\varrho_2}{\varrho_1\varrho_2}\right)\min_{k\in R}\left[\sigma k + (\varrho_2-\varrho_1)\frac{g}{k}\right]. \tag{4.1}$$

The preceeding analysis leads us to the conclusion that in the context of a viscous denser fluid, the corresponding critical velocity is lower than the classical value V_{KH}. In fact

$$V_{\substack{\text{critical with}\\ \text{viscosity}}} = \left(\frac{\varrho_{\text{viscous}}}{\varrho_{\text{viscous}} + \varrho_{\text{inviscid}}}\right)^{1/2} V_{KH}. \tag{4.2}$$

In his original paper Kelvin [15] stated that "Observation shows the sea to be ruffled by a wind of much smaller velocity than the critical value" and this he accorded to the effects of viscosity. For an air/water interface, $\varrho_{\text{air}}/\varrho_{\text{water}}$ is very small and so it is clear that Kelvin's observation is unlikely to be explainable in terms of viscosity but is probably a product of the uncontrolled nature of the experiment. More recent experiments of Francis [16] in which air was blown over lubricating oil in a wind tunnel produced no detectable variations from the Chandrasekhar formula. This is not surprising as again $\varrho_{\text{air}}/\varrho_{\text{oil}}$ is very small and so any predicted discrepancies would be within experimental error. We might consider enhancing this ratio by experimenting with fluids of comparable density but the advantages are not immediately clear since for such fluids, the Kelvin velocity will be substantially lower not to mention losses on the grounds of decreased surface tension.

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