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A model of non-equilibrium turbulence with an asymmetric stress. Application to the problems of thermal convection

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A critical review of hydrodynamical models with asymmetric stress tensor is given. Particular attention is focused on the balance law of angular momentum as the necessary element for a correct description of the internal motions of turbulent oriented eddies. On the basis of this analysis a non-equilibrium turbulence model is proposed that is shown to be close to the hydrodynamic equations with intrinsic rotation and helical turbulence.

We employ this model in the study of the initial stage of thermal convection in a horizontal layer of a rotating non-equilibrium turbulent fluid that is heated from below. Linearizing the balance equations of mass, momentum, angular momentum and energy yields the boundary value problem, from which the general properties of the spectrum are determined. In the case of the horizontal layer with equilibrium boundary conditions on free boundaries we study the influence of the rotation and turbulent motion on the convective instability.

1 Introduction

The study of the formation and evolution of large-scale structures in rotating and/or shear turbulent flows [1]–[8] is of great interest both from fundamental as well as applied points of view. Extensive analytical and numerical investigations were devoted to the helical large-scale structures supported by small-scale turbulent motions [3]–[8]. The description of such structures is based on the helicity, defined as the pseudo-scalar $\langle \mathbf{v} \cdot (\nabla \times \mathbf{v}) \rangle$. When the helicity is non-zero the flow does not possess reflectional symmetry. The Reynolds stresses corresponding to such flows are not symmetric. Since new interesting effects are associated with such asymmetric Reynolds terms it is natural to pay attention to asymmetric continuum models.

There exists a generalization of classical continuum mechanics, the asymmetric mechanics [9]–[11] often called the theory of Cosserat continua, 1909 –

it was first introduced into the theory of elasticity. To our knowledge, asymmetric stress tensors in fluid dynamics were first introduced in Ref. [12]. That paper contains the criticism of (i) Newton's fundamental hypothesis on internal friction in fluids and gases and (ii) of the Navier-Stokes equations, which are obtained with the help of the mentioned hypothesis. According to Newton's hypothesis, the fluid internal friction is determined by the *defectus lubricatus* and proportional to the first derivatives of the velocity with respect to the spatial coordinates. In a strict sense, such a representation is valid only for fluids consisting of particles that may be considered as material points. If one considers the fluid particles as objects with finite sizes which can rotate, then it is possible to obtain more general expressions for the internal viscous forces by taking into account in addition to the "usual" internal viscosity, the particle rotation associated with the fluid deformation [12]. In [13] the aforementioned fluid particles are called "subelemental" particles, which "may be atoms or molecules, or collections of these, or some larger aggregates, as is the case for turbulent eddies". On the basis of these ideas [12] and by formulating the balance laws of mass, momentum, angular momentum and energy in [14] more accurate forms of the equations for the hydrodynamics of a fluid with an intrinsic rotation are represented.

Perhaps, the first attempt to describe a turbulent fluid as a medium with a vortical structure on the balance laws of momenta was made in [15], however this idea remained unnoticed during many years. A similar description of turbulent flows was stimulated by the development of asymmetric hydrodynamics [9]–[11]. For example, in [16] a turbulent fluid is considered as a subclass of micropolar fluids, but this description does not take into account any change of vortices during their motion.

Nikolaevskii in [17], [18] takes an intermediate point of view between the approaches taken in [15], [16] and the traditional models. In order to lay asymmetric hydrodynamics of turbulent flows on a solid foundation the author proposes to consider a turbulent fluid as a medium with a heterogeneous distribution of hydrodynamic fields and to let all quantities that are described by general balance laws be mean values averaged over some elementary macrovolume. Such fluid elements are small compared to the significant spatial scales of the problem but sufficiently large to contain the turbulent eddy microstructure. The asymmetric stress effects enter through the employed averaging procedure. They are not present in the Euler or Navier-Stokes equations from which the derivation begins; however, these equations are regarded as sufficient for the description of the fluid flow at any point on the microvolume scale. According to [17], [18], the very choice of the averaging procedure may contain a tacit assumption about the stress asymmetry beyond that already implicitly contained in the Navier-Stokes equations. The equations deduced in [17], [18] are formally identical to the conventional ones, however with the Reynolds stress defined as a surface integral $\tau_{ij} = -\langle \varrho v_i' v_j' \rangle$ where $\mathbf{v}'$ denotes an average of the velocity over the surface normal to the j-axis. This Reynolds stress is in general not symmetric, and the angular momentum intrinsic to the volume element is non-zero. These considerations are represented in the review [13] along with the examples of other flows with significant orientational effects (liquid crystals, suspensions).

Besides, the author of [13] alludes to the fact that "Reynolds in his original papers drew a careful distinction between spatial and temporal averages, and recognized the possibility that the stress tensor might be asymmetric".

A thermodynamical theory of turbulence is proposed in [19]–[21]; it also describes the anisotropy of turbulent flows by means of an asymmetric stress tensor. In contrast to the usual description of a continuum the authors of [19] introduce two independent kinematic fields: the ordinary particle velocity $\mathbf{v}$ and the director velocity $\mathbf{w}$. This means that they consider the fluid continuum as a directed medium with each of its material points being endowed with an additional independent kinematical vector field, a director $\mathbf{d}$ having the physical dimension of a length. In asymmetric continuum mechanics the fields of internal angular momenta or internal angular velocities are introduced.

Multiplying $(2.6)_2$ in [19] by $\mathbf{x}$ and $(2.6)_3$ by $\mathbf{d}$ and subtracting the resulting equations from $(2.6)_4$ yields the equality

$$\int_P \mathbf{d} \times \mathbf{k}\, dv = 0,$$

where $\mathbf{k}$ is the intrinsic director force per unit volume, dv is a volume element and P is any part of a directed medium under consideration. Since the volume P is arbitrary, the above relationship says that the intrinsic director force $\mathbf{k}$ is either parallel to the director $\mathbf{d}$ or $\mathbf{k} \equiv \mathbf{0}$. Thus, equations (2.6) in [19] are not linearly independent, and the independent invariants whose number equals the number of independent governing equations are mass, momentum, angular momentum and energy, as in asymmetric mechanics.

The internal angular momentum $\mathbf{M}$ of the present paper corresponds to "the moment of momentum" of [19]–[21], i. e., $\mathbf{d} \times (y_1\mathbf{v} + y_2\mathbf{w})$, where $y_1\mathbf{v} + y_2\mathbf{w}$, is the director momentum per unit mass; the coefficients y_1, y_2 depend on the position, and for their determination it is necessary to introduce constitutive relations. The function $\mathbf{M}$, having the meaning of an effective intrinsic internal angular momentum of turbulent oriented eddies in every point of the continuum, is treated as the independent variable which can be determined by the conservation law alone. As will be shown later on, the scale parameter of oriented turbulent eddies is defined by the usual and topological internal invariant of turbulence, i. e.,

$$\kappa^{-1} \sim \int |\mathbf{M}|^2\, dv \Big/ \int \left(\mathbf{M}^* \cdot (\nabla \times \mathbf{u}) - \mathbf{M} \cdot (\nabla \times \mathbf{u}^*)\right)\, dv.$$

This is the ratio of the energy associated with the intrinsic motion of the oriented eddies to the part of their energy linked to the vortical field of the large-scale motion, the latter corresponding to the typical spatial scales that are much larger than the maximum turbulent scales. In our model the quantity κ is a free parameter.

The internal structure of the continuum is manifest through the formation of turbulent oriented eddies due to the mentioned "link" mechanism. Along with the term "oriented eddy" we shall use the term "mole" to emphasize the physical reality of the oriented eddies and distinguish them from the mathematical notion "vortex". Furthermore, we shall employ the term "non-equilibrium turbulence"; by this we mean turbulent states of fluid elements in which intrinsic angular velocities of moles deviate from local angular velocities of the corresponding mean flow. At such conditions the Reynolds stress tensor is not symmetric. The corresponding mathematical structure of the theory is that of asymmetric mechanics. In our model, we introduce two new coefficients, a rotational viscosity and a diffusivity of the diffusion of oriented eddies, that arise in the equation for angular momenta of such eddies.

We have no intention to belittle the importance of the conventional descriptions of turbulence that are based on the Navier-Stokes equations and are successfully applied to problems of fluid dynamics; however we would like to make two remarks: First, the turbulence models as noted in [22] need not only be Galilean invariant; the Reynolds stresses and the higher turbulence correlations based on an ensemble average are frame indifferent tensor relations, i. e., they are form invariant under time dependent rotations and translations of the spatial frame of reference, i. e. the group $\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x}+\mathbf{b}(t)$, $t^* = t+c$, where c is any constant, $\mathbf{b}(t)$ is any time-dependent vector, and $\mathbf{Q}(t)$ is any time-dependent proper orthogonal tensor; the Reynolds stress transport equations are only form invariant under the group of translations, i. e. $x_k^* = A_{ki}x_i + b_k(t)$ where A_{ki} is any *constant* proper orthogonal tensor.

Second, according to Noether's theorem [23], the conservation law of angular momentum is the consequence of the invariance of continuous transformations in space-time under the group of spatial rotations; classically, this law is identically satisfied, however, if one includes an additional parameter such as e. g. internal angular momentum associated with oriented eddies it is necessary to include this balance law as an independent statement.

In section 2 we describe the method for obtaining the governing equations of our model based on a generalization of the Boussinesq hypothesis to the turbulent stress-mean velocity field relationship. In section 3 the equations of thermal convection for a horizontal layer of a rotating non-equilibrium turbulent fluid are obtained. These equations are used to determine the general properties of the disturbance spectrum. In section 4 the boundary-value problem for the linear form of the mentioned equations is solved. Finally, we represent the solution for laminar fluids.

2 Description of the non-equilibrium turbulent model

Consider a shear flow of a turbulent fluid containing oriented eddies (moles) with a typical spatial scale $\lambda \leq \delta$, where δ is the spatial scale of the shear flow. It is not difficult to show that the conventional description of turbulence based on the Reynolds averaging procedure of the Navier-Stokes equations does not take into account the internal (intrinsic) motions of these moles. In fact, the averaging time period T for a mean function of a turbulent field, for example, for the velocity,

$$\langle \mathbf{v}(\mathbf{r}, t)\rangle = \frac{1}{T}\int_{t-T/2}^{t+T/2} \mathbf{v}(\mathbf{r}, t)\, dt$$

must satisfy the following conditions: the period T is (i) long as compared to typical temporal scales of turbulent fluctuations and (ii) short as compared to a typical time interval of the essential change of the mean flow characteristics. The latter can be evaluated as $\delta/\langle v\rangle$, where $\langle v\rangle$ is the transverse component of the mean shear flow velocity. Thus, a typical spatial scale of fluctuations is small with respect to δ, and the behavior of eddies having the typical sizes $\lambda \leq \delta$ is ignored.

If one takes into account the intrinsic motions of the moles, then the Reynolds stress will have an asymetric form. This can be shown as follows. Consider a fluid volume element $\Delta V = \Delta x \Delta y \Delta z$ with linear size $\lambda \leq \delta$. If this volume element rotates about the y-axis, then

$$\Delta I_y \dot{\omega}_y = (\tau_{xz}\Delta y \Delta z)\Delta x - (\tau_{zx}\Delta x \Delta y)\Delta z = (\tau_{xz} - \tau_{zx})\Delta V,$$

where $\dot{\boldsymbol{\omega}} = (\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z)$ is the angular acceleration of the considered fluid volume, and ΔI_y is its moment of inertia with respect to the y-axis. Since $\Delta I \sim \lambda^5$, $\Delta V \sim \lambda^3$, we have $\tau_{xz} - \tau_{zx} \sim \dot{\omega}_y \lambda^2 \neq 0$, because λ is some finite quantity. Due to the orientational properties of large-scale moles [13], the differences of mean tangential Reynolds stresses will also be non-zero, i.e., $\langle v_i' v_j'\rangle - \langle v_j' v_i'\rangle \neq 0$, for $i \neq j$.

The Reynolds stress tensor can be represented as the sum of symmetric and asymmetric parts, viz.,

$$\langle v_i' v_j'\rangle = \frac{1}{2}\left(\langle v_i' v_j'\rangle + \langle v_j' v_i'\rangle\right) + \frac{1}{2}\left(\langle v_i' v_j'\rangle - \langle v_j' v_i'\rangle\right).$$

In equilibrium, the angular velocity of the internal motion of moles at any arbitrary point equals the observed angular velocity of the time-averaged flow, $\langle \boldsymbol{\omega}'\rangle = \lambda^{-2}\langle \mathbf{M}\rangle = \langle \boldsymbol{\Omega}\rangle$, where $\langle \boldsymbol{\Omega}\rangle = (1/2)\langle \nabla \times \mathbf{v}\rangle$, and $\langle v_i' v_j'\rangle = \langle v_j' v_i'\rangle$. A deviation from the equilibrium state is obviously the deviation of the difference $\langle \boldsymbol{\Omega}\rangle - \lambda^2\langle \mathbf{M}\rangle$ from zero. Now we must introduce constitutive equations relating the Reynolds stress to mean flow characteristics. The linear relationship between the symmetric part of $\langle v_i' v_j'\rangle$ and the mean strain-rate tensor is known as the Boussinesq hypothesis.

In this paper we introduce the following constitutive equation

$$\langle v_i' v_j'\rangle - \langle v_j' v_i'\rangle = \gamma(\Omega_{ij} - \lambda^{-2} M_{ij}). \tag{2.1}$$

It can be regarded as a generalization of the Boussinesq hypothesis. $\Omega_{ij} = (1/2)\varepsilon_{ijl}\Omega_l$, $M_{ij} = (1/2)\varepsilon_{ijl}M_l$ are the antisymmetric pseudo tensors dual to $\boldsymbol{\Omega}$ and $\mathbf{M}$, ε_{ijl} being the Levi-Cività tensor. The quantity γ plays the role of a rotational viscosity, which later will be considered to be a constant. Making use of the representation (2.1) the momentum equation for a turbulent fluid can be written as

$$\langle\varrho\rangle\frac{D\langle v_i\rangle}{Dt} = -\frac{\partial\langle p\rangle}{\partial x_i} + \mu\frac{\partial^2\langle v_i\rangle}{\partial x_j\partial x_j} + \frac{\partial}{\partial x_j}\left(-\frac{1}{2}\langle\varrho\rangle(\langle v_i'v_j'\rangle + \langle v_j'v_i'\rangle)\right) + \\ + \langle\varrho\rangle\left(\frac{\gamma}{4}\frac{\partial^2\langle v_i\rangle}{\partial x_j\partial x_j} + \frac{\lambda^{-2}\gamma}{2}(\nabla\times\langle\mathbf{M}\rangle)_i\right). \tag{2.2}$$

Here $\langle\varrho\rangle$, $\langle p\rangle$ are the averaged density and pressure, respectively; μ is the dynamic coefficient of molecular viscosity, and the two last terms are caused by the asymmetric parts of $\langle v_i'v_j'\rangle$which are absent in the conventional Reynolds equations.

In the absence of external long-range force fields the function $\mathbf{M}$ changes owing to both the asymmetric part of the stress tensor and a diffusion through the surface of a fluid volume considered

$$\frac{D\langle\mathbf{M}\rangle}{Dt} = \gamma(\langle\mathbf{\Omega}\rangle - \kappa\langle\mathbf{M}\rangle) + \eta\Delta\langle\mathbf{M}\rangle, \tag{2.3}$$

where η is a coefficient of the diffusion of moles.

Equations (2.2), (2.3) can also be obtained by a formal time averaging procedure of the equations of fluid dynamics with internal rotation [14], by analogy with the averaging of the Navier-Stokes equations. According to [14], the internal energy of such a medium equals

$$E = E_0(S,\varrho) + (1/2)\lambda^{-2}M^2 - \mathbf{M}\cdot\mathbf{\Omega},$$

where S is the entropy. The equilibrium value of the internal angular momentum $\mathbf{M}$ can be determined from the condition $\partial E/\partial\mathbf{M} = \mathbf{0}$, thus $\mathbf{M} = \mathbf{\Omega}/\kappa$, and $E_0 = E(S,\varrho) - (1/2)\lambda^{-2}M^2$.

The physical processes in open non-equilibrium systems (non-equilibrium turbulence, in particular) are described by methods of nonlinear thermodynamics [24]. These processes give rise to a change of the type of fluid symmetry, hence to a corresponding change of the number and type of conservation laws. As known, homogeneity and isotropy of small-scale components can be violated. Apart from the ordinary cases characterized by the direct energy cascade there are situations where the inverse energy cascade exists, for example, in large-scale turbulent convection, supported by small-scale helical turbulent motions [7]. The stress tensor of a turbulent fluid can be written as

$$\sigma_{ij} = -A\delta_{ij} + B\pi_{ij}, \tag{2.4}$$

where A is the pressure, B is an "eddy" viscosity, and π_{ij} the viscous stress tensor.

In real turbulent flows (for example, boundary layers, wakes, jets), the role played by large-scale turbulence is very essential. Such turbulent flows are under the influence of pressure gradients, caused by external forces. Therefore, the turbulent moles are exposed to the $\nabla\times\langle\mathbf{v}\rangle$-field, having pseudo-vector properties. From a physical point of view this means, that there exists an internal angular momentum exchange. A new dissipative process and new added terms in the total stress tensor correspond to such an interaction mechanism, e. g. according to (2.2)

$$\sigma_{ij} = -p\delta_{ij} + (\mu + \mu_T)\pi_{ij} + (\gamma/2)(\lambda^{-2}M_{ij} - \Omega_{ij}). \tag{2.5}$$

Here M_{ij} and Ω_{ij} are antisymmetric tensors, duals to the pseudo-vectors of the turbulent moles' angular momenta $M_l = (1/2)\varepsilon_{ijl}M_{ij}$ and the angular velocity of the mean flow $\Omega_l = (1/2)\varepsilon_{ijl}\Omega_{ij}$ respectively; μ_t is a turbulent "shear" viscosity and λ is the mean turbulent scale.

Now, consider the pseudo vectors M_l, Ω_l dual to the tensors M_{ij}, Ω_{ij} and make use of the following transformation rule of the antisymmetric pseudo tensor ε_{ijk}

$$\varepsilon^0_{\alpha\beta\gamma} = \frac{\Delta}{|\Delta|}\varepsilon_{ijl}\frac{\partial y_\alpha}{\partial x_i}\frac{\partial y_\beta}{\partial x_j}\frac{\partial y_\gamma}{\partial x_l},$$

where Δ is the determinant of the matrix $\|\partial y_\alpha/\partial x_i\|$ and $y_\alpha = y_\alpha(x_j)$. By simple algebra one may then deduce the relations

$$\begin{aligned}\frac{\Delta}{|\Delta|}\frac{\partial y_\alpha}{\partial x_i}\frac{\partial y_\beta}{\partial y_j}\frac{\partial y_\gamma}{\partial y_l}\epsilon_{ijl}M_l &= \frac{1}{2}\epsilon_{ijl}\epsilon^0_{\alpha\beta\gamma}M_{ij},\\ \frac{\Delta}{|\Delta|}\frac{\partial y_\alpha}{\partial x_i}\frac{\partial y_\beta}{\partial x_j}\frac{\partial y_\gamma}{\partial x_l}\epsilon_{ijl}\Omega_l &= \frac{1}{2}\epsilon_{ijl}\epsilon^0_{\alpha\beta\gamma}\Omega_{ij}.\end{aligned} \tag{2.6}$$

Let the new coordinate system be such that $\varepsilon_{ijl}\varepsilon^0_{\alpha\beta\gamma} = 1$ (for $i \neq j \neq 1$, $\alpha \neq \beta \neq \gamma$). Then, since $\Delta/|\Delta|$ denotes the sign of Δ the quantity

$$\frac{\Delta}{|\Delta|}\frac{\partial y_\alpha}{\partial x_i}\frac{\partial y_\beta}{\partial x_j}\frac{\partial y_\gamma}{\partial x_l}\gamma = d, \tag{2.7}$$

is a pseudo-scalar function. Substituting (2.6) and (2.7) into (2.5) yields

$$\sigma_{ij} = -p\delta_{ij} + (\mu + \mu_t)\pi_{ij} + d(\lambda^{-2}M_l - \Omega_l)\varepsilon_{ijl}. \tag{2.8}$$

This expression agrees with the total stress for the steady, isotropic and non-parity-invariant turbulence, [5], [6].

In the special case of equilibrium between the angular momenta of the moles and the angular velocity field $\lambda^{-2}\langle\mathbf{M}\rangle = \langle\mathbf{\Omega}\rangle$, the turbulent moles lose their individuality, become identical to each other, and the stress tensor (2.8) reduces to (2.4).

It is known [25], that experimental data collected at the outer part of a boundary layer are very well described by an asymmetric stress function, called the “function of wake”. The non-equilibrium turbulence concept points at the physical reasons for the appearance of such asymmetric functions in some types of shear flows. It arises because of pseudo-vector properties of both the angular momenta of the turbulent moles and the angular velocity of the mean flow. The loss of equilibrium between them gives rise to the asymmetric part of the stress tensor (2.8).

3 Equations of thermal convection

Consider a horizontal layer of an incompressible fluid heated from below and uniformly rotating about the vertical axis with steady angular velocity $\mathbf{\Omega} = \Omega_{\mathbf{e}}$ ($\mathbf{e} = \{0, 0, 1\}$ is the unit vector in the z-direction). We confine attention to the case $|\mathbf{\Omega}| l \ll g^2$ where l is a typical horizontal length, g is the gravity acceleration, so that the centrifugal acceleration can be ignored.

Suppose now that in this fluid a statistically uniform steady, nonequilibrium turbulence (in the mentioned sense) with typical spatial and temporal scales λ and $\tau = \lambda^2/\gamma$ is excited. Let a large-scale ($L \gg l$) disturbance of velocity or temperature be given; we study the possibility of its growth due to interactions with the turbulent field.

The governing equations for the mean (convective) values of velocity $\langle \mathbf{u} \rangle$ and temperature $\langle \Theta \rangle$ in the Boussinesq approximation take the form

$$\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + (\langle \mathbf{u} \rangle \cdot \nabla)\langle \mathbf{u} \rangle = -\frac{1}{\varrho}\nabla p + \left(\nu + \frac{\gamma}{4}\right)\Delta\langle \mathbf{u} \rangle + \beta g \mathbf{e} \langle \Theta \rangle - \\ - 2(\mathbf{\Omega} \times \langle \mathbf{u} \rangle) + \frac{\kappa\gamma}{2}(\nabla \times \langle \mathbf{M} \rangle), \tag{3.1}$$

$$\frac{\partial \langle \mathbf{M} \rangle}{\partial t} + (\langle \mathbf{u} \rangle \cdot \nabla)\langle \mathbf{M} \rangle = \gamma\left(\frac{1}{2}\nabla \times \langle \mathbf{u} \rangle - \kappa \langle \mathbf{M} \rangle\right) + \eta \Delta \langle \mathbf{M} \rangle + \langle \mathbf{m} \rangle, \tag{3.2}$$

$$\frac{D\langle T \rangle}{Dt} + \nabla \langle q \rangle = 0, \tag{3.3}$$

$$\nabla \langle \mathbf{u} \rangle = 0, \quad \mathbf{e} = (0, 0, 1), \tag{3.4}$$

where $\kappa = \lambda^{-2}$; β is the coefficient of thermal expansion; the coefficient of shear viscosity is renormalized, e. g. $\nu + \nu_t \to \nu$; T is the total temperature, which we represent as the sum of some reference temperature $T^0(z)$ and a deviation $\Theta(r, t)$, e. g. $T(r, t) = T^0(z) + \Theta(r, t)$. In what follows we choose this reference temperature in the form $T^0(z) = T_0 - Az$, $A =$ const by analogy with the mechanical equilibrium of a laminar fluid. The last term on the right-hand side of (3.2) is caused by an averaged moment $\varrho\langle \mathbf{m} \rangle$ of the gravity pulsation $\varrho' \mathbf{g}$.

$$\varrho \langle \mathbf{m} \rangle = \mathbf{g} \times \langle \varrho' \mathbf{R} \rangle, \tag{3.5}$$

where **R** is the instantaneous moment arm with respect to the center of mass of the mole; it is of the order of the spatial scale of the turbulent mole, so that $|\mathbf{R}| \sim \lambda$.

The equation of state takes the form $\varrho = \varrho_0(1 - \beta T)$, hence

$$\varrho' = -\varrho_0\beta\Theta', \quad (\beta > 0). \tag{3.6}$$

Substituting (3.6) into (3.5) yields

$$\varrho\langle\mathbf{m}\rangle = -\varrho\beta(\mathbf{g} \times \langle\Theta'\mathbf{R}\rangle). \tag{3.7}$$

In the linear case we can assume a linear relation between $\mathbf{e}\langle\Theta'\mathbf{R}\rangle$ and the angular momentum of turbulent moles, implying

$$\varrho\langle\mathbf{m}\rangle = \varrho\beta g k_0\langle\mathbf{M}\rangle, \tag{3.8}$$

where k_0 is a phenomenological constant characteristic of the fluid properties.

The heat flux density in the balance law of energy (3.3) consists of two parts, viz.,

$$\mathbf{q} = \chi\nabla T + \langle\mathbf{u}'\Theta'\rangle \tag{3.9}$$

each associated with molecular and turbulent heat transfer, respectively. Here χ is the coefficient of thermal conductivity. The angular momentum of a turbulent mole equals

$$\mathbf{M} = \mathbf{u}' \times \mathbf{R}, \tag{3.10}$$

from which one obtains

$$\mathbf{R} \times \mathbf{M} = \mathbf{u}'R^2. \tag{3.11}$$

The averaged value of R^2 is λ^2, where λ is a free parameter of the model considered. Representing the angular momentum of a mole by the sum $\mathbf{M} = \langle\mathbf{M}\rangle + \mathbf{M}'$ and substituting (3.10) and (3.11) into (3.9) yields the following formula for the turbulent component of heat flux

$$\begin{aligned}\langle\mathbf{u}'\Theta'\rangle &= \kappa\left\langle\left(\mathbf{R} \times (\langle\mathbf{M}\rangle + \mathbf{M}')\right)\Theta'\right\rangle = \\ &= \kappa\langle(\mathbf{R} \times \langle\mathbf{M}\rangle)\Theta'\rangle + \kappa\langle(\mathbf{R} \times \mathbf{M}')\Theta'\rangle.\end{aligned} \tag{3.12}$$

On the right hand side of (3.12), the first, nonlinear, term will be ignored and thus

$$\mathbf{q} = \chi\nabla T + \chi_1\nabla\Theta. \tag{3.13}$$

We now consider the stability of a layer of a turbulent fluid with respect to disturbances of the mean (steady) turbulent field. The spatial scale of these disturbances is assumed to be much larger than the typical spatial scale of energy-containing (oriented) turbulent eddies. For an initial stage of the evolution we shall set $\langle u\rangle \ll u'' \sim \langle (u^j j')^2\rangle^{1/2}$ and ignore the nonlinear terms in the equations describing the large-scale disturbances.

Substituting relations (3.9) and (3.13) into equations (3.1)–(3.4) and linearizing them with respect to the mean large-scale velocity and temperature yields

$$\frac{\partial\langle\mathbf{u}\rangle}{\partial t} = -\frac{1}{\varrho}\nabla p + \left(\nu + \frac{\gamma}{\Delta}\right)\Delta\langle\mathbf{u}\rangle + \beta g\mathbf{e}\langle\Theta\rangle - 2(\boldsymbol{\Omega}\times\langle\mathbf{u}\rangle) + \frac{\kappa\gamma}{2}(\nabla\times\langle\mathbf{M}\rangle), \tag{3.14}$$

$$\frac{\partial\langle\mathbf{M}\rangle}{\partial t} = \gamma\left(\frac{1}{2}(\nabla\times\langle\mathbf{u}\rangle) - \kappa\langle\mathbf{M}\rangle\right) + \eta\Delta\langle\mathbf{M}\rangle - \beta g k_0\langle\mathbf{M}\rangle, \tag{3.15}$$

$$\frac{\partial\langle\Theta\rangle}{\partial t} = A(\mathbf{e}\cdot\langle\mathbf{u}\rangle) + \chi\Delta\langle\Theta\rangle, \tag{3.16}$$

$$\nabla\langle\mathbf{u}\rangle = 0, \tag{3.17}$$

where $\chi + \chi_1$ has been replaced by χ. Note that in equilibrium eqs. (3.14)–(3.17) reduce to the classical equations for thermal convection of a rotating fluid [26]

$$\begin{aligned}\frac{\partial\langle\mathbf{u}\rangle}{\partial t} &= -\frac{1}{\varrho}\nabla p + \nu\Delta\langle\mathbf{u}\rangle + \beta g\mathbf{e}\langle\Theta\rangle - 2\Omega(\mathbf{e}\times\langle\mathbf{u}\rangle),\\ \frac{\partial\langle\Theta\rangle}{\partial t} &= -A\mathbf{e}\langle\mathbf{u}\rangle + \chi\Delta\langle\Theta\rangle,\\ \nabla\langle\mathbf{u}\rangle &= 0,\end{aligned}$$

since the effect of turbulent small-scale motions is taken into account in the coefficients ν and χ only. For the problems of thermal convection in the Earth's atmosphere such an equilibirum is possible in a comparatively small part of the near ground layer. In the main part of the atmosphere turbulence must most probably be considered as being non-equilibriated, even though this difference from equilibrium is probably quite small. For the instability analysis we confine our attention to the case of a small non-equilibrium state of turbulence.

Let us assume, that $\nu \sim \eta \sim \chi \sim \lambda u'$, and introduce also the dimensionless parameter $\gamma \to \gamma/\nu$. Eqs. (3.14)–(3.17) can then be rewritten in the following dimensionless form

$$\frac{\partial\langle\mathbf{u}\rangle}{\partial t} = -\nabla p + \left(1 + \frac{\gamma}{4}\right)\Delta\langle\mathbf{u}\rangle + \mathrm{Ra}\langle\Theta\rangle\mathbf{e} - \Omega(\mathbf{e}\times\langle\mathbf{u}\rangle) + \gamma(\nabla\times\langle\mathbf{M}\rangle), \tag{3.18}$$

$$\frac{\partial\langle \mathbf{M}\rangle}{\partial t} = \gamma\kappa\left(\frac{1}{4}(\nabla\times\langle \mathbf{u}\rangle) - \langle \mathbf{M}\rangle\right) + \Delta\langle \mathbf{M}\rangle + \mathrm{Ra}\langle \mathbf{M}\rangle, \tag{3.19}$$

$$\frac{\partial\langle \mathbf{\Theta}\rangle}{\partial t} = \langle \mathbf{u}\rangle\cdot e + \Delta\langle\Theta\rangle, \tag{3.20}$$

$$\nabla\langle \mathbf{u}\rangle = 0. \tag{3.21}$$

Here the scale values of length, time, velocity, temperature, pressure, angular velocity and angular momentum are L (the layer depth), $t_0 = L^2/\nu$, $u_0 = \nu/L$, $T_0 = AL$, $p_0 = \varrho_0\nu^2/L^2$, $\Omega_0 = (2L^2/\nu)^{-1}$, $M_0 = 2\nu\kappa^{-1}L^{-2}$. Besides, $k_0 = AL^2/\nu$, and the Rayleigh number $\mathrm{Ra} = (g\beta AL^4)/\nu^2$. If the fluid layer is immersed in a heat conductive massif, it is necessary to close eqs. (3.18) – (3.21) by the following equation for the massif temperature

$$\frac{\nu}{\chi_m}\frac{\partial\Theta_m}{\partial t} = \Delta\Theta_m, \tag{3.22}$$

where χ_m is the thermal conductivity coefficient of the massif. At the fluid-massif boundary the velocity must vanish, and the temperature and heat flux must be continuous. If this boundary is smooth (no roughness), the angular momentum of the turbulent moles must also vanish. Besides, we assume that in the massif far from the fluid the temperature disturbances decay to zero.

Eqs. (3.18) – (3.21) subject to some boundary conditions define a boundary-value problem. Let the velocity components, mole angular momentum components, the pressure, the temperature deviations be proportional to time as follows

$$\{\langle \mathbf{u}\rangle, \langle \mathbf{M}\rangle, p, \Theta, \Theta_m\} \sim \exp(-\omega t) \tag{3.23}$$

ω is the growth rate of small disturbances. Substitute (3.23) into (3.18) – (3.21); this yields the following set of linear differential equations for the disturbance amplitudes

$$-\omega\mathbf{u} = -\nabla p + \left(1+\frac{\gamma}{4}\right)\Delta\mathbf{u} + \mathrm{Ra}\,\Theta\mathbf{e} + \Omega(\mathbf{e}\times\mathbf{u}) + \gamma(\nabla\times\mathbf{M}), \tag{3.24}$$

$$-\omega\mathbf{M} = \gamma\left(\frac{1}{4}\kappa(\nabla\times\mathbf{u}) - \kappa\mathbf{M}\right) + \Delta\mathbf{M} + \mathrm{Ra}\,\mathbf{M}, \tag{3.25}$$

$$-\omega\Theta = \mathbf{u}\cdot\mathbf{e} + \Delta\Theta, \tag{3.26}$$

$$\nabla\mathbf{u} = 0, \tag{3.27}$$

$$-\omega(\nu/\chi_m)\Theta_m = \Delta\Theta_m. \tag{3.28}$$

The boundary conditions for these dimensionless amplitude functions that must hold along the boundary S between the fluid and the solid massif are,

$$\mathbf{u} = \mathbf{0},\quad \mathbf{M} = \mathbf{0},\quad \Theta = \Theta_m,\quad (\chi/\chi_m)\frac{\partial\Theta}{\partial n} = \frac{\partial\Theta_m}{\partial n}. \tag{3.29}$$

The boundary-value problem (3.24) – (3.29) is an eigenvalue problem. In case of a closed domain (for example, a plane layer) the spectrum is discrete and can be determined.

Some general properties of the spectrum that can be deduced without concretizing the domain can be determined by a method, described in [27]. Notice that the special case of the horizontal plane layer had been examined previously in [28]. For the sake of simplicity we take homogeneous boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{M} = \mathbf{0}, \quad \Theta = \mathbf{0},$$

corresponding to the case when the heat conductivity of the massif is much larger than that of the fluid.

Multiply equations (3.24) – (3.26) by complex-conjugate functions $\mathbf{u}^*$, $\mathbf{M}^*$ and Θ^* respectively, and integrate the resulting equations over the fluid volume. Taking into account that for an incompressible fluid and uniform boundary conditions

$$\begin{aligned}
&\nabla p \cdot \mathbf{u}^* = \nabla(p\mathbf{u}^*), \\
&\mathbf{u}^* \cdot \Delta\mathbf{u} = \mathbf{u}^* \cdot (\nabla \times (\nabla \times \mathbf{u})) = \nabla \times \big(\mathbf{u}^* \cdot (\nabla \times \mathbf{u})\big) - (\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{u}^*), \\
&\int (\mathbf{u}^* \cdot \Delta\mathbf{u})\, dV = -\int |\nabla \times \mathbf{u}|^2\, dV, \\
&\int (\mathbf{M}^* \cdot \Delta\mathbf{M})\, dV = -\int |\nabla \times \mathbf{M}|^2\, dV
\end{aligned}$$

and

$$\int \nabla(p\mathbf{u}^*)\, dV = \int p\mathbf{u}^*\, dS = 0,$$

one obtains

$$\begin{aligned}
-\omega \int |\mathbf{u}|^2\, dV &= -\Big(1 + \frac{\gamma}{4}\Big) \int |\nabla \times \mathbf{u}|^2\, dV + \mathrm{Ra} \int \Theta\mathbf{u}^* \cdot \mathbf{e}\, dV \\
&\quad + \Omega \int (\mathbf{e} \times \mathbf{u}) \cdot \mathbf{u}^*\, dV + \gamma \int \mathbf{u}^* \cdot (\nabla \times \mathbf{M})\, dV, \\
-\omega \int |\mathbf{M}|^2\, dV &= \frac{\gamma\kappa}{4} \int \mathbf{M}^* \cdot (\nabla \times \mathbf{u})\, dV - \kappa\gamma \int |\mathbf{M}|^2\, dV \\
&\quad - \int |\nabla \times \mathbf{M}|^2\, dV + \mathrm{Ra} \int |\mathbf{M}|^2\, dV \\
-\omega \int |\Theta|^2\, dV &= -\int |\nabla\Theta|^2\, dV + \int \Theta^*\mathbf{u} \cdot \mathbf{e}\, dV. \qquad (3.30)
\end{aligned}$$

Subtracting the complex-conjugate relations from (3.30) yields

$$(\omega^* - \omega)\int |\mathbf{u}|^2 dV = \mathrm{Ra}\int (\Theta\mathbf{u}^*\cdot\mathbf{e} - \Theta^*\mathbf{u}\cdot\mathbf{e})\,dV +$$
$$+\,\Omega\left(\int (\mathbf{e}\times\mathbf{u})\cdot\mathbf{u}^*\,dV - \int (\mathbf{e}\times u^*)\cdot\mathbf{u}\,dV\right) +$$
$$+\,\gamma\int \big(\mathbf{u}^*\cdot(\nabla\times\mathbf{M}) - \mathbf{u}\cdot(\nabla\times\mathbf{M}^*)\big)\;dV, \qquad (3.31)$$

$$(\omega^* - \omega)\int |\mathbf{M}|^2\,dV = \frac{\gamma\kappa}{4}\int \mathbf{M}^*\cdot\big(\nabla\times\mathbf{u}) - \mathbf{M}\cdot(\nabla\times\mathbf{u}^*)\big)\;dV, \qquad (3.32)$$

$$(\omega^* - \omega)\int |\Theta|^2\,dV = -\int (\Theta\mathbf{u}^*\cdot\mathbf{e} - \Theta^*\mathbf{u}\cdot\mathbf{e})\,dV. \qquad (3.33)$$

The last term in (3.32) equals zero. Multiplying now equation (3.33) by Ra and adding the result to (3.31) yields

$$(\omega^* - \omega)\int \big(|\mathbf{u}|^2 + \mathrm{Ra}\,|\Theta|^2\big)\;dV = -2\Omega\int (\mathbf{u}^*\times\mathbf{u})\cdot\mathbf{e}\,dV +$$
$$+\,\gamma\int \big(\mathbf{u}^*\cdot(\nabla\times\mathbf{M}) - \mathbf{u}\cdot(\nabla\times\mathbf{M}^*)\big)\;dV.$$

Multiplying this equation by $\kappa/4$ and adding the emerging equation to (3.32) we obtain

$$\omega^* - \omega = \frac{1}{A_+}\left[\frac{\Omega\kappa}{2}\int (\mathbf{u}^*\times\mathbf{u})\mathbf{e}\,dV\right.$$
$$+\,\frac{\gamma\kappa}{4}\int \big(\mathbf{M}^*\cdot(\nabla\times\mathbf{u}) - \mathbf{M}\cdot(\nabla\times\mathbf{u}^*) -$$
$$\left. -\mathbf{u}^*\cdot(\nabla\times\mathbf{M}) - \mathbf{u}\cdot(\nabla\times\mathbf{M}^*)\big)\;dV\right]$$
$$= \frac{i}{A_+}\left[\kappa\Omega\int (\mathrm{Im}\,\mathbf{u}\times\mathrm{Re}\,\mathbf{u})\cdot\mathbf{e}\,dV\right. +$$
$$+\,\frac{\gamma\kappa}{2}\int \big(\mathrm{Re}\,\mathbf{u}\cdot(\nabla\times\mathrm{Im}\,\mathbf{M}) -$$
$$\mathrm{Im}\,\mathbf{M}\cdot(\nabla\times\mathrm{Re}\,\mathbf{u}) - \mathrm{Im}\,\mathbf{u}\cdot(\nabla\times\mathrm{Re}\,\mathbf{M})$$
$$\left. +\,\mathrm{Re}\,\mathbf{M}\cdot(\nabla\times\mathrm{Im}\,\mathbf{u})\big)\;dV\right], \qquad (3.34)$$

where

$$A_+ = \int |\mathbf{M}|^2\, dV + \frac{\kappa}{4} \int \left(|\mathbf{u}|^2 + \mathrm{Ra}\, |\Theta|^2\right)\, dV.$$

Return now to eqs. (3.30) and add to every equation its complex-conjugate; then one obtains

$$\begin{aligned}
-(\omega^* + \omega) \int |\mathbf{u}|^2\, dV = \mathrm{Ra} \int (\Theta \mathbf{u}^* \cdot \mathbf{e} + \Theta^* \mathbf{u} \cdot \mathbf{e})\, dV + \\
+ \gamma \int \left(\mathbf{u}^* \cdot (\nabla \times \mathbf{M}) + \mathbf{u}(\nabla \times \mathbf{M}^*)\right)\, dV - \\
- 2\left(1 + \frac{\gamma}{4}\right) \int |\nabla \times \mathbf{u}|^2\, dV,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
-(\omega^* + \omega) \int |\mathbf{M}|^2\, dV = \frac{\gamma\kappa}{4} \int \left(\mathbf{M}^* \cdot (\nabla \times \mathbf{u}) + \mathbf{M} \cdot (\nabla \times \mathbf{u}^*)\right)\, dV - \\
- 2\kappa\gamma \int |\mathbf{M}|^2\, dV - 2 \int |\nabla \times \mathbf{M}|^2\, dV + \\
+ 2\,\mathrm{Ra} \int |\mathbf{M}|^2\, dV,
\end{aligned} \tag{3.36}$$

$$-(\omega^* + \omega) \int |\Theta|^2\, dV = \int (\Theta^* \mathbf{u} \cdot \mathbf{e} + \Theta \mathbf{u}^* \cdot \mathbf{e})\, dV - 2 \int |\nabla \Theta|^2\, dV. \tag{3.37}$$

Multiplying (3.37) by the Rayleigh number Ra and subtracting (3.35), yields

$$\begin{aligned}
&(\omega^* + \omega) \left(\int |\mathbf{u}|^2\, dV - \mathrm{Ra} \int |\Theta|^2\, dV\right) = \\
&= -\gamma \int \left(\mathbf{u}^* \cdot (\nabla \times \mathbf{M}) + \mathbf{u} \cdot (\nabla \times \mathbf{M}^*)\right)\, dV \\
&+ 2\left(1 + \frac{\gamma}{4}\right) \int |\nabla \times \mathbf{u}|^2\, dV - 2\,\mathrm{Ra} \int |\nabla \Theta|^2\, dV.
\end{aligned}$$

Multiplying this equation by $\kappa/4$ and subtracting (3.36), one obtains

$$\omega^* + \omega = \frac{1}{A_-}\left[2\kappa\gamma \int |\mathbf{M}|^2\,dV + \frac{\kappa}{2}\left(1+\frac{\gamma}{4}\right)\int |\nabla\times\mathbf{u}|^2\,dV - \right.$$

$$- \frac{\kappa\gamma}{4}\int\Big(\mathbf{u}^*\cdot(\nabla\times\mathbf{M}) + \mathbf{u}\cdot(\nabla\times\mathbf{M}^*) + \mathbf{u}\cdot(\nabla\times\mathbf{M}^*) +$$

$$+ \mathbf{M}^*\cdot(\nabla\times\mathbf{u}) + \mathbf{M}\cdot(\nabla\times\mathbf{u}^*)\Big)\,dV$$

$$\left. + 2\int |\nabla\times\mathbf{M}|^2\,dV - 2\,\mathrm{Ra}\int |\mathbf{M}|^2\,dV - \frac{\kappa}{2}\,\mathrm{Ra}\int |\nabla\Theta|^2\,dV\right], \qquad (3.38)$$

where

$$A_- = \int |\mathbf{M}|^2\,dV + \frac{\kappa}{4}\int\left(|\mathbf{u}|^2 - \mathrm{Ra}\,|\Theta^2\right)\,dV.$$

The third term in the numerator of this expression can be reduced to the following form

$$-\frac{\kappa}{2}\int\left(\mathbf{u}^*\cdot(\nabla\times\mathbf{M}) + \mathbf{u}\cdot(\nabla\times\mathbf{M}^*) + \mathbf{M}\cdot(\nabla\times\mathbf{u}^*)\right)\,dV =$$

$$= -\frac{\kappa}{2}\int\Big(\mathrm{Re}\,\mathbf{M}\cdot(\nabla\times\mathrm{Re}\,\mathbf{u}) + \mathrm{Im}\,\mathbf{M}\cdot(\nabla\times\mathrm{Im}\,\mathbf{u}) +$$

$$+ \mathrm{Re}\,\mathbf{u}\cdot(\nabla\times\mathrm{Re}\,\mathbf{M}) + \mathrm{Im}\,\mathbf{u}\cdot(\nabla\times\mathrm{Im}\,\mathbf{M})\Big)\,dV.$$

This term does not exceed the sum of the first two terms in [...] of (3.38). In fact, the equilibrium condition in dimensionless variables has the form $(1/4)(\nabla\times\mathbf{u}) = \mathbf{M}$, and in case of small non-equilibrium turbulence this expression can be used for the estimation of the amplitudes of the mole angular momentum $\mathbf{M}$, hence

$$2\kappa\gamma\int |\mathbf{M}|^2\,dV > \frac{\kappa\gamma}{2}\int (\mathrm{Re}\,\mathbf{M}\cdot(\nabla\times\mathrm{Re}\,\mathbf{u})$$
$$+ \mathrm{Im}\,\mathbf{M}\cdot(\nabla\times\mathrm{Im}\,\mathbf{u}))\,dV,$$

$$\frac{\kappa}{2}\left(1+\frac{\gamma}{4}\right)\int |\nabla\times\mathbf{u}|^2\,dV > \frac{\kappa\gamma}{2}\int (\mathrm{Re}\,\mathbf{u}\cdot(\nabla\times\mathrm{Re}\,\mathbf{M})$$
$$+ \mathrm{Im}\,\mathbf{u}\cdot(\nabla\times\mathrm{Im}\,\mathbf{M}))\,dV.$$

Therefore, the sum of the first three integrals in (3.38) is always positive.

The expressions (3.34) and (3.38) allow to draw some inferences on the real and imaginary parts of instability increments. Let a fluid be heated from below. In this case the Rayleigh number Ra is positive, since the equilibrium temperature gradient A is positive. The denominator A on the rhs. of expression (3.34) is positive, and the sum of the terms in [...] is real and in general non-zero. As can be seen from (3.34), this non-zero value is conserved at $\Omega = 0$. Therefore, when the turbulent fluid is heated from below, the increments of normal disturbances are complex, and so they decay or grow with oscillations. In the absence of turbulence and rotation all normal disturbances at $A > 0$ change monotonically [26]. Hence, turbulence removes the monotonicity principle for normal disturbances. If it is absent, non-zero rotation at $A > 0$ leads to oscillations; this is confirmed by calculations and experiments for a plane layer of arbitrary domain.

Now, consider heating from above ($A < 0$, $\mathrm{Ra} < 0$). According to formula (3.38), for the decrement real part $\mathrm{Re}\,\omega = (\omega+\omega^*)/2$, all integrals containing the Rayleigh number are positive at $\mathrm{Ra} < 0$. The sum of the remaining integrals is also positive, and $\mathrm{Re}\,\omega > 0$. Therefore, in the case of moderate non-equilibrium turbulence all normal disturbances decay at heating from above, and equilibrium is stable with respect to large-scale disturbances.

4 Stability of a turbulent rotating fluid

Recall that we consider the stability of a fluid layer with respect to large scale disturbances of the mean steady turbulent field. The typical spatial scale of these disturbances is much larger than the spatial scale of the energy-containing oriented turbulent eddies. The estimates mentioned above show that during an initial stage of evolution the effect of the nonlinear (with respect to the mean flow velocity) term in the averaged equations can be ignored.

From eqs. (3.18) – (3.21) one can eliminate the pressure p and horizontal components of $\langle \mathbf{u} \rangle$, $\nabla \times \langle \mathbf{u} \rangle$, $\langle \mathbf{M} \rangle$, $\nabla \times \langle \mathbf{M} \rangle$ and then obtains

$$
\begin{aligned}
\frac{\partial}{\partial t}\Delta u_3 &= \left(1+\frac{\gamma}{4}\right)\Delta\Delta u_3 + \mathrm{Ra}\,\Delta_1\Theta - \Omega\frac{\partial F_3}{\partial x_3} + \gamma\Delta G_3,\\
\frac{\partial F_3}{\partial t} &= \left(1+\frac{\gamma}{4}\right)\Delta F_3 + \Omega\frac{\partial u_3}{\partial x_3} - \gamma\Delta M_3,\\
\frac{\partial}{\partial t}\Delta M_3 &= \frac{1}{4}\kappa\gamma\Delta F_3 - \kappa\gamma\Delta M_3 + \Delta\Delta M_3 + \mathrm{Ra}\,\Delta M_3,\\
\frac{\partial G_3}{\partial t} &= -\frac{1}{4}\kappa\gamma\Delta u_3 - \kappa\gamma G_3 + \Delta G_3 + \mathrm{Ra}\,G_3,\\
\frac{\partial\Theta}{\partial t} &= \Delta\Theta + u_3, \qquad \nabla\mathbf{u} = \mathbf{0}.
\end{aligned} \tag{4.1}
$$

Here $\Delta_1 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ and the brackets $\langle\ \rangle$ are omitted; furthermore

$$F_3 \equiv (\nabla \times \mathbf{u})_3, \quad G_3 \equiv (\nabla \times \mathbf{M})_3.$$

To formulate the boundary conditions, we consider the layer boundaries as free surfaces, at which the tangential stresses vanish. We further suppose that these surfaces are not distorted by the convective flow and remain flat. As mentioned above the temperature disturbances also vanish at these boundaries. In such a case the equilibrium condition

$$\frac{1}{4}(\nabla \times \langle \mathbf{u} \rangle) = \langle \mathbf{M} \rangle \tag{4.2}$$

applies, and the asymmetric parts of the components of the stress tensor caused by non-equilibrium turbulence vanish, e. g. expression (2.5) reduces to (2.4). Therefore, as in the classical case we obtain

$$u_3 = 0, \quad \partial u_1/\partial x_3 = \partial u_2/\partial x_3 = 0, \quad \Theta = 0, \text{ at } x_3 = 0 \text{ and } x_3 = 1.$$

Making use of the boundary conditions for the velocity yields

$$\frac{\partial^2 u_3}{\partial x_3^2} = 0, \text{ at } x_3 = 0 \text{ and } x_3 = 1.$$

The function F at the free surfaces must obey the condition $F_3' = 0$, because

$$F_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = i(k_1 u_2 - k_2 u_1), \text{ and } \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0.$$

This condition means that because of (4.2) $M_3' = 0$ at $x_3 = 0, 1$. Hence, the necessary boundary conditions for the horizontal layer with free flat surfaces have the following form

$$u_3 = 0, \quad \frac{\partial^2 u_3}{\partial x_3^2} = 0, \quad \Theta = 0, \quad F_3' = 0, \quad M_3' = 0, \quad \text{at } x_3 = 0, 1. \tag{4.3}$$

Eigenfunctions for the boundary-value problem (4.1), (4.3) are simple harmonics

$$u = a \cdot \sin n\pi x_3, \quad \Theta = b \cdot \sin n\pi x_3, \quad F = c \cdot \cos n\pi x_3,$$
$$G = d \cdot \sin n\pi x_3, \quad M = f \cdot \cos n\pi x_3, \quad (n = 1, 2, 3, \dots).$$

The dispersion relation for $n = 1$ turns out to be

$$0 = -(\omega - K^2)K^2 + \frac{K^2 B}{\omega - K^2 + \mathrm{Ra} - \kappa\gamma} + \frac{k^2 \mathrm{Ra}}{\omega - K^2} - \\ - \frac{(\Omega\pi)^2 \left(D - (\omega - K^2)K^2\right)}{(\omega - K^2)\left(D - (\omega - K^2)K^2\right) + BK^2},$$

where

$$B = \frac{1}{4}\kappa\gamma K^2, \quad D = \kappa\gamma K^2 - K^2 \mathrm{Ra}, \quad K^2 = k_1^2 + k_2^2 + \pi^2, \quad k^2 = k_1^2 + k_2^2.$$

This is an algebraic equation of the fifth power with respect to ω. In order to find the neutral stability curves determining the critical Rayleigh number as a function of the wavenumber k, we assume $\omega = 0$ and obtain the following cubic equation for the Rayleigh number

$$\mathrm{Ra}^3 + r\,\mathrm{Ra}^2 + s\,\mathrm{Ra} + t = 0, \tag{4.4}$$

where

$$r = -\frac{1}{k^2\delta}\left(\delta^2 K^6 + k^2\left(2\delta K^2 + \kappa\gamma(\delta+1)\right) + (\Omega\pi)^2\right),$$

$$s = \frac{1}{k^2\delta}\left(K^6\delta\left(2\delta K^2 + \kappa\gamma(\delta+1)\right) + k^2(\kappa\gamma + K^2)(\kappa\gamma + \delta K^2) - \frac{1}{4}\kappa\gamma K^6\delta + \right.$$

$$\left. + 2(\kappa\gamma + K^2)(\Omega\pi)^2\right),$$

$$t = -\frac{1}{k^2\delta}\left(K^6(\kappa\gamma + \delta K^2)\left(\delta(\kappa\gamma + K^2) - \frac{1}{4}\kappa\gamma\right) + (\Omega\pi)^2(\kappa\gamma + K^2)^2\right),$$

$$\delta = 1 + \frac{\gamma}{4}.$$

The solution of (4.4) consists of three branches $\mathrm{Ra}(k)$, corresponding to its three roots. In order to choose the physically relevant branch, consider the limit $\kappa \to 0$, $\gamma \to 0$. Then equation (4.4) can be represented as

$$\left(\mathrm{Ra} - \frac{K^6 + (\Omega\pi)^2}{k^2}\right) \cdot \left(\mathrm{Ra} - K^2\right)^2 = 0. \tag{4.5}$$

The first root corresponds to the neutral curve of the classical convection in a horizontal layer of a rotating fluid without turbulence. The two other roots correspond to a branch, having the meaning of the neutral curve on an internal turbulent scale level and satisfying the equilibrium condition (4.2). In fact, in case of equilibrium on a level of turbulent scales it is necessary to add the equilibrium condition (4.2) to eqs. (4.1), viz,

$$0 = \frac{1}{4}F_3 - M_3. \tag{4.6}$$

From eqs. (4.1), (4.6) one obtains

$$0 = -(\omega - K^2)K^2 - \kappa\gamma K^2 + \kappa\gamma K^2 - K^2\,\mathrm{Ra}$$

and at $\omega = 0$ we have

$$\mathrm{Ra} = K^2 \tag{4.7}$$

which was to be proved. (4.7) implies that for long wavelengths any equilibrium on the level of turbulent scales is absent at $\mathrm{Ra} > \pi^2$. It means, that at practically interesting values of the Rayleigh numbers the turbulence is of non-equilibrium type.

The analysis of the considered limit case of the dispersion relation (4.4) shows that large-scale equilibrium is determined by that branch $\mathrm{Ra}(k)$, which associates asymptotically with the first root of (4.5).

The neutral curves corresponding to changes of parameters γ and κ at $\Omega = 0$ are depicted in Figs. 1, 2. The lowest curve corresponds to the asymptotic case $\gamma \to 0$, $\kappa \to 0$ and coincides with the classical neutral curve for a horizontal layer in the absence of turbulence [26]. From the neutral curves corresponding to $\gamma \neq 0$, $\kappa \neq 0$ one can see that nonequilibrium turbulent fluids are more stable with respect to long wave disturbances, when turbulence is in equilibrium on the domain boundaries. This conclusion can be interpreted as follows: In a fluid with developed small-scale turbulence there is a possibility to transfer a part of the heat that is entering from outside and which is a source of long wave disturbances, to inner degrees of freedom which are caused by the angular momentum of turbulent moles. Growth rates for the mentioned disturbances that lead to a loss of large-scale equilibrium, become smaller because the total energy is conserved in the considered system at the same rate of heating on the boundaries and corresponding value of the Rayleigh number.

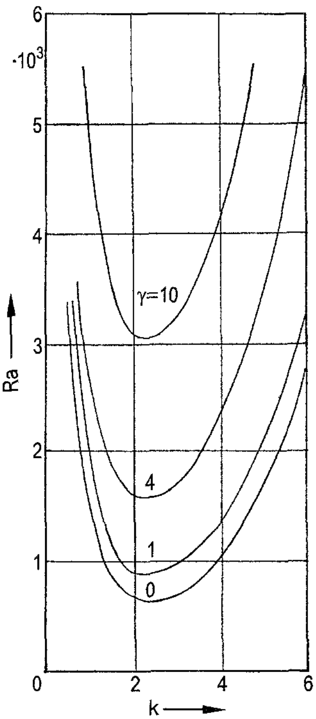

Fig. 1.

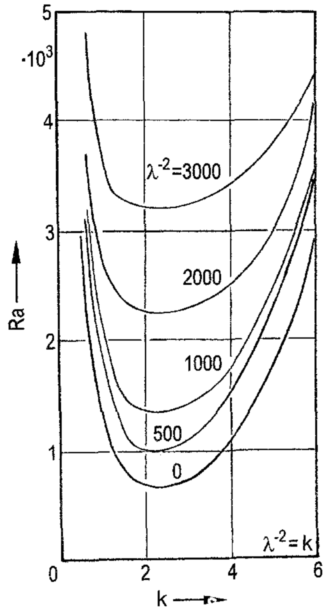

Fig. 2.

It is interesting to note that the instability domain becomes more narrow with increasing parameter γ (Fig. 1) and more gently sloping in the vicinity of the minimum of the neutral curve with increasing parameter κ or decreasing turbulent scale γ (Fig. 2). In other words, the critical Rayleigh number "scatters" over wave number.

In Fig. 3 the depencence of the critical (minimum) Rayleigh number Ra_m on the rotation parameter Ω^2 is displayed for various values of the parameter κ and for $\gamma = 1$. The lowest curve coincides with that for convection in a rotating horizontal layer of a nonturbulent fluid [28]. At $\Omega^2 < 10^2$ the dependence of Ra_m on the rotation parameter is rather weak and the values of Ra_m increase uniformly with increasing parameter κ, as mentioned earlier. At $\Omega^2 > 10^2$ all curves tend asymptotically to a limit curve, that corresponds to the non-turbulent case. Such a behavior can be explained by gradually increasing the predominated influence of the rotation energy of the fluid layer as a whole in comparison with the total energy of the rotating turbulent moles. Therefore, both rotation of the layer as a whole and rotation of individual turbulent moles exert a stabilizing influence on the instability development and slow down the beginning of the thermal convection. At small and moderate values of Ω^2 the second process prevails.

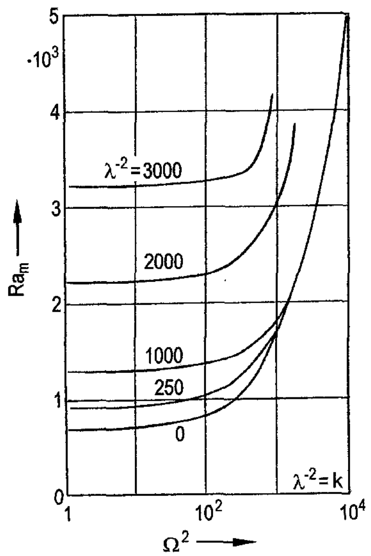

Fig. 3.

5 Conclusion remarks

On the basis of the model of non-equilibrium turbulent fluid with intrinsic motions of turbulent finite-size eddies or moles (proposed by the authors) we have considered the thermal convection of a horizontal layer of a rotating turbulent fluid heated from below. Even though the conventional models of fluids (and conventional Euler and Navier-Stokes equations), in which the fluid has no structure and remains structureless when it flows, remain adequate for the solution of many problems of fluid dynamics, there are some situations for which the main properties of the flows are determined by asymmetric stress tensors. For example, such situations occur in atmospheric turbulence processes, in the mechanics of liquid biological systems, in shock wave interactions in turbulent boundary layers, and so on.

We have shown that the intrinsic motions of finite-size turbulent moles and the orientation of their angular momenta in the field of axial forces give rise to stresses which have to be taken into account as antisymmetric additions to the Reynolds stress tensors. The corresponding mathematical methods are similar to the methods of generalized mechanics. In particular, the balance law of angular momentum is no longer identically satisfied, and it must be included in the governing equations as a basic balance statement.

The typical scale of oriented turbulent eddies is defined by the ratio of the conventional and topological internal invariants of turbulence. This ratio characterizes the relation of the energy associated with intrinsic motions of oriented eddies to the part of their energy that is linked to the vortical field of large-scale motions, and corresponds to the typical spatial scale.

For the problem of thermal convection it is shown that a rotation of the horizontal layer and independent motions of turbulent moles exert a stabilizing influence on the convective instability, and thus slow down the beginning of the thermal convection process. Such an effect occurs in the considered case of equilibrium boundary conditions and free boundaries of the horizontal layer heated from below. Finally, we note that the theory can be extended to the case of non-equilibrium conditions on one of the plane layer boundaries, corresponding to atmospheric processes, and we intend to consider such cases in future studies.

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