Acta Mechanica 24, 117—128 (1976)

On the Frame Dependence of Electric Current and Heat Flux in a Metal

By

I. Müller, Düsseldorf

With 6 Figures

(*Received January 20, 1975*)

Summary — Zusammenfassung

On the Frame Dependence of Electric Current and Heat Flux in a Metal. The ideas and methods of the kinetic theory of metal electrons are used to show that Ohm's law of electric conduction and Fourier's law of heat conduction in a metal both contain frame dependent terms. It follows that the principle of material objectivity does not hold rigorously.

Zur Bezugssystem-Abhängigkeit des elektrischen und des Wärmestroms in einem Metall. Die Ideen und Methoden der kinetischen Theorie der Metallelektronen werden verwandt, um zu zeigen, daß das Ohmsche Gesetz der elektrischen Leitung und das Fouriersche Gesetz der Wärmeleitung in einem Metall systemabhängige Terme enthalten. Es folgt, daß das Prinzip der materiellen Objektivität nicht in Strenge gilt.

1. Introduction

This paper makes use of the simple ideas underlying the kinetic theory of metal electrons to derive approximate expressions for the relations between electric current and electric field and between heat flux and temperature gradient in a metal. In continuum mechanics and thermodynamics such relations are called constitutive relations and, according to the principle of material frame indifference, these relations should be independent of frame. It is shown here that the kinetic theory of metal electrons does not support the principle of material frame indifference; indeed, both Ohm's law of electric conduction and Fourier's law of heat conduction are shown to be dependent on frame. The frame dependence of these laws is due to the action of the Coriolis force upon the electrons in free flight between collisions. Formally, the effect of the Coriolis force is similar to the effect of a magnetic field on the free flying electrons, and therefore both the frame dependent parts of Ohm's law and of Fourier's law are governed by the Hall coefficient. It is true though that the Hall effect is much bigger then the Coriolis effect because of the large specific charge of an electron.

To my knowledge the first authors to remark upon the frame dependence of constitutive relations in the kinetic theory were Chapman and Cowling (see [1], p. 266) who observed without comment that in a gas the Burnett equations for stress and heat flux show a dependence on the antisymmetric part of the velocity gradient. In [2] TRUESDELL found the same dependence and rejects the corresponding terms as improbable; he suggests that there may have been an error in the kinetic theory analysis or that such terms are cancelled by higher iterates. This latter proposition was found invalid when MÜLLER [3] showed that even the *exact* equations of transfer, which can be derived in the case of Maxwellian molecules, contain frame dependent terms. Müller exhibited the consequence of one such term to the heat flux of a rigidly rotating gas and showed that the effect produced by it is negligible under normal conditions. EDELEN and MCLENNAN [4] recently rediscovered the dependence of the Burnett equations for stress and heat flux on the antisymmetric part of the velocity gradient and discussed its implication upon the principle of material frame indifference. Even more recently WANG [5] has criticized MÜLLER [3] and EDELEN and MCLENNAN [4] because their analyses are not rigorous and, of course, it is quite true that the iterative schemes in the kinetic theory are in want of a rigorous basis. On the other hand, these schemes offer suggestive approximations and it is generally accepted that they lead to the statistical version of macroscopic constitutive equations.

The present paper has the purpose to exhibit yet another case of frame dependence of constitutive relations. In linking this frame dependence to the well observed Hall effect the paper seems to make a convincing argument for the proposition that the *principle of material frame indifference is only approximately true* in a metal.

Further research is clearly indicated to determine where this approximate principle ceases to be reliable.

2. The Collision Equation for Metal Electrons and Some Moments of the Distribution Function

a) The Collision Equation

The kinetic theory of metal electrons is based upon the idea that the electrons in a metallic body move like free particles which occasionally collide with a lattice ion but not with each other. The ions are assumed to be rigid spheres of radius s which are at rest at their lattice points; their (uniform and constant) density will be denoted by n_0. The electrons are assumed to be mass points of mass m, and due to the large mass ratio of ions and electrons it is reasonable to consider the energy of an electron to be unchanged by a collision. Therefore, the velocities c_i and c_i' of an electron before and after a collision are related by

$$c_i' = c_i - 2e_i(c_n e_n), \tag{2.1}$$

when e_k is the unit vector from the center of the ion to the point of impact of the electron. The mean free path between two collisions will be denoted by l; it is defined as $l = \frac{1}{n_0 \pi s^2}$.

The state of the "electron gas" is characterized by the phase density $f(x_i, c_i, t)$ of electrons at the point x_i, time t and with velocity c_i. Thus

$$f(x_i, c_i, t)\, dx_1\, dx_2\, dx_3\, dc_1\, dc_2\, dc_3 = f\, d\boldsymbol{x}\, d\boldsymbol{c}$$

is the expected number of electrons in an element $dx\,dc$ of phase space. This phase density obeys the collision equation[1]

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} + \dot{c}_i \frac{\partial f}{\partial c_i} = \frac{1}{\pi l} \int_0^{2\pi} \int_0^{\pi} c(f' - f) \cos\theta \sin\theta \, d\theta \, d\varepsilon, \tag{2.2}$$

where c is the magnitude of the electron velocity, θ is the angle between e_i and c_i and ε is the angle of the plane spanned by c_i and c_i' with an arbitrary fixed plane through $c_i \cdot f$ and f' are the values of the phase density for the velocities c_i and c_i' respectively.

The components x_i, c_i and $\dot{c}_i$ of position, velocity and acceleration respectively are referred to coordinate axes in a non-inertial observer frame. The corresponding quantities referred to coordinate axes in an inertial frame are denoted by x_i^*, c_i^* and $\dot{c}_i^*$ and we have the transformation formulae

$$\begin{aligned} x_i &= O_{ij}(t)\, x_j^* + b_i(t), \\ c_i &= O_{ij} c_j^* + W_{ij}(x_j - b_j) + \dot{b}_i, \\ \dot{c}_i &= O_{ij} \dot{c}_j^* + 2W_{ij}(c_j - \dot{b}_j) - W_{ik}^2(x_k - b_k) + \dot{W}_{ik}(x_k - b_k) + \ddot{b}_i, \end{aligned} \tag{2.3}$$

where $O_{ij}(t)$ is a time dependent proper orthogonal matrix and $b_i(t)$ is a time dependent vector. W_{ij} is defined by $\dot{O}_{ik} O_{jk}$ and represents the antisymmetric matrix of angular velocity of the non-inertial observer frame with respect to an inertial frame.

If in the inertial frame we have an electro-magnetic field with components E_i^*, B_i^*, the electrons of velocity c_i^* are subject to the Lorentz force $-e(E_i^* + \varepsilon_{ijk} c_j^* B_k^*)$, and their acceleration is $\dot{c}_i^* = -\frac{e}{m}(E_i^* + \varepsilon_{ijk} c_j^* B_k^*)$. Insertion of this value into $(2.3)_3$ leads to the expression

$$\begin{aligned} \dot{c}_i = -\frac{e}{m}(E_i + \varepsilon_{ijk} c_j B_k) + 2W_{ij}(c_j - \dot{b}_j) - W_{ik}^2(x_k - b_k) \\ + \dot{W}_{ik}(x_k - b_k) + \ddot{b}_i \end{aligned} \tag{2.4}$$

for the acceleration of the electrons in the non-inertial frame. In Eq. (2.4) E_i and B_i denote the components of the electro-magnetic field in the non-inertial frame and use has been made of the fact that the components of the Lorentz force $-e(E_n^* + \varepsilon_{njk} c_j^* B_k^*)$ transform into

$$-e(E_i + \varepsilon_{ijk} c_j B_k) = -O_{in} e(E_n^* + \varepsilon_{njk} c_j^* B_k^*)$$

under the transformation $(2.3)_1$ (e.g. see [7], p. 672).

We denote the velocity-independent part of the acceleration in (2.4) by k_i and write

$$\dot{c}_i = k_i + \left(2W_{ij} - \frac{e}{m} \varepsilon_{ijk} B_k\right) c_j. \tag{2.5}$$

[1] E.g. see [6], p. 311 ff. As far as convenient I follow [6] in notation and argument.

Thus we have for the collision equation

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} + \left[k_i + \left(2W_{ij} - \frac{e}{m}\,\varepsilon_{ijk}B_k\right)c_j\right]\frac{\partial f}{\partial c_i} = \frac{1}{\pi l}\int_0^{2\pi}\int_0^{\pi} c(f' - f)\cos\theta\sin\theta\, d\theta\, d\varepsilon, \tag{2.6}$$

where k_i is given by the equation

$$k_i = -\frac{e}{m}E_i - 2W_{ij}\dot{b}_j - W_{ij}^2(x_j - b_j) + \dot{W}_{ik}(x_k - b_k) + \ddot{b}_i. \tag{2.7}$$

b) *Moments of the Distribution Function*

From the definition of the phase density it follows that

$$\varrho \equiv \int mf\, d\boldsymbol{c} \tag{2.8}$$

is the density of the electron gas; the integration extends over all velocity components from $-\infty$ to ∞. The quantities

$$\varrho v_i \equiv \int mc_i f\, d\boldsymbol{c}, \tag{2.9}$$

$$P_{ij} \equiv \int mc_i c_j f\, d\boldsymbol{c}, \tag{2.10}$$

$$q_i \equiv \int \frac{m}{2} c^2 c_i f\, d\boldsymbol{c}, \tag{2.11}$$

are the mass flux, momentum flux and energy flux respectively due to the motion of the electrons. Also the electric current carried by the electrons is given

$$J_i = -\frac{e}{m}\varrho v_i = -\int ec_i f\, d\boldsymbol{c}. \tag{2.12}$$

Since the lattice ions are supposed to be at rest according to the model exploited here, they do not contribute to the electric current nor to the flux of energy and therefore q_i and J_i as defined in (2.11) and (2.12) represent the energy flux and the electric current respectively in the metal. The objective of this paper is the calculation of approximate expressions for q_i and J_i.

3. Sommerfeld's Iteration for the Phase Density

a) *Equilibrium Phase Densities*

The equilibrium phase density of the electrons is known from statistical mechanics of systems of identical Fermions:

$$f_0(c) = \frac{2m^3}{h^3}\,\frac{1}{\frac{1}{\zeta}\,e^{\frac{mc^2}{2kT}} + 1}, \tag{3.1}$$

where h is Planck's constant, T is the temperature and ζ is a function of ϱ and T whose form follows from the requirement

$$\varrho = \frac{2m^3}{h^3} \int \frac{m}{\frac{1}{\zeta} e^{\frac{mc^2}{2kT}} + 1} \, d\mathbf{c}. \tag{3.2}$$

While the phase density f_0 makes the collision production on the right hand side of the collision equation (2.6) vanish, it does not necessarily satisfy the collision equation. This will only be the case, if the fields ϱ and T, or ζ and T satisfy the conditions

$$\frac{\partial \zeta}{\partial t} = O, \quad \frac{\partial T}{\partial t} = O, \quad \frac{kT}{m} \frac{\partial \ln \zeta}{\partial x_i} = k_i, \quad \frac{\partial T}{\partial x_i} = O. \tag{3.3}$$

Therefore, in equilibrium density and temperature must be time-independent and the temperature field must be uniform, while a density gradient may exist when it is balanced by a conservative field k_i. In particular, however, if E_i is chosen so as to make k_i in (2.7) vanish, the density field must also be uniform in equilibrium.

b) Approximate Phase Density in Non-Equilibrium

When the fields of ζ and T do not conform to the conditions (3.3), the collision equation cannot have the solution (3.1). For that case, following Sommerfeld I lay down the expansion

$$f = f_0 + U_k c_k + U_{kl} c_k c_l + U_{klm} c_k c_l c_m + \cdots \tag{3.4}$$

for the solution of the collision equation. Here f_0 is the phase density of "local equilibrium", i.e., it has the form (3.1) but with ζ and T being unrestricted functions of x_n and t; the coefficients U_k, U_{kl}, U_{klm}, ... in (3.4) form symmetric and traceless tensor functions of x_n, t and the magnitude c of the velocity. These functions are determined by an iterative scheme of which I describe the first two steps:

First Step: The left hand side of the collision equation (2.6) is calculated by use of the term f_0 in (3.4), while its right hand side is calculated by use of the two terms $f_0 + U_k c_k$ in (3.4). An easy calculation shows that

$$\frac{1}{\pi l} \int_0^{2\pi} \int_0^{\pi} c(c_k' - c_k) \cos\theta \sin\theta \, d\theta \, d\varepsilon = -\frac{1}{l} c c_k \tag{3.5}$$

and therefore this first step gives an expression for $U_k c_k$ in terms of first derivatives of the fields ζ and T.

Second Step: The left hand side of the collision equation (2.6) is calculated by use of the two terms $f_0 + U_k c_k$ — the latter of which has been calculated in the first step — while its right hand side is calculated by use of the three terms $f_0 + U_k c_k + U_{kl} c_k c_l$. One shows easily that

$$\frac{1}{\pi l} \int_0^{2\pi} \int_0^{\pi} c(c_k' c_l' - c_k c_l) \cos\theta \sin\theta \, d\theta \, d\varepsilon = \frac{2}{l} c \left(c_k c_l - \frac{1}{3} c^2 \delta_{kl} \right) \tag{3.6}$$

and therefore the second step gives an expression for $U_{kl} c_k c_l$ in terms of first and second derivatives of the fields ζ and T.

Further steps follow the same prescription and some reflection makes it clear that this iterative scheme provides us with an expansion of f into terms of increasing powers of the mean free path l.

With the prescription thus given the actual calculation of $U_k c_k$ and $U_{kl} c_k c_l$ is a straightforward matter even though it is somewhat tedious. Therefore, I restrict the attention to the case where the phase density is time-independent and even in that case I only list the result: With the definitions

$$A_n \equiv \frac{\left(\frac{1}{\zeta} e^{\frac{mc^2}{2kT}} \right)^n}{\left(\frac{1}{\zeta} e^{\frac{mc^2}{2kT}} + 1 \right)^{n+1}} \quad \text{and} \quad F_i = \frac{kT}{m} \frac{\partial \ln \zeta}{\partial x_i} - k_i , \tag{3.7}$$

and if one ignores terms of third order in the mean free path l, the phase density comes out as

$$\begin{aligned} f = \frac{2m^3}{h^3} \Bigg\{ & A_0 - l A_1 \frac{c_k}{c} \frac{m}{kT} \left(F_k + \frac{c^2}{2} \frac{\partial \ln T}{\partial x_k} \right) \\ & - \frac{1}{2} l^2 \Bigg[(A_1 - 2A_2) \frac{c_k c_l}{c^2} \frac{m^2}{k^2 T^2} \left(F_k + \frac{c^2}{2} \frac{\partial \ln T}{\partial x_k} \right) \left(F_l + \frac{c^2}{2} \frac{\partial \ln T}{\partial x_l} \right) \\ & \qquad - A_1 \frac{c_k c_l}{c^2} \frac{m}{kT} \left\{ \frac{\partial F_k}{\partial x_l} - F_k \frac{\partial \ln T}{\partial x_l} - \frac{c^2}{2} \left(\frac{\partial \ln T}{\partial x_k} \frac{\partial \ln T}{\partial x_l} - \frac{\partial^2 \ln T}{\partial x_k \, \partial x_l} \right) \right\} \\ & \qquad + A_1 \frac{c_k c_l}{c^4} \frac{m}{kT} k_k \left(F_l - \frac{c^2}{2} \frac{\partial \ln T}{\partial x_l} \right) \\ & \qquad - A_1 \frac{1}{c^2} \frac{m}{kT} k_k \left(F_k + \frac{c^2}{2} \frac{\partial \ln T}{\partial x_k} \right) \\ & \qquad + A_1 \frac{c_l}{c^2} \frac{m}{kT} \left(2 W_{lk} - \frac{e}{m} \varepsilon_{lkn} B_n \right) \left(F_k + \frac{c^2}{2} \frac{\partial \ln T}{\partial x_k} \right) \Bigg] \Bigg\} . \end{aligned} \tag{3.8}$$

This expression I consider as a sufficiently good approximation of the phase density for the purpose at hand which is the calculation of the electric current J_i and of the flux of energy q_i.

4. The Laws of Ohm and Fourier for Electrical Conduction and Heat Conduction

a) *The Electric Current and the Flux of Energy*

The phase density (3.8) is now introduced into the integrands of the integrals (2.12) and (2.11) which define the electric current and the flux of energy respectively. Inspection shows that only the second and the last term on the right hand side of (3.8) contribute to J_i and q_i and we obtain

$$J_i = e\,\frac{2m^3}{h^3}\,l\,\frac{4\pi}{3}\,\frac{m}{kT}\int_0^\infty \left[\delta_{ik} + \frac{l}{c}\left(W_{ik} - \frac{e}{2m}\,\varepsilon_{ikn}B_n\right)\right] c^3 A_1 \left(F_k + \frac{c^2}{2}\,\frac{\partial \ln T}{\partial x_k}\right) dc\,,$$

$$q_i = -\frac{m}{2}\,\frac{2m^3}{h^3}\,l\,\frac{4\pi}{3}\,\frac{m}{kT}\int_0^\infty \left[\delta_{ik} + \frac{l}{c}\left(W_{ik} - \frac{e}{2m}\,\varepsilon_{ikn}B_n\right)\right] c^5 A_1 \left(F_k + \frac{c^2}{2}\,\frac{\partial \ln T}{\partial x_k}\right) dc\,. \tag{4.1}$$

With the definition

$$I_n = \int_0^\infty c^n A_1\, dc \qquad (n = 2, 3, \ldots, 7) \tag{4.2}$$

we may rewrite the Eqs. (4.1) in the forms

$$J_i = e\,\frac{2m^3}{h^3}\,l\,\frac{4\pi}{3}\,\frac{m}{kT}\left[\left\{I_3\delta_{ik} + I_2 l\left(W_{ik} - \frac{e}{2m}\,\varepsilon_{ikn}B_n\right)\right\} F_k + \frac{1}{2}\left\{I_5\delta_{ik} + I_4 l\left(W_{ik} - \frac{e}{2m}\,\varepsilon_{ikn}B_n\right)\right\}\frac{\partial \ln T}{\partial x_k}\right],$$

$$q_i = -\frac{m}{2}\,\frac{2m^3}{h^3}\,l\,\frac{4\pi}{3}\,\frac{m}{kT}\left[\left\{I_5\delta_{ik} + I_4 l\left(W_{ik} - \frac{e}{2m}\,\varepsilon_{ikn}B_n\right)\right\} F_k + \frac{1}{2}\left\{I_7\delta_{ik} + I_6 l\left(W_{ik} - \frac{e}{2m}\,\varepsilon_{ikn}B_n\right)\right\}\frac{\partial \ln T}{\partial x_k}\right]. \tag{4.3}$$

b) *Ohm's Law*

Let us consider a metal in which the temperature and the electron density are uniform and which is subject to an electric field. In such a situation we have

$$F_k \equiv \frac{e}{m}\,E_k' = \frac{e}{m}\,E_k + 2W_{ij}\dot{b}_j - W_{ij}^2(x_j - b_j) - \dot{W}_{ij}(x_j - b_j) + \ddot{b}_i$$

and Eq. $(4.3)_1$ reduces to Ohm's law of stationary electrical conduction[2]

$$J_i = \frac{8\pi}{3}\,\frac{m^3e^2}{h^3}\,\frac{l}{kT}\left\{I_3\delta_{ik} - I_2\,\frac{e}{2m}\,l\left(\varepsilon_{ikn}B_n - 2\,\frac{m}{e}\,W_{ik}\right)\right\} E_k'\,. \tag{4.4}$$

The factor of E_i' on the right hand side of (4.4) is called the electric conductivity which we denote by σ, and the factor of $(\boldsymbol{B}\times\boldsymbol{E}')$ is called the Hall coefficient which we denote by S:

$$\sigma \equiv \frac{8\pi}{3}\,\frac{m^3e^2}{h^3}\,\frac{l}{kT}\,I_3\,, \qquad S = -\frac{4\pi}{3}\,\frac{m^2e^3}{h^3}\,\frac{l^2}{kT}\,I_2\,. \tag{4.5}$$

---

[2] E_k' is the part of the total electric field E_k which creates a current, the rest of E_k merely counterbalances the force $2W_{ij}\dot{b}_j - W_{ij}^2(x_j - b_j) - \dot{W}_{ij}(x_j - b_j) + \ddot{b}$; so that the density of electrons is kept uniform.

The whole matrix which relates J_i to E_k' is called the tensor of electric conductivity. While this tensor is often considered to be a constitutive quantity, inspection of (4.4) shows that it depends on frame through its dependence on the angular velocity matrix W_{ik}. To be sure though, the dependence on frame will generally be negligible because of the large value of the specific charge of electrons.

c) *Fourier's Law*

Elimination of F_k between the two Eqs. (4.3) leads to an expression for the flux of energy in terms of the electric current and the temperature gradient, viz.

$$\begin{aligned} q_i = &-\frac{1}{2}\frac{m}{e}\frac{I_5}{I_3}\left[\delta_{ik} + \left(\frac{I_4}{I_5} - \frac{I_2}{I_3}\right) l \left(W_{ik} - \frac{e}{2m}\,\varepsilon_{ikm}B_m\right)\right] J_k \\ &- \frac{m}{2}\frac{2m^3}{h^3}\, l\, \frac{4\pi}{3}\frac{m}{2kT}\left[\left(I_7 - \frac{I_5^2}{I_3}\right)\delta_{ik}\right. \\ &\left. + \left(I_6 - \frac{2I_4I_5}{I_3} + \frac{I_2I_5^2}{I_3^2}\right) l \left(W_{ik} - \frac{e}{2m}\,\varepsilon_{ikn}B_n\right)\right]\frac{\partial \ln T}{\partial x_k}. \end{aligned} \tag{4.6}$$
[3]

In particular, when there is no electric current the Eq. (4.6) reduces to Fourier's law of heat conduction which relates the energy flux q_i to the temperature gradient:

$$\begin{aligned} q_i = &-\frac{2\pi}{3}\frac{m^5}{h_3}\frac{l}{kT^2}\left[\left(I_7 - \frac{I_5^2}{I_3}\right)\delta_{ik}\right. \\ &\left. - \left(I_6 - \frac{2I_4I_5}{I_3} + \frac{I_2I_5^2}{I_3^2}\right)\frac{e}{2m}\, l\left(\varepsilon_{ikn}B_n - 2\frac{m}{e}W_{ik}\right)\right]\frac{\partial T}{\partial x_k}. \end{aligned} \tag{4.7}$$

The factor of $\frac{\partial T}{\partial x_i}$ on the right hand side of (4.7) is the negative heat conductivity which we denote by $\varkappa$. The factor of $(\boldsymbol{B} \times \operatorname{grad} T)$ governs the effect of the magnetic field on the flux of energy, and we denote it by K:

$$\varkappa = \frac{2\pi}{3}\frac{m^5}{h^3}\frac{l}{kT^2}\left(I_7 - \frac{I_5^2}{I_3}\right), \quad K = -\frac{\pi}{3}\frac{m^4 e}{h_3}\frac{l^2}{kT^2}\left(I_6 - \frac{2I_4I_5}{I_3} + \frac{I_2I_5^2}{I_3^2}\right). \tag{4.8}$$

The whole matrix which relates q_i to $-\frac{\partial T}{\partial x_k}$ is called the tensor of heat conductivity and we conclude from (4.7) that this tensor is frame dependent, because it depends on the angular velocity matrix W_{ik} of the frame.

Therefore, the kinetic theory of metal electrons does not support the view of continuum thermodynamics according to which the flux of energy q_i is related to the field of temperature in a manner solely dependent on material.

---

[3] In the derivation of (4.6) from (4.3) terms of higher than second order in the mean free path were neglected for consistency with the approximation (3.8) of the phase density.

5. A Suggestive Interpretation of the Frame Dependence and of the Magnetic Field Dependence of Current and Energy Flux

a) Currents and Energy Fluxes under Lorentz- and Coriolis Forces

A well-known argument for the visualization of the Hall effect runs as follows: We consider first a metal plate as drawn in Fig. 1 which is subject to an electric field in the direction indicated there and we focus the attention to a small volume element whose linear dimensions are of the order of magnitude of a mean free path of the electrons. A blow-up of this element is shown in Fig. 2 and 3. Fig. 2

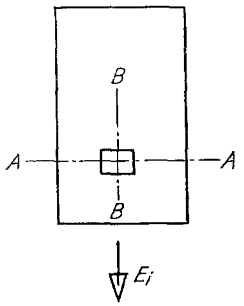

Fig. 1

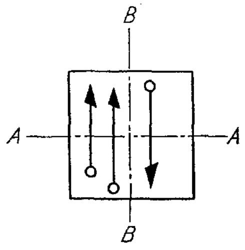

Fig. 2

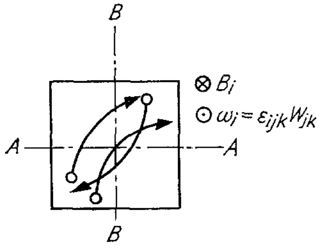

Fig. 3

is appropriate to the case when there is no magnetic field and the metal is at rest in an inertial frame; it shows schematically the paths of some electrons between collisions and these paths are straight lines. More electrons move upward than downward because of the electric field and therefore we have a net charge transport, or a current across the plane $A-A$ but no current across the plane $B-B$; this illustrates the first term on the right hand side of Eq. (4.4). Fig. 3 shows the paths of the same electrons, but now the metal rotates with respect to an inertial frame in the plane of the plate, or is subject to a perpendicular magnetic field, or both. The electron paths are curved under the influence of the Lorentz force and the Coriolis force and as a result there is a charge transport, or current, across the plane $B-B$, i.e. perpendicular to the electric field. This argument offers a suggestive interpretation of the Hall current which is represented by the second term on the right hand side of Eq. (4.4).

Let us now leave the case of electrical conduction and consider heat conduction instead. Fig. 4 shows the same metal plate as Fig. 1; however, now there is no electric field but a temperature gradient in the indicated direction and the mean velocity v_i of the electrons is zero. Fig. 5 shows a blow-up of a little element of the plate and straight electron paths appropriate to the case when the plate is at rest in an inertial frame and when there is no magnetic field. Now the numbers of electrons going up or down through the plane $A-A$ is equal but the upward bound electrons carry a higher energy in the mean than the downward bound electrons. There results a net flux of energy across the plane $A-A$, but no flux of energy across the plane $B-B$, and indeed by (4.7) we expect q_i to be parallel to $\frac{\partial T}{\partial x_i}$ in this case. Fig. 6 shows the same situation when either a magnetic field is present or the plate rotates or both. Now there is also a flux of energy across the plane B—B which in (4.7) is represented by the second term on the right hand side.

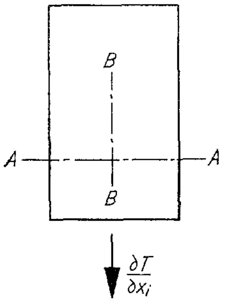

Fig. 4

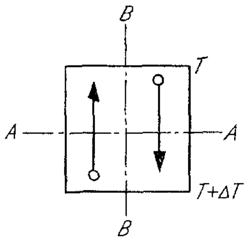

Fig. 5

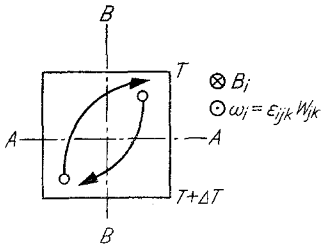

Fig. 6

We note that the current perpendicular to the electric field or the energy flux perpendicular to the temperature gradient can either be created by a magnetic field or by the rotation of the frame. In general of course both effects may be present and one may cancel the other, although it takes an extremely high angular velocity to offset the influence of a magnetic field for which the Hall effect is observable.

b) The Law of Wiedemann-Franz

The coefficients σ and S in Ohm's law (4.4) and the coefficients $\varkappa$ and K in Fourier's law (4.7) can be made more explicit by evaluation of the integrals I_n which were defined in (4.2).

The evaluation of the integrals I_n is facilitated by the fact that the electron gas in a metal represents a strongly degenerate Fermi system. Under the assumption of complete degeneration — where $\zeta = \zeta^0 \gg 1$ — one can easily prove that[4]

$$I_n^c = \frac{1}{m}\left(\frac{2}{m}\right)^{\frac{n-1}{2}} (kT)^{\frac{n+1}{2}} (\ln \zeta^0)^{\frac{n-1}{2}}, \quad \text{where} \quad \ln \zeta^0 = \frac{3^{2/3}h^2}{8\pi^{2/3}m^{5/3}} \frac{\varrho^{2/3}}{kT} \tag{5.1}$$

and the index c on I_n denotes complete degeneration.

Thus according to (4.5) the electric conductivity and the Hall coefficient come out as

$$\sigma = \frac{16\pi}{3}\frac{me^2}{h^3} l(kT \ln \zeta^0), \qquad S = -\frac{4\pi}{3}\frac{me^3}{h^3}\sqrt{\frac{2}{m}}\, l^2 (kT \ln \zeta^0)^{1/2}. \tag{5.2}$$

Insertion of (5.1) into the coefficients $\varkappa$ and K in (4.8) shows that both these coefficients vanish in a completely degenerate gas so that there can be no flux

[4] E.g., see [6], p. 262ff. While there is no actual calculation of I_n in [6], the method of determining similar integrals of this type in the case of a completely degenerate and a strongly degenerate gas is explained there.

of energy in that case. For the calculation of these coefficients it is therefore necessary that we take account of the fact that, while the electron gas is strongly degenerate, the degeneration is not complete. For the strongly degenerate gas one can expand $\frac{I_n}{I_n^c}$ in a series of increasing powers of $\frac{1}{\ln \zeta^0}$ of which we only need the first two non-vanishing terms; these read[5]

$$I_n = I_n^c \left(1 + [(n-1)(n-3) - (n-1)] \frac{\pi^2}{24} \frac{1}{(\ln \zeta^0)^2}\right). \tag{5.3}$$

Thus we obtain for $\varkappa$ and K according to (4.8):

$$\varkappa = \frac{16\pi^3}{9} \frac{mk^2}{h^3} lT(kT \ln \zeta^0), \qquad K = -\frac{4\pi^2}{9} \frac{mk^2e^3}{h^3} l^2T(kT \ln \zeta^0)^{1/2}. \tag{5.4}$$

Comparison of $(5.4)_1$ and $(5.2)_2$ shows that

$$\frac{\varkappa}{\sigma} = \frac{\pi^2}{3} \frac{k^2}{e^2} T \tag{5.5}$$

so that the ratio of the heat conductivity to the electric conductivity is proportional to the absolute temperature with a universal factor of proportionality. This result is known as the law of Wiedemann-Franz and its explanation was a major success for the kinetic theory of metal electrons. It may be noteworthy that the coefficient K in the flux of energy and the Hall coefficient S in the electric current have the same ratio, i.e.

$$\frac{K}{S} = \frac{\pi^2}{3} \frac{k^2}{e^2} T$$

according to $(5.4)_2$ and $(5.2)_2$.

Thus we conclude that the frame dependence of the flux of energy is governed by the Hall coefficient.

Acknowledgement

This work was done while I was supported in part by a grant from the National Science Foundation to the Johns Hopkins University in Baltimore, Md. USA.

References

[1] Chapman, S., and T. G. Cowling: The Mathematical Theory of Non-Uniform Gases. Cambridge University Press. 1961.

[2] Truesdell, C.: On the Differential Equations of Slip Flow. Proceeding of the National Academy of Science **34**, 342 (1948).

[3] Müller, I.: On the Frame Dependence of Stress and Heat Flux. Arch. Rat. Mech. Anal. **45**, 241 (1972).

[4] Edelen, D. G. B., and T. A. McLennan: Material Indifference: A Principle or a Convenience, Int. J. Engng. Sci. **11**, 813 (1973).

[5] E.g., see again [6], p. 265ff.

[5] WANG, C. C.: On the Concept of Frame Indifference in Continuum Mechanics and in the Kinetic Theory of Gases (to be published).
[6] SOMMERFELD, A.: Vorlesungen über Theoretische Physik, Bd. V, Thermodynamik und Statistik. Wiesbaden: Dietrichsche Verlagsbuchhandlung. 1952.
[7] TRUESDELL, C., and R. TOUPIN: The Classical Field Theories (Handbuch der Physik III/1). Berlin—Göttingen—Heidelberg: Springer. 1960.

Prof. Ingo Müller
Institut für theoretische Physik
Universität Düsseldorf
Universitätsstraße 1
D-4000 Düsseldorf
Federal Republic of Germany