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Analysis of a Thick Plate With a Circular Hole Resting on a Smooth Rigid Bed and Subjected to Axisymmetric Normal Load

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With 7 Figures

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Summary — Zusammenfassung

Analysis of a Thick Plate With a Circular Hole Resting on a Smooth Rigid Bed and Subjected to Axisymmetric Normal Load. A solution for the stresses and displacements in an radially infinite thick plate having a circular hole, one face of which resting on a smooth rigid bed and the other face subjected to axisymmetric normal loading is given. The solution is obtained in terms of Fourier-Bessel series and integral for the Love's stress function. Numerical results are presented for one particular ratio of thickness of plate to the hole radius and loading. It is also shown that the Poisson's ratio has a predominant effect on certain stresses and displacements. The solution would be useful in the stress analysis of bolted joints.

Analyse einer dicken Platte mit einem kreisförmigen Loch, auf einer ebenen, starren Unterlage mit achsensymmetrischer Vertikalbelastung. Eine Lösung für die Spannungen und Verschiebungen in einer radial, unendlich ausgedehnten, dicken Platte mit einem kreisförmigen Loch, wobei eine Seite auf einer ebenen, starren Unterlage aufliegt, die andere Seite durch eine achsensymmetrische Vertikallast belastet ist, wird angegeben. Die Lösung wird in Form von Fourier-Bessel-Reihen und Integralen der Loveschen Spannungsfunktion angegeben. Numerische Ergebnisse werden für ein bestimmtes Verhältnis der Plattendicke zum Lochradius sowie zur Belastung angegeben. Es wird auch gezeigt, daß das Poissonsche Verhältnis einen besonderen Einfluß auf bestimmte Spannungen und Verschiebungen hat. Die Lösung ist anwendbar für die Spannungsermittlung von Bolzenverbindungen.

Introduction

Pressure distribution and contact area between two bolted or rivetted plates is of practical interest particularly in the study of heat transfer across the joint and in electric contacts. In such connections, the plates will be in contact in the vicinity of the bolt or rivet and would be separated beyond this region. A mathematical solution to this problem poses some difficulty because of mixed boundary conditions across the joint. However, the separation of the plates along the joint depends on the thickness of the plates and for thick plates there could be perfect contact between the two plates along the joint. In such a case, if the two plates are of same thickness and are made of the same material then the pressure distribution between the plates could be obtained by using a single plate model in which the mid-plane would correspond to the joint. Fernlund [1] has obtained a solution for the single plate model which is approximate. Recently the authors [2]

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have given an exact solution to this problem. On the other hand if the bolted connection is made up of two plates having different thickness or material properties, then a single plate model solution cannot be used. In such a case each plate may be considered independently and if a solution for this can be written, then the solution for the bolted connection can be obtained by using the continuity conditions along the joint. To achieve this it is necessary to obtain the solution for a single plate with an asymmetric load. The solution for this case is given here and applied to the particular case of a bolted joint in which the elastic properties of the plates are such that one of the plate could be considered rigid. Numerical results have been obtained for a plate having a thickness to radius ratio 2. A study of the effect of Poisson's ratio has also been made by obtaining the stresses and displacements for two different values of Poisson's ratio.

Statement of the Problem

An elastic radially infinite plate of thickness '2h' having a circular hole of radius 'a' is considered to be resting on a smooth rigid bed. An axisymmetric normal load of intensity 'q' is considered to be acting over a circular region of radius 'b' (Fig. 1). The plate material is considered as homogeneous and isotropic. The solution to the problem has been obtained in terms of Fourier-Bessel series and integral for the Love stress function.

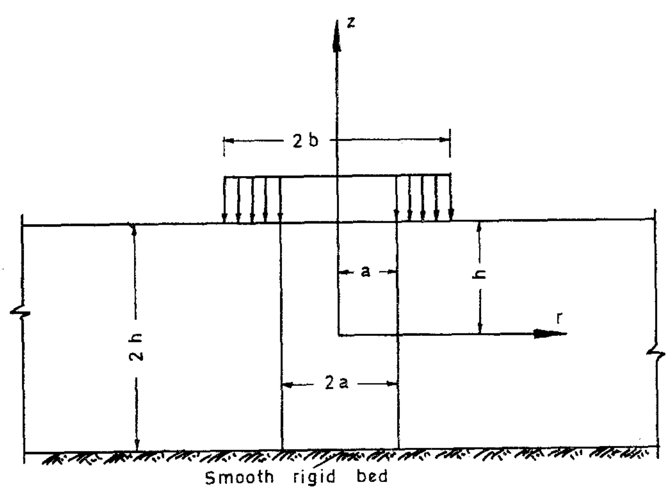

Fig. 1. Thick plate with a hole resting on a smooth rigid bed

Basic Equations

The Love's stress function of for an axisymmetric problem must satisfy the differential equation [3],

$$\nabla^2\nabla^2\varphi = 0 \tag{1}$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$$

the stresses and displacements are obtained from

$$\begin{aligned}
\sigma_r &= \frac{\partial}{\partial z}\left[\nu \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial r^2}\right] \\
\sigma_\theta &= \frac{\partial}{\partial z}\left[\nu \nabla^2 \varphi - \frac{1}{r}\frac{\partial \varphi}{\partial r}\right] \\
\sigma_z &= \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2}\right] \\
\tau_{rz} &= \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2}\right] \\
u &= -\frac{1}{2G}\frac{\partial^2 \varphi}{\partial r\, \partial z}; \qquad w = \frac{1}{2G}\left[2(1-\nu)\nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2}\right]
\end{aligned} \tag{2}$$

where ν and G are the Poisson's ratio and shear modulus respectively.

Boundary Conditions

The boundary conditions for the problem are (Fig. 1)

$$r = a, \qquad \sigma_r = \tau_{rz} = 0, \tag{3}$$

$$z = +h, \qquad \sigma_z = f(r) = \begin{cases} q & a \leqq r \leqq b, \\ 0 & r > b, \end{cases} \tag{4}$$

$$\begin{aligned} &\tau_{rz} = 0, \\ z = -h, \qquad &w = \tau_{rz} = 0. \end{aligned} \tag{5}$$

Solution

Love's stress function is selected as follows

$$\begin{aligned}
\varphi = &\sum_{m=1,2,3}^{\infty} \frac{A_m \sin \alpha_m z}{\alpha_m^3 K_1(\alpha_m a)}\left[\alpha_m r K_1(\alpha_m r) - \left\{\frac{\alpha_m a K_0(\alpha_m a)}{K_1(\alpha_m a)} - 2(1-\nu)\right\} K_0(\alpha_m r)\right] \\
&+ \int_0^\infty \frac{F(\beta)\, V_0(\beta r)}{\beta^3 \cosh \beta h}\left[\beta z \cosh \beta z - (\beta h \coth \beta h + 2\nu)\sinh \beta z\right] d\beta \\
&+ \sum_{n=1,3,5}^{\infty} \frac{C_n \cos \gamma_n z}{\gamma_n^3 K_1(\gamma_n a)}\left[\gamma_n r K_1(\gamma_n r) - \left\{\frac{\gamma_n a K_0(\gamma_n a)}{K_1(\gamma_n a)} - 2(1-\nu)\right\} K_0(\gamma_n r)\right] \\
&+ \int_0^\infty \frac{E(\beta)\, V_0(\beta r)}{\beta^3 \sinh \beta h}\left[\beta z \sinh \beta z - (\beta h \tanh \beta h + 2\nu)\cosh \beta z\right] d\beta
\end{aligned} \tag{6}$$

where

$$\alpha_m = m\pi/h, \qquad \gamma_n = n\pi/2h$$

and

$$V_0(\beta r) = J_0(\beta r)\, Y_1(\beta a) - Y_0(\beta r)\, J_1(\beta a).$$

It may be seen that the stress function given by Eq. (6) satisfies the biharmonic Eq. (1). Substituting for into Eq. (2), the expressions for stresses and displacements can be obtained in terms of the undetermined coefficients A_m, $F(\beta)$, C_n and $E(\beta)$.

Applying the boundary condition $w\ (z = -h) = 0$ we get

$$E(\beta) = (\tanh^2 \beta h)\, F(\beta). \tag{7}$$

Applying the boundary condition $\sigma_r\ (r = a) = 0$ and separating the symmetric and antisymmetric parts, the following two equations are obtained.

$$A_m \alpha_m a \left[1 + \frac{2(1-\nu)}{\alpha_m^2 a^2} - \frac{K_0^2(\alpha_m a)}{K_1^2(\alpha_m a)}\right] = \int_0^\infty \frac{(-1)^m F(\beta)\, 4m^2\pi^2\beta h \tanh \beta h\, V_0(\beta a)}{(m^2\pi^2 + \beta^2 h^2)^2}\, d\beta \tag{8}$$

and

$$C_n \gamma_n a \left[1 + \frac{2(1-\nu)}{\gamma_n^2 a^2} - \frac{K_0^2(\gamma_n a)}{K_1^2(\gamma_n a)}\right] = -\int_0^\infty \frac{(-1)^{n+1}\, 4n^2\pi^2\beta h \tanh \beta h F(\beta)\, V_0(\beta a)}{\left(\frac{n^2\pi^2}{4} + \beta^2 h^2\right)^2}\, d\beta. \tag{9}$$

Applying the boundary condition

$$\sigma_z\ (z = \pm h) = f(r) = \int_a^\infty \beta K(\beta)\, V_0(\beta r)\, d\beta.$$

We get

$$F(\beta) = \frac{\beta K(\beta)}{T(\beta h)} - \sum_{m=1,2,3,..}^{\infty} \frac{(-1)^m\, 4\alpha_m \beta^2 A_m}{\pi(\alpha_m^2 + \beta^2)^2 B(\beta a)\, T(\beta h)} + \sum_{n=1,3,5,..}^{\infty} \frac{(-1)^{n+1} 4\gamma_n \beta^2 C_n}{\pi(\gamma_n^2 + \beta^2)^2 B(\beta a)\, T(\beta h)} \tag{10}$$

where

$$B(\beta a) = J_1^2(\beta a) + Y_1^2(\beta a)$$

and

$$T(\beta h) = (1 + \tanh^2 \beta h)\left(1 + \frac{4\beta h}{\sinh 4\beta h}\right)$$

For the loading considered here

$$K(\beta) = \int_a^b \frac{q r V_0(\beta r)\, dr}{B(\beta a)} = \frac{bq}{\beta}\, \frac{V_1(\beta b)}{B(\beta a)} \tag{11}$$

where

$$V_1(\beta b) = J_1(\beta b)\, Y_1(\beta a) - Y_1(\beta b)\, J_1(\beta a).$$

Substituting for $F(\beta)$ and $E(\beta)$ from Eq. (10) and Eq. (7), putting $\beta a = x$, $h/a = \varrho$, $b/a = \delta$, we get the following two sets of equations in A_m and C_n.

$$A_m \alpha_m a \left[1 + \frac{2(1-\nu)}{\alpha_m^2 a^2} - \frac{K_0^2(\alpha_m a)}{K_1^2(\alpha_m a)}\right] + \sum_{s=1,2,3}^{\infty} (-1)^{m+s}\, 16m^2\pi^2 s\varrho^4 A_s I(m, s) + \sum_{n=1,3,5,..}^{\infty} (-1)^{m+n}\, 128m^2\pi^2 n\varrho^4 C_n I(m, n) = (-1)^m\, 4m^2\pi^2 \varrho\delta q P(m) \tag{12}$$

and

$$C_n\gamma_n a\left[1+\frac{2(1-\nu)}{\gamma_n^2a^2}-\frac{K_0^2(\gamma_n a)}{K_1^2(\gamma_n a)}\right]+\sum_{k=1,3,5}^{\infty}(-1)^{n+k}\,512n^2\pi^2k\varrho^4C_kI(n,k)$$
$$+\sum_{m=1,2,3}^{\infty}(-1)^{n+m}\,64n^2\pi^2m\varrho^4A_mI(n,m)=(-1)^n\,16n^2\pi^2\varrho\delta qP(n) \tag{13}$$

where

$$I(m,s)=\int_0^\infty\frac{x^3\tanh\varrho x\,V_0(x)\,dx}{(m^2\pi^2+\varrho^2x^2)^2\,(s^2\pi^2+\varrho^2x^2)^2\,B(x)\,T(\varrho x)}$$

$$I(m,n)=I(n,m)=\int_0^\infty\frac{x^3\tanh\varrho x\,V_0(x)\,dx}{(m^2\pi^2+\varrho^2x^2)^2\,(n^2\pi^2+4\varrho^2x^2)^2\,B(x)\,T(\varrho x)}$$

$$P(m)=\int_0^\infty\frac{x\tanh\varrho x\,V_0(x)\,V_1(x\delta)\,dx}{(m^2\pi^2+\varrho^2x^2)^2\,B(x)\,T(\varrho x)}$$

$$I(n,k)=\int_0^\infty\frac{x^3\tanh\varrho x\,V_0(x)\,dx}{(n^2\pi^2+4\varrho^2x^2)^2\,(k^2\pi^2+4\varrho^2x^2)^2\,B(x)\,T(\varrho x)}$$

$$P(n)=\int_0^\infty\frac{x\tanh\varrho x\,V_0(x)\,V_1(x\delta)\,dx}{(n^2\pi^2+4\varrho^2x^2)^2\,B(x)\,T(\varrho x)}$$

where

$$\begin{aligned}V_0(x)&=J_0(x)\,Y_1(x)-Y_0(x)\,J_1(x),\\ V_1(x\delta)&=J_1(x\delta)\,Y_1(x)-Y_1(x\delta)\,J_1(x),\\ B(x)&=J_1^2(x)+Y_1^2(x),\\ T(\varrho x)&=(1+\tanh^2\varrho x)\left(1+\frac{4\varrho x}{\sinh 4\varrho x}\right)\end{aligned}$$

Eqs. (12) and (13) yield two sets of interrelated linear equations for determining the two sets of coefficients A_m and C_n. Generally a finite number of terms in both the series in m and n are taken and the coefficients A_m and C_n are determined by solving Eqs. (12) and (13). The stresses and displacements can then be determined by making use of relations given in Eqs. (10) and (7) for $F(\beta)$ and $E(\beta)$.

Numerical Results

Numerical results for the stresses and displacements have been obtained for a plate having thickness to radius of the hole ratio $(h/a)=2$ and loading $(b/a)=1.5$, for two different values of Poisson's ratio namely 0.3 and 0.495. It may be observed that for formulating the simultaneous equations (Eqs. (12) and (13)) and for determination of stresses and displacements, a large number of infinite integrals are to be evaluated. These infinite integrals have been evaluated numerically using Weddle's formula. The upper limit for the integrals was fixed such that the difference in the value of the integral for any two values is less than 1×10^{-7}. A detailed study on the convergence of the series was made and it was found that the convergence was fairly good with the consideration of 20 terms in the series. The object of selecting two Poisson's ratio, viz., 0.3 and 0.495 for the numerical study is twofold: Firstly to study the effect of Poisson's ratio on stresses and displacements, secondly to find out the utility of three-dimensional photoelastic experimental study on the problem using stress freezing technique (Poisson's ratio of the photoelastic material in such a case will be around 0.495).

The variation of radial (σ_r), tangential (σ_θ), vertical (σ_z) and shear (τ_{rz}) stresses as well as the radial (u) and vertical (w) displacement along different sections

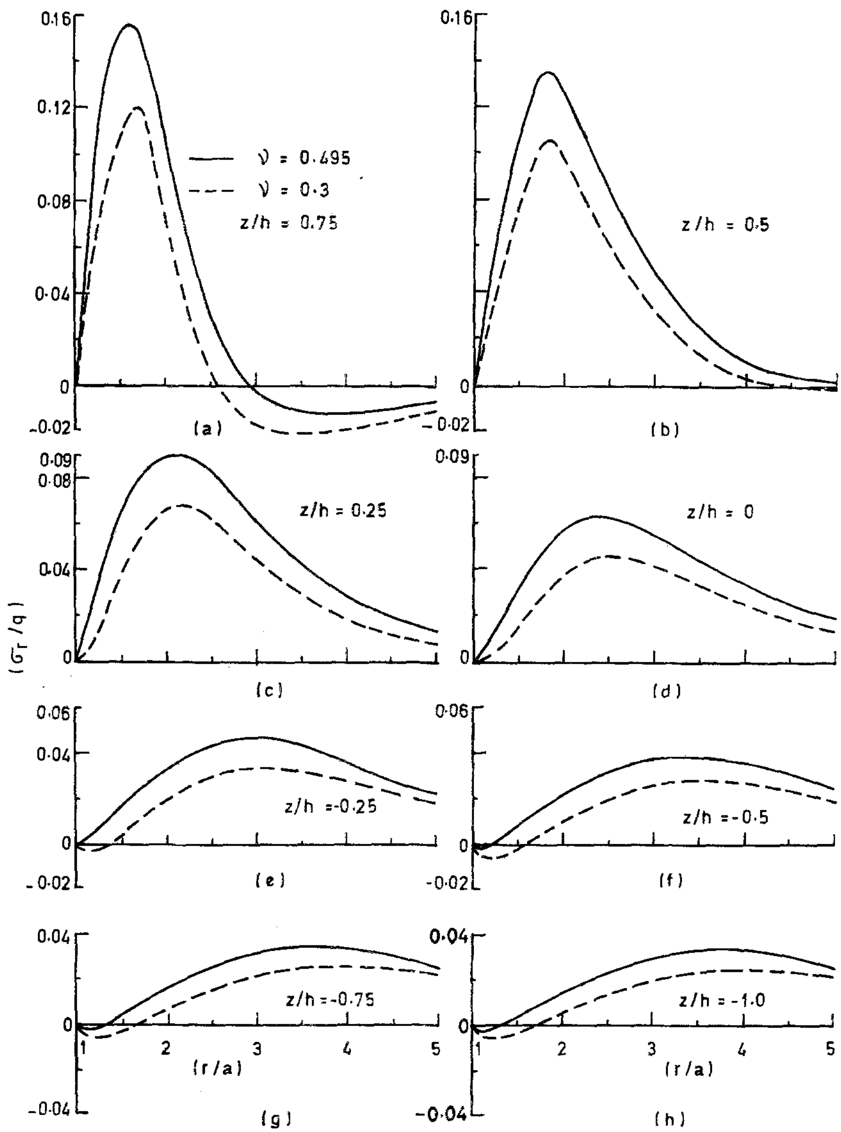

Fig. 2. Variation of radial (σ_r) stress distribution with Poisson's ratio along different horizontal sections (z/h), ($h/a = 2.0$, $b/a = 1.5$)

are shown in Figs. 2 to 7. It may be observed from Figs. 2 to 7 that the effect of Poisson's ratio is considerable on radial, tangential stresses as well as radial displacements while on the vertical and shear stresses as well as vertical displacement the effect is negligible. From this it may be concluded that the photoelastic results for the vertical and shear stresses can be used for a prototype plate without much error.

It is interesting to note that the boundary condition used at the bottom of the plate ($Z = -h$) namely $w = 0$ and $\tau_{rz} = 0$ corresponds to mid-plane symmetry conditions for a symmetrically loaded thick plate with a hole [2]. Hence it is possible to obtain the same numerical results from the case of a symmetrically

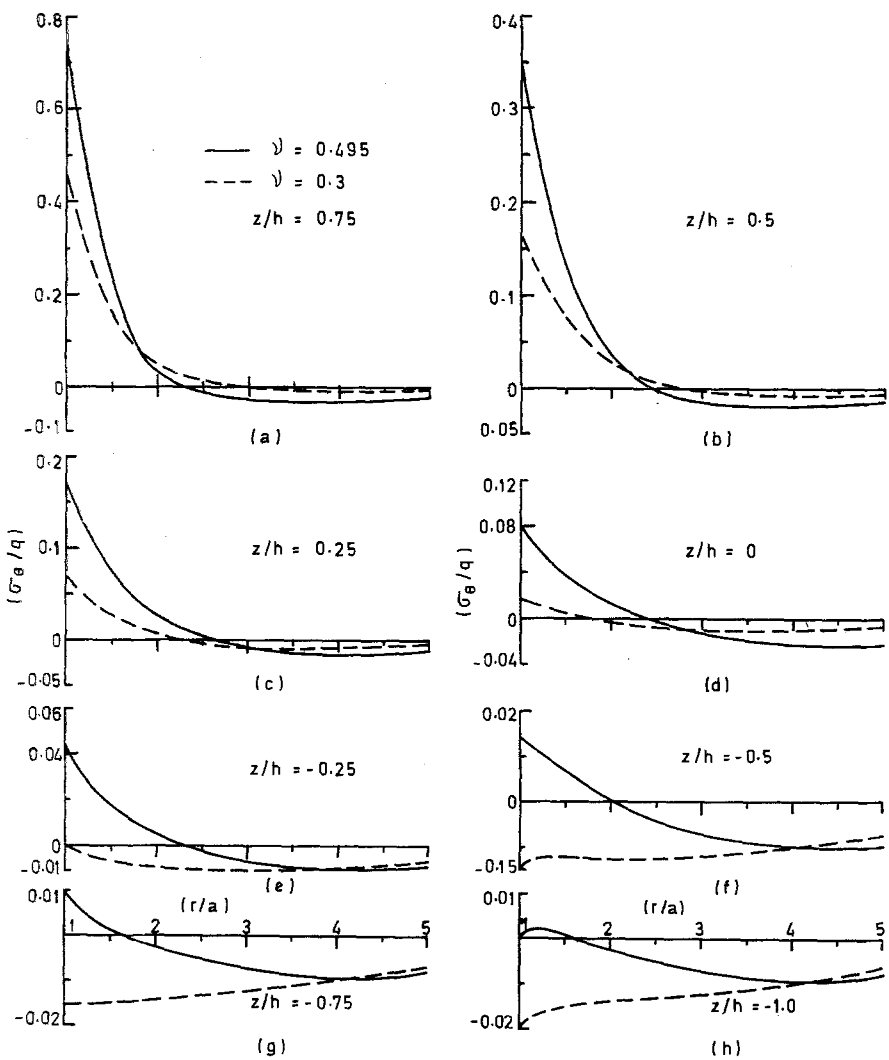

Fig. 3. Variation of tangential (σ_θ) stress distribution with Poisson's ratio along different horizontal sections (z/h), ($h/a = 2.0$, $b/a = 1.5$)

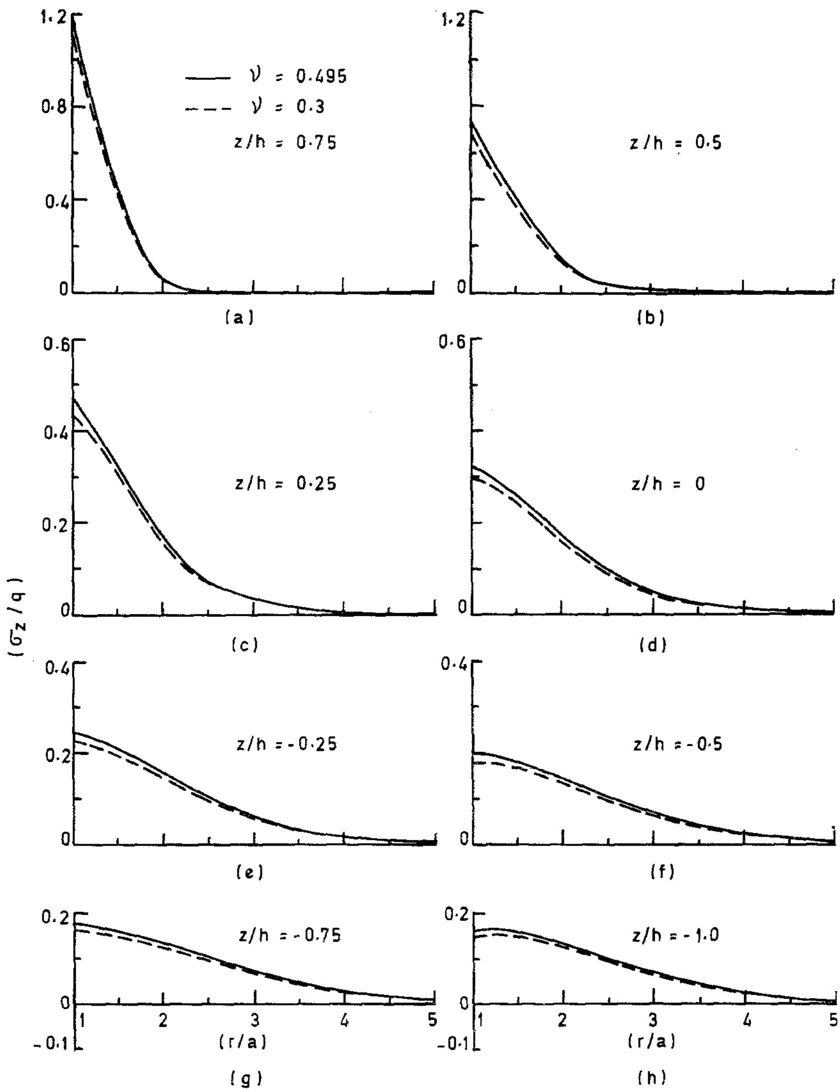

Fig. 4. Variation of vertical (σ_z) stress distribution with Poisson's ratio along different horizontal sections (z/h), ($h/a = 2.0$, $b/a = 1.5$)

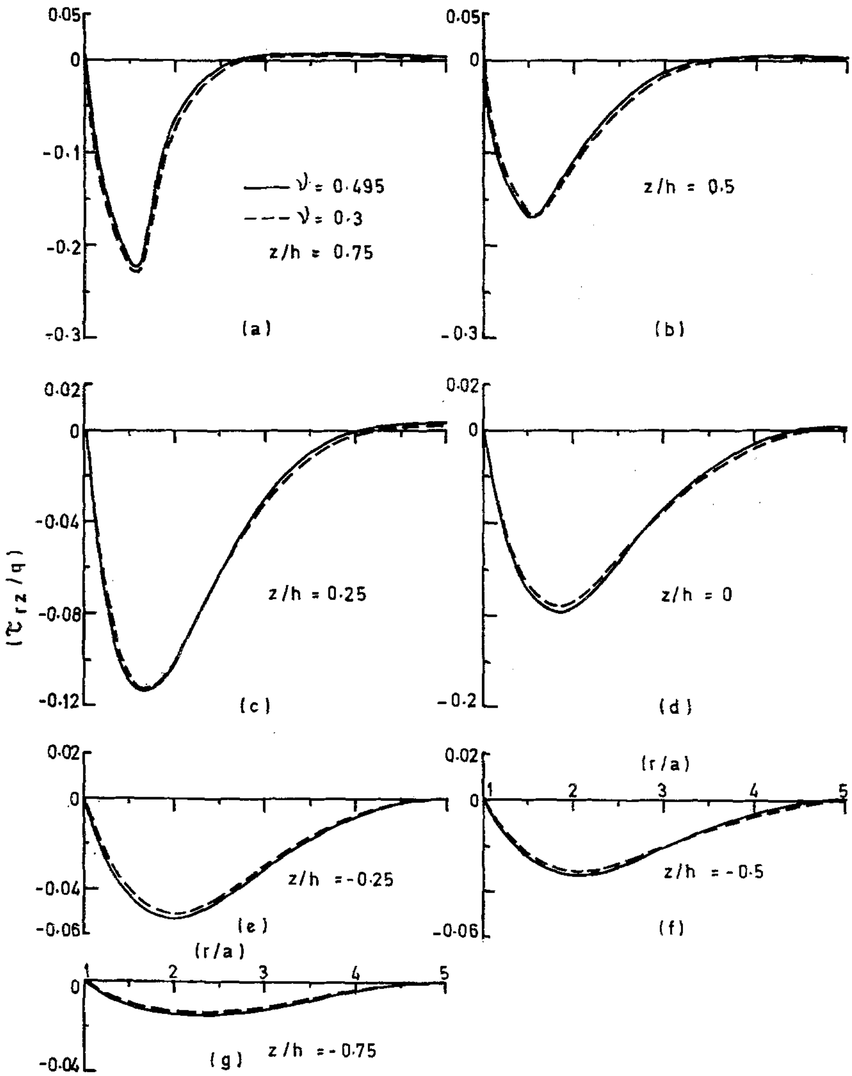

Fig. 5. Variation of shear (τ_{rz}) stress distribution with Poisson's ratio along different horizontal sections (z/h), ($h/a = 2.0$, $b/a = 1.5$)

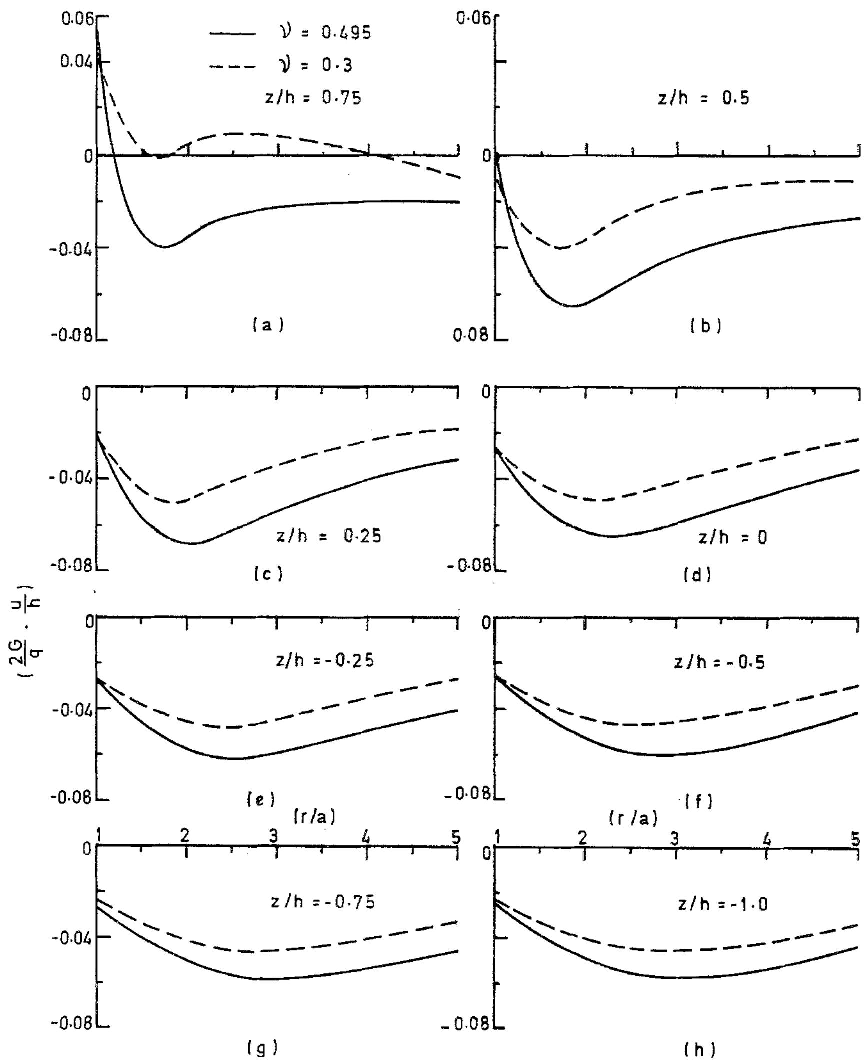

Fig. 6. Variation of radial (u) displacement with Poisson's ratio along different horizontal sections (z/h), ($h/a = 2.0$, $b/a = 1.5$)

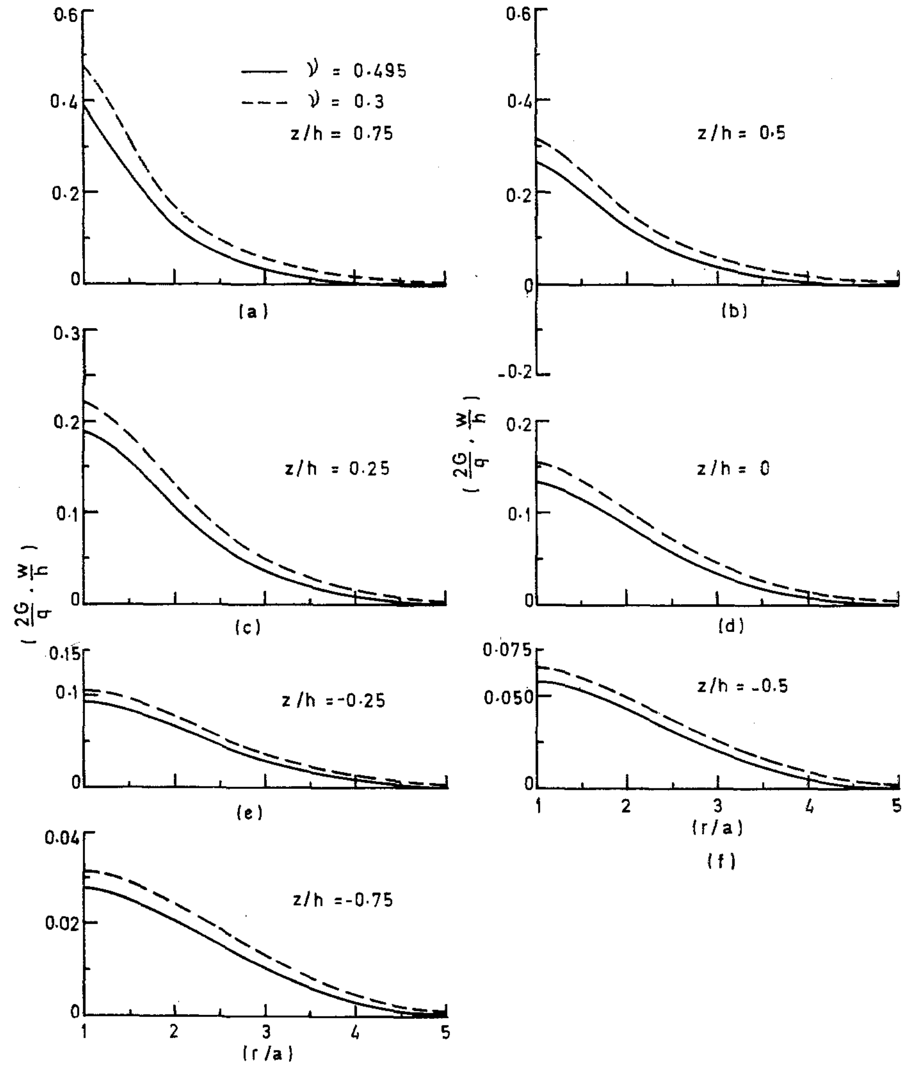

Fig. 7. Variation of vertical (w) displacement with Poisson's ratio along different horizontal sections (z/h), ($h/a = 2.0$, $b/a = 1.5$)

loaded plate by considering the thickness of the plate equal to twice that of the plate on a rigid bed. On the other hand if the plate is supported in such a way that the above boundary conditions cannot be realised then the solution to the problem should be approached in the way given in this paper.

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