Schrödinger Operators
with Singular Magnetic Vector Potentials

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§ 1. Introduction

In this paper we wish to consider domains of essential self-adjointness for Schrödinger operators

$$H = - \sum_{j=1}^{n} (\partial_j - i\, a_j)^2 + V \tag{1}$$

where $\partial_j = \partial/\partial x_j$ and where a_j, V are measurable functions on $\mathbb{R}^n$ with additional conditions which will vary from theorem to theorem. Throughout we employ the notation:

$$D_j = \partial_j - i\, a_j$$
$$V_+ = \max(V, 0)$$
$$V_- = \max(-V, 0).$$

The interest in operators of the form (1) comes, of course, from the theory of non-relativistic quantum mechanics but we emphasize that for all operators of direct physical interest, we know that $C_0^\infty(\mathbb{R}_n)$ is a core due to the work of Kato [8], Stummel [19] and Ikebe-Kato [6]. We are interested in the mathematical question of determining minimal conditions which assure essential self-adjointness on C_0^∞.

Let us begin by summarizing the "general principles" which have been discovered in the study of operators of the form (1). The first two are classical going back to the work of Weyl and others on the case $n=1$:

(1) Almost all theorems are stated in terms of separate hypotheses on a, V_+, V_-. This is an empirical fact. In case $\vec{a}=0$, this is partly expressed by the recent theorem of Davies and Faris [3, 5] that if V_- is $-\Delta$ bounded with bound less than 1 and $-\Delta+V_+$ is essentially self-adjoint on $D(-\Delta) \cap D(V)$, then $-\Delta+V$ is essentially self-adjoint on $D(-\Delta) \cap D(V)$.

(2) A global condition is needed on V_-. Explicitly, a suitable local average of V_- should grow no faster than $|x|^2$ at ∞.

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(3) [2, 19] Only local conditions are needed on V_+.

(4) [6] Only local conditions are needed on a.

(5) [19, 1, 4, 18] Call p n-*canonical* if $p=2$ when $n\leq 3$, $p>2$ when $n=4$ and $p=\frac{1}{2}n$ if $n\geq 5$. Local L^p (or, at least, local weak L^p [18]) estimates are needed on V_-.

(6) [15, 9] Only local L^2 conditions are needed on V_-.

In this paper, we wish to explore the question of minimal local regularity conditions on $\vec{a}$, the magnetic vector potential. Since the work of Ikebe and Kato [6], it has been standard to suppose that $\vec{a}\in C^1$. At first sight, it does not appear that one can weaken this much, for terms of the form $(\partial_j a_j)\, \Phi$ appear when $(\partial_j - i\, a_j)^2\, \Phi$ is expanded. For this to be in L^2 when $\Phi\in C_0^\infty$, it seems natural to demand that a_j be in $(H^1)_{\mathrm{loc}}$, the local Sobolev space of functions with locally L^2 (distributional) derivatives of first order. However, this view does not take into account the fact that the a_k are not "physical" but only $\vec{b}=\mathrm{curl}\,\vec{a}$ is "physical", i.e. it ignores the freedom of gauge transformations. To be more explicit, we make two remarks. First, we have the formal relation

$$\exp\bigl(i\,\lambda(x)\bigr)\left[-\sum_{j=1}^n D_j^2 + V \right] \exp\bigl(-i\,\lambda(x)\bigr)= -\sum_{j=1}^n \tilde{D}_j^2 + V$$

where $\tilde{D}_j = \partial_j - i\,\tilde{a}_j$ with $\tilde{a}_j = a_j + \partial_j\lambda$. Secondly, the only term in $(-\sum D_j^2 + V)\,\Phi$ which involves derivatives of a is $(\mathrm{div}\,\vec{a})\,\Phi$ and $\mathrm{div}\,\vec{a}$ is not invariant under the gauge transformation $a\to\tilde{a}$. In fact, it is quite common in quantum theory to work in the *Coulomb Gauge* where $\mathrm{div}\,\vec{a}=0$. Most of our results (§§ 2, 5) will impose the condition $\mathrm{div}\,\vec{a}=0$.

If $\mathrm{div}\,\vec{a}=0$, $\vec{a}\in(L^4)_{\mathrm{loc}}$ and $V\in(L^2)_{\mathrm{loc}}$, then we define the operator H on C_0^∞ by:

$$H\Phi = -\Delta\Phi + 2i\sum_{j=1}^n a_j\,\partial_j\,\Phi + (a^2 + V)\,\Phi. \tag{2}$$

It is a simple excercise to see that (2) defines a symmetric operator since $\mathrm{div}\,\vec{a}=0$.

Our main results concerning the operator (2) when $\mathrm{div}\,\vec{a}=0$ appear in § 5. They depend on an idea of Kato [9] and a purely technical argument found in §§ 3, 4. To illustrate the simplicity of working in Coulomb gauge, we first show that when $\vec{a}\in L^4(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$ and $V\in L^2(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$, then $\mathrm{div}\,\vec{a}=0$ and a perturbation theory argument proves that (2) is self-adjoint on $D(-\Delta)$. This argument, which appears in § 2 is included solely for motivational purpose and may be skipped by the reader interest in our main results. In § 6, we briefly consider results when $\mathrm{div}\,\vec{a}\neq 0$.

We conclude this appendix by saying a few words about the definition of Schrödinger operators with singular magnetic potentials by the method of quadratic forms [13, 14]. Suppose $V \geq 0$, $V \in (L^1)_{\text{loc}}$. Define a quadratic form on $C_0^\infty \times C_0^\infty$ by

$$h(\Psi, \Phi) = \sum_{j=1}^{n} \langle (\partial_j - i\, a_j)\, \Psi, (\partial_j - i\, a_j)\, \Phi \rangle + \langle v^{\frac{1}{2}}\, \Psi, v^{\frac{1}{2}}\, \Phi \rangle$$

h is a positive closable quadratic form (because $\partial_j - i\, a_j$ and $v^{\frac{1}{2}}$ are closable operators) whose closure is thus the form of a unique self-adjoint operator H [10, 11]. Thus, if we only seek a "natural" meaning for H and don't care about operator cores, $\vec{a} \in (L^2)_{\text{loc}}$ is all that is required. This was first commented by E. Nelson (unpublished) who based his comments on the use of stochastic and path integrals rather than forms.

§ 2. Coulomb Gauge, I: Theorem with a Global Hypotheses

The Kato-Rellich theory of regular perturbations can be applied to prove a simple result on Schrödinger operators with singular magnetic fields:

Theorem 1. *Let $\vec{a}$ be an $\mathbb{R}^3$-valued function on $\mathbb{R}^3$ with $\vec{a} \in L^4 + L^\infty$ and* div $\vec{a} = 0$. *Let $V \in L^2 + L^\infty$. Then $H = - \sum_{j=1}^{3} (\partial_j - i\, a_j)^2 + V$ is essentially self-adjoint on C_0^∞ (and self-adjoint on $D(-\Delta)$).*

Proof. Since $H = - \Delta + 2 i\, \vec{a} \cdot \vec{V} + a^2 + V$ and $a^2 \in L^2 + L^\infty$, we need only prove an estimate of the form:

$$\|(\vec{a} \cdot \vec{V})\, \Phi\| \leq \alpha \|\Delta \Phi\| + \beta \|\Phi\| \tag{3}$$

for some $\alpha < 1$ and all $\Phi \in C_0^\infty$. On the one hand for any ε:

$$\|\partial_j \Phi\|_2 \leq \varepsilon \|\Delta \Phi\|_2 + \varepsilon^{-1} \|\Phi\|_2 \tag{4}$$

and on the other hand for any ε, these is a c so that

$$\|\partial_j \Phi\|_4 \leq \varepsilon \|\Delta \Phi\|_2 + c \|\Phi\|_2. \tag{5}$$

(4) follows from the Plancherel theorem and the bound

$$\|p_j \Phi\|^2 \leq \|p_j^2 \Phi\| \, \|\Phi\| \leq \varepsilon^2 \|p_j \Phi\|^2 + \varepsilon^{-2} \|\Phi\|^2 \leq \varepsilon^2 \|p^2 \Phi\|^2 + \varepsilon^{-2} \|\Phi\|^2.$$

(5) comes from the Plancherel theorem, the Hausdorff-Young inequality and $\|p_j \Phi\|_{\frac{4}{3}} \leq \|p_j(1+\alpha p^2)^{-1}\|_4 [\|\Phi\|_2 + \alpha \|p^2 \Phi\|_2]$. (3) follows immediately from (4) and (5). ☐

Remarks. 1. In some ways, it is more natural to place restrictions on $\vec{a}$ and $a^2 + V$ rather than $\vec{a}$ and V. One sees that it is then only necessary to have $a^2 + V \in L^2 + L^\infty$ and $\vec{a} \in L^p + L^\infty$ with $p > 3$, for the proof to go through.

2. It is quite easy to discuss scattering theory for operators of the type dealt with in Theorem 1 (see also [7]). For example, by using Cook's method, one can prove $s\text{-}\lim e^{iHt} e^{-iH_0 t}$ exist if $V \in L^2 + L^r$ ($2 \leq r < 3$) and $\vec{a} \in L^4 \cap L^s$ ($s < 3$).

3. The condition div $\vec{a} = 0$ may be replaced with div $\vec{a} \in L^2 + L^\infty$.

§ 3. Kato's Inequality and the Reduction to Condition K_p

In Kato's recent discussion of Schrödinger operators with $(L^2)_{\text{loc}}$ potentials [9], a major role is played by:

Proposition *(Kato's Inequality).* *If* $u \in (L^1)_{\text{loc}}$ *and* $-\Delta u \in (L^1)_{\text{loc}}$, *then*

$$\Delta |u| \geq \text{Re}((\text{sgn } u) \Delta u) \tag{6}$$

where (6) *is understood as a distributional inequality and* sgn u *is defined by:*

$$(\text{sgn } u)(x) = \begin{cases} 0 & \text{if} \quad u(x) = 0 \\ u^*(x)/|u(x)| & \text{if} \quad u(x) \neq 0. \end{cases}$$

In our extension of Kato's method, we make heavy use of the fact that in applications u is in $(L^2)_{\text{loc}}$ rather than merely in $(L^1)_{\text{loc}}$.

Definition. Let $p \in [4, \infty]$ and let $q = (1 - p^{-1})^{-1}$. We say that a function $\vec{a}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ obeys *condition K_p* if and only if

(i) $\vec{a} \in (L^p)_{\text{loc}}$.

(ii) If $u \in (L^2)_{\text{loc}}$ and $-\Delta u + 2i \text{ div}(\vec{a} u) \in (L^1)_{\text{loc}}$, then grad $u \in (L^q)_{\text{loc}}$.

Lemma 1. *Suppose* div $\vec{a} = 0$. *If* $p^{-1} + q^{-1} = 1$, *if* $\vec{a} \in (L^p)_{\text{loc}}$, $u \in (L^q)_{\text{loc}}$ *and* grad $u \in (L^q)_{\text{loc}}$, *then*

$$\text{div}(\vec{a} u) = \vec{a} \text{ grad } u.$$

Proof. Let j_δ be an approximate identity, i.e. $j_\delta(x) = \delta^{-1} j(x/\delta)$ where $j \in C_0^\infty$, $j \geq 0$ and $\int j(x) dx = 1$. Let $a^\delta = a * j_\delta$ so that div $a^\delta = 0$ and let $u^\varepsilon = u * j_\varepsilon$. Then by Lebnitz' rule,

$$\text{div}(\vec{a}^\delta u^\varepsilon) = \vec{a}^\delta \text{ grad } u^\varepsilon. \tag{7}$$

Using the given L^p estimates and Holders inequality, it is easy to prove that the two sides of (7) converge in $(C_0^\infty)'$ to div$(\vec{a} u)$ and $\vec{a}$ grad u as δ, $\varepsilon \to 0$. ☐

Lemma 2. *Suppose $\vec{a}$ obeys condition K_p, that $\operatorname{div} \vec{a} = 0$ and that $-\Delta u + 2i \operatorname{div}(\vec{a} u) \in (L^1)_{\mathrm{loc}}$. Then $-\Delta u \in (L^1)_{\mathrm{loc}}$.*

Proof. This follows from Lemma 1. $\square$

Theorem 2. *Let $\vec{a}$ obey condition K_p for some $p \in [4, \infty]$ and suppose $\operatorname{div} \vec{a} = 0$. Let $\tilde{H}$ be the operator from $(L^2)_{\mathrm{loc}}$ to $(C_0^\infty)'$ given by:*

$$\tilde{H} u = -\Delta u + 2i \sum_{j=1}^{n} \partial_j (a_j u) + a^2 u.$$

Then for any $u \in (L^2)_{\mathrm{loc}}$ with $H u \in (L^1)_{\mathrm{loc}}$ we have

$$\Delta |u| \geqq - \operatorname{Re}\left[(\operatorname{sgn} u)(\tilde{H} u)\right] \tag{8}$$

as a distributional inequality.

Proof. Following Kato [9], we first prove a modified version of (8) when u is smooth and then obtain (8) by a double limiting proceedure. So let u be C^∞ and define $u_\varepsilon = (|u|^2 + \varepsilon^2)^{\frac{1}{2}}$. Since $u_\varepsilon^2 = u^* u + \varepsilon^2$:

$$\begin{aligned} 2 u_\varepsilon \, \partial_j u_\varepsilon &= u^* \, \partial_j u + (\partial_j u)^* \, u \\ &= u^* (D_j u) + (D_j u)^* \, u \\ &= 2 \operatorname{Re}\left(u^* (D_j u) \right). \end{aligned} \tag{9}$$

In particular, since $u_\varepsilon \geqq |u|$, we obtain $|\partial_j u_\varepsilon| \leqq |D_j u|$. Letting

$$\|\vec{v}\| = \left(\sum_{i=1}^{n} |v_i|^2 \right)^{\frac{1}{2}},$$

we obtain

$$\|\operatorname{grad} u_\varepsilon\| \leqq \|\vec{D} u\|. \tag{10}$$

Moreover, by (8),

$$\begin{aligned} 2 \partial_j (u_\varepsilon \, \partial_j u_\varepsilon) &= (\partial_j u)^* (D_j u) + u^* (\partial_j D_j u) + \text{complex conjugate} \\ &= (D_j u)^* (D_j u) + u^* (D_j^2 u) + \text{complex conjugate}. \end{aligned}$$

Summing over j, and using $\operatorname{div} \vec{a} = 0$, we obtain

$$\|\operatorname{grad} u_\varepsilon\|^2 + u_\varepsilon \Delta u_\varepsilon = \|D u\|^2 - \operatorname{Re}\left[u^* (\tilde{H} u) \right]. \tag{11}$$

Define

$$\operatorname{sgn}_\varepsilon u \equiv \frac{u^*}{u_\varepsilon}.$$

Then (10) and (11) imply

$$\Delta u_\varepsilon \geqq - \operatorname{Re}\left((\operatorname{sgn}_\varepsilon u)(\tilde{H} u) \right). \tag{19}$$

Now suppose that $u \in (L^2)_{\mathrm{loc}}$ and $\tilde{H} u \in (L^1)_{\mathrm{loc}}$. Let j_δ be an approximate identity and let $u^\delta \equiv u * j_\delta$. By Lemmas 1 and 2, $\operatorname{div}(\vec{a} u) = \vec{a} \operatorname{grad} u$,

$\Delta u \in (L^1)_{\text{loc}}$ and grad $u \in (L^q)_{\text{loc}}$. Thus $\tilde{H} u^\delta \to \tilde{H} u$ in $(L^1)_{\text{loc}}$. By passing to a subsequence we can also be certain that $u^\delta \to u$ pointwise a. e. so $\text{sgn}_\varepsilon(u^\delta) \to \text{sgn}_\varepsilon(u)$ pointwise a. e. As a result, for any $\varphi \in C_0^\infty$:

$$\int \varphi \left[\text{sgn}_\varepsilon(u^\delta)(\tilde{H} u^\delta) - \text{sgn}_\varepsilon(u)(\tilde{H} u)\right]$$
$$= \int \varphi \left[\text{sgn}_\varepsilon(u^\delta) - \text{sgn}_\varepsilon(u)\right](\tilde{H} u) + \int \varphi \left[\text{sgn}_\varepsilon(u^\delta)\right](\tilde{H} u^\delta - \tilde{H} u)$$

goes to zero (by the dominated convergence theorem). We conclude that $(\text{sgn}_\varepsilon u^\delta) \tilde{H} u^\delta$ converges to $(\text{sgn}_\varepsilon u) \tilde{H} u$ in $(C_0^\infty)'$. Moreover, since

$$\left| \sqrt{|a|^2 + \varepsilon^2} + \sqrt{|b|^2 + \varepsilon^2} \right| = \left| \int_{|a|}^{|b|} x (x^2 + \varepsilon^2)^{-\frac{1}{2}} \, dx \right| \leqq |b - a|.$$

$u_\varepsilon^\delta \to u_\varepsilon$ in $(L^1)_{\text{loc}}$ and so in $(C_0^\infty)'$. Thus (12) holds for u since it holds for u^δ. Taking $\varepsilon \to 0$, (8) follows. $\square$

Theorem 3. *Let $\tilde{a}$ obey condition K_p for some $p \in [4, \infty]$ and suppose that* div $\tilde{a} = 0$. *Let $V_+ \in L^2(\mathbb{R}^n)_{\text{loc}}$, $V_- \in L^p(\mathbb{R}^n)$ with p n-canonical. Then the operator H from C_0^∞ to L^2 defined by*

$$H \varphi = -\Delta \varphi + 2i \sum_{i=1}^n a_i \partial_i \varphi + a^2 \varphi + V \varphi$$

is essentially self-adjoint.

Proof. Since V_- is Δ-bounded with arbitrarily small bound, we can find E_0 with $\|V_-(-\Delta + E_0)^{-1}\| < 1$. We will prove that $H + E_0$ has a dense range. If not, we can find $u \in L^2$ with $\langle u, (H + E_0) \varphi \rangle = 0$ for all $\varphi \in C_0^\infty$. Since the operator $\tilde{H}$ of Theorem 2 is defined with distributional derivatives,

$$\tilde{H} u + V u + E_0 u = 0.$$

Since $V \in (L^2)_{\text{loc}}$, we conclude that $\tilde{H} u = -Vu - E_0 u \in (L^1)_{\text{loc}}$ so Theorem 2 is applicable. We conclude that

$$\Delta |u| \geqq -\text{Re}\left((\text{sgn } u)(-Vu - E_0 u)\right) \geqq V|u| + E_0|u| \geqq -V_-|u| + E_0|u|.$$

Thus

$$(-\Delta + E_0)|u| \leqq V_-|u|. \tag{13}$$

But both sides of (13) lie in $\mathscr{S}(\mathbb{R}^3)'$ and $(-\Delta + E_0)^{-1}$ is a positivity preserving map of $\mathscr{S}$ into $\mathscr{S}$ and so of $\mathscr{S}'$ into $\mathscr{S}'$. Thus

$$|u| \leqq (-\Delta + E_0)^{-1} V_-|u|$$

so that

$$\|u\| \leqq \|(-\Delta + E_0)^{-1} V_-\| \|u\|.$$

Since $\|(-\Delta + E_0)^{-1} V_-\| = \|[V_-(-\Delta + E_0)^{-1}]^*\| < 1$, $\|u\| = 0$. $\square$

Remarks. 1. $V_- \in L^p$ was not essential. All that was needed is that $V_- \in (L^2)_{\mathrm{loc}}$ and that V_- is $-\Delta$-bounded with bound strictly less than 1.

2. By following arguments in [9] or [16], one can prove a whole class of other results when a obeys K_p. For example there are theorems when $V = O(-r^{2-\varepsilon})$ at ∞ [9] or $O(-cr^{-2})$ at 0 for suitable c [16].

§ 4. Investigation of Condition K_p

In order to apply Theorem 3, we need to know when condition K_p holds. If $n < 4$, the situation is very simple:

Lemma 3. *If $n \leqq 3$ and $\vec{a}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $\vec{a} \in \left(L^4(\mathbb{R}^n) \right)_{\mathrm{loc}}$, then $\vec{a}$ obeys condition K_4.*

Proof. We must show that if $u \in (L^2)_{\mathrm{loc}}$ and $f \equiv -\Delta u + 2i \operatorname{div}(\vec{a}u) \in (L^1)_{\mathrm{loc}}$ then $\operatorname{grad} u \in (L^{\frac{4}{3}})_{\mathrm{loc}}$. Without loss, we need only prove $\operatorname{grad} u$ is in $(L^{\frac{4}{3}})_{\mathrm{loc}}$ near 0. Let φ be a function which is one in a neighborhood of 0. Let g be defined by:

$$g = (1-\Delta)^{-1}[(f+u)\varphi] - 2i \sum_{j=1}^{n} [(1-\Delta)^{-1}\partial_j](a_j u \varphi)$$

where $(1-\Delta)^{-1}$ and $[(1-\Delta)^{-1}\partial_j]$ are the usual integral operators on L^1. Then g is in L^1 and

$$(1-\Delta)g = (f+u)\varphi - 2i \operatorname{div}(\vec{a}u\varphi)$$

so that $(1-\Delta)(g-u) = 0$ in a neighborhood of 0. By the elliptic regularity theorem (see, e.g. [12]) $g-u$ is C^∞ near zero so we need only prove that $\operatorname{grad} g \in L^{\frac{4}{3}}$ to conclude that $\operatorname{grad} u$ is in $(L^{\frac{4}{3}})_{\mathrm{loc}}$ near 0. Now

$$\partial_i g = [\partial_i(1-\Delta)^{-1}](f+u)\varphi - 2i \sum_{j=1}^{n} [\partial_i(1-\Delta)^{-1}\partial_j](a_j u \varphi). \qquad (14)$$

Since $n \leqq 3$, $\partial_i(1-\Delta)^{-1}$ is a convolution operator with a function in $L^{\frac{4}{3}}$ so the first term is in $L^{\frac{4}{3}}$ by Young's inequality. Moreover, on each L^p $(p \neq 1, \infty)$ $\partial_i(1-\Delta)^{-1}\partial_j$ is a bounded (singular) integral operator (see, e.g. [17]). Since $a_j u \varphi \in L^{\frac{4}{3}}$ by Hölder's inequality, the second term in (14) is in $L^{\frac{4}{3}}$. Thus $\operatorname{grad} g \in L^{\frac{4}{3}}$. $\square$

The following result is all we have been able to prove when $n \geqq 4$; as we will explain, we feel this is far from optimal if $n > 4$:

Lemma 4. *If $p > n \geqq 4$ and $\vec{a}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ so that $\vec{a} \in (L^p)_{\mathrm{loc}}$, then $\vec{a}$ obeys condition K_p.*

Proof. As in the proof of Lemma 2, we need only prove that

$$[\partial_i(1-\Delta)^{-1}](f+u)\varphi - 2i \sum_{j=1}^{n} (\partial_i(1-\Delta)^{-1}\partial_j)(a_j u \varphi) \qquad (15)$$

is in $(L^q)_{loc}$ with $q = p/p-1$ when $u \in (L^2)_{loc}$, $\varphi \in C_0^\infty$ and $f \in (L^1)_{loc}$. Since $a_j \varphi \in L^p$, $a_j u \varphi \in L^r$ with $r = 2p/p+2$ by Hölder's inequality. Since $p > 4$, $r > 4/3$ and $q < 4/3$ so $L^r \subset (L^q)_{loc}$. Since $\partial_i(1-\Delta)^{-1}\partial_j$ is bounded from L^r to L^r, the second term in (15) is in $(L^q)_{loc}$. Moreover $\partial_i(1-\Delta)^{-1}$ is convolution with a function in L^s for any $s < n/n-1$, so by Young's inequality, the first term is in L^s for any $s < n/n-1$. But since $p > n$, $q < n/n-1$, so the first term in (15) is in $(L^q)_{loc}$. $\square$

We conjecture that if $p > 4$ and $\vec{a} \in L^p_{loc}(\mathbb{R}^n)$, then a obeys condition K_p. This conjecture is based on the fact that if $-\Delta u \in (L^1)_{loc}$ and $u \in (L^2)_{loc}$, then grad $u \in (L^q)_{loc}$ for any $q > 4/3$. In fact (L. Tartar, private communication), if $-\Delta u \in (L^1)_{loc}$ and $u \in (L^2)_{loc}$, then grad $u \in (L^{4/3}_W)_{loc}$, the weak $L^{4/3}$ space (but not in general in $L^{4/3}$).

§ 5. Coulomb Gauge, II: Theorem with a Local Hypothesis

By combining Theorem 3 and Lemmas 3, 4, we obtain

Theorem 4. *Let* $a_j \in L^q(\mathbb{R}^n)_{loc}$, $j = 1, \ldots, n$. *Let* $\sum_{j=1}^{n} \partial_j a_j = 0$. *Suppose* $q > n$; $q \geqq 4$. *Let* $V_+ \in (L^2(\mathbb{R}^n))_{loc}$ *and* $V_- \in L^p$ *with* p *n-canonical. Then:*

$$-\sum_{j=1}^{n} (\partial_j - i\, a_j)^2 + V$$

is essentially self-adjoint on C_0^∞.

Remarks. 1. By a good deal of extra work, Lemma 3 can be extended to the case $p = n$ if $n \geqq 5$ so that the condition in Theorem 4 becomes $a_j \in (L^q)_{loc}$ with $\frac{1}{2}q$ n-canonical.

2. If our conjecture at the end of § 4 is correct, $q > n$, $q \geqq 4$ can be replaced with $q \geqq 4$ if $n \leqq 3$ and $q > 4$ if $n \geqq 4$.

§ 6. Other Gauges

We first note that all the arguments in § 3 remain valid if div $\vec{a} \in L^2_{loc}$ rather than div $\vec{a} = 0$. Thus:

Theorem 5. *The conclusion of Theorem 4 remain valid if the hypothesis* $\sum_{j=1}^{n} \partial_j a_j = 0$ *is replaced with* $\sum_{j=1}^{n} \partial_j a_j \in (L^2)_{loc}$.

In particular we recover the Ikebe-Kato C^1 result and a result when $\vec{a} \in (H^1)_{loc}$. Our attitude about other gauges is better expressed by the following which suggests that C_0^∞ is the "wrong" domain to consider:

Theorem 6. *Let* $a_j \in L^q(\mathbb{R}^n)$ *with* $q > n$ *and* $q \geq 4$. *Let* $V_+ \in \left(L^2(\mathbb{R}^n)\right)_{\mathrm{loc}}$
and $V_- = 0$. *Let* H *be the operator* $- \sum\limits_{j=1}^{n} (\partial_j - i\, a_j)^2 + V$ *defined as a quadratic form. Then there exists a measurable function* λ *so that*

1) *For every* $\varphi \in C_0^\infty$, $e^{-i\lambda(.)}\, \varphi(.) \in D(H)$
2) $\{e^{-i\lambda} \varphi \,|\, \varphi \in C_0^\infty\}$ *is an operator core for* H.

Remark. By more work, it may be possible to extend this result to allow $a_j \in \left(L^q(\mathbb{R}^n)\right)_{\mathrm{loc}}$ and $V_- \neq 0$.

Proof. Let $\lambda = \sum\limits_{j=1}^{n} [(-\Delta)^{-1} \partial_j]\, a_j$ where $(-\Delta)^{-1}\partial_j$ is the operator defined by the theory of Reisz potentials and transforms [17]. Then $\lambda \in L^q(\mathbb{R}^n)$ and $\partial_i\, \lambda \in L^q(\mathbb{R}^n)$, again by the theory of Reisz potentials. Let $\hat{a}_j = a_j + \partial_j \lambda$. Then $\sum\limits_{j=1}^{n} \partial_j \hat{a}_j = \sum\limits_{j=1}^{n} \partial_j a_j + \Delta \lambda = 0$. As a result the operator $\hat{H} = - \sum\limits_{j=1}^{n} (\partial_j - i\, \hat{a}_j)^2 + V$ is essentially self-adjoint on C_0^∞ by Theorem 4. If we can show that

$$H = e^{-i\lambda}\, \hat{H}\, e^{+i\lambda} \tag{16}$$

we will obtain (1) and (2) and so prove the theorem. But it is easy to prove that if $\varphi \in C_0^\infty$, then $e^{i\lambda} \varphi \in D(\overline{\partial_j - i\, \hat{a}_j \restriction C_0^\infty})$ (by approximiting λ with smooth functions) and that

$$e^{-i\lambda}(\partial_j - i\, \hat{a}_j)\, e^{i\lambda} \varphi = (\partial_j - i\, a_j)\, \varphi. \tag{17}$$

(17) immediatly implies the operator equality

$$e^{-i\lambda}\, \hat{D}_j\, e^{i\lambda} = D_j \tag{18}$$

where $D_j = \overline{\partial_j - i\, a_j \restriction C_0^\infty}$; (18) implies the equality of the quadratic forms

$$e^{-i\lambda}\left(\sum_{j=1}^{n} \hat{D}_j^*\, \hat{D}_j + V\right) e^{+i\lambda} = \sum_{j=1}^{n} D_j^*\, D_j + V$$

which is (16).   □

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