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Elementary Abelian p-groups as Automorphism Groups of Infinite Groups. I

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1. Background

If G is a group we will write Aut G for the group of all automorphisms of G and Inn G for the normal subgroup of all inner automorphisms of G. Many authors have studied the relationship between the structure of G and that of Aut G, in particular when the latter is finite. This paper is a further contribution to this study.

The first results on groups whose automorphism groups are finite were published by Baer in a paper [2] in which he proved that a torsion group has finite automorphism group only if it is finite. Baer also proved that a group with only a finite number of endomorphisms is finite. In 1962 Alperin [1] characterized finitely generated groups with finitely many automorphisms as finite central extensions of cyclic groups. Nagrebeckiĭ [9] discovered in 1972 the important result that in any group with finitely many automorphisms the elements of finite order form a finite subgroup. This of course generalizes Baer's original result. Robinson [10] has given another proof of Nagrebeckiĭ's Theorem as well as obtaining information on the primes dividing the order of the maximal torsion subgroup. He also characterized the center of a group whose automorphism group is finite and gave a general method for constructing examples.

On the other hand there seems to be little hope of obtaining a useful classification of groups whose automorphism groups are finite, even in the abelian case. Indeed, it has been shown be several authors that torsion-free abelian groups with only one non-trivial automorphism – the involution $x \mapsto x^{-1}$ – are relatively common (de Groot [5], Fuchs [4], Corner [3]).

However, Hallett and Hirsch have adopted a different approach, asking which finite groups can occur as the automorphism groups of torsion-free abelian groups. They have established the following definitive result [7, 8]:

Proposition 1.1. *Suppose that A is a finite group. Then there is a torsion-free*

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abelian group G such that $\operatorname{Aut} G \simeq A$ *if and only if A is a subdirect product of finitely many of the following groups:*

(i) *cyclic groups of orders* 2, 4, *or* 6,
(ii) *the quaternion group of order* 8,
(iii) *the dicyclic group of order* 12,
(iv) *the binary tetrahedral group of order* 24,

and A satisfies the condition that if it has elements of order 12, *then it has at least one element of order* 2 *that is not a sixth power.*

2. Statement of Results

It follows from the result of Hallett and Hirsch that an elementary abelian 2-group of any finite rank can occur as the automorphism group of an infinite abelian group. Since an infinite abelian group always has an automorphism of order 2, no elementary abelian p-group may arise in this fashion as long as p is odd. It remains to determine which elementary abelian p-groups can occur as the automorphism groups of infinite non-abelian groups. This is a first step in determining which finite abelian groups can occur as the automorphism groups of infinite groups. We will prove the following result:

Main Theorem. *Suppose that G is an infinite non-abelian group and that* $\operatorname{Aut} G$ *is an elementary abelian p-group of rank n.*

(i) *If* $p=2$, *then* $n \geqq 3$.
(ii) *If* $p=3$, *then* $n \geqq 8$.
(iii) *If* $p>3$, *then* $n=8$, *or* $n \geqq 10$ *and n is composite.*

It will be shown in a subsequent paper that any elementary abelian p-group not eliminated by this result actually does occur as the automorphism group of uncountably many non-isomorphic infinite non-abelian groups.

We will also derive various miscellaneous results concerning the structure of a group whose automorphism group is an elementary abelian p-group, as well as showing that such groups always have an outer automorphism.

This paper is an excerpt from the author's doctoral dissertation.

3. General Results

Throughout the sequel G will denote an infinite non-abelian group with center C such that $\operatorname{Aut} G$ is finite. By the Theorem of Nagrebeckiĭ T, the torsion subgroup of C, is finite and hence is a summand of C. Therefore, in additive notation, we may write $C=F \oplus T$ where F is torsion-free. We will denote the factor group G/C by Q. Of course Q is isomorphic with $\operatorname{Inn} G$. Since Q and T are finite, F is non-trivial.

Note that for an arbitrary group G with center C and central quotient group $Q=G/C$ we have the following embedding

$$\operatorname{Hom}(Q_{ab}, C) \rightarrowtail \operatorname{Aut} G \tag{3.1}$$

where $Q_{ab} = Q/Q'$ and $\operatorname{Hom}(Q_{ab}, C)$ is isomorphic with the group of all automorphisms of G inducing the identity on both Q and C.

Suppose that Q is abelian. Then G', the commutator subgroup of G, lies in C and the following identities hold for all x, y, and z in G and for all integers n (see Scott [11], p. 57):

$$\begin{aligned} &[x, yz] = [x, y][x, z], \\ &[x, y]^n = [x, y^n], \\ &(xy)^n = x^n y^n [y, x]^{n(n-1)/2}, \end{aligned} \tag{3.2}$$

where $[x, y] = x^{-1} y^{-1} x y$ is the commutator of x and y. Consequently, if Q if an elementary abelian p-group, then so is G', for $[x, y]^p = [x, y^p] = 1$ (since y^p lies in C for any y in G). It follows for $p > 2$ that

$$(xy)^p = x^p y^p. \tag{3.3}$$

Now suppose that Q is a finite abelian p-group. Then G may be generated modulo C by independent generators $a_1, \dots, a_m$ where $a_i^{p^{n(i)}}$ lies in C for some natural number $n(i)$. It would be convenient to denote the operation of C additively, but since G is not abelian it would be more appropriate to denote the operation of G multiplicatively. These two contradictory aims will be reconciled as follows: operations between the elements $a_1, \dots, a_m$ will be denoted multiplicatively; operations between elements of C will be denoted additively. Since $a_i^{p^{n(i)}}$ is an element of C we will write

$$a^{p^{n(i)}} = f_i + e_i$$

where f_i lies in F and e_i lies in T. In spite of a certain ambiguity arising from this notation it is the most convenient for our purposes and will be used throughout the sequel.

Lemma 3.1. *Let G be an infinite group such that* $\operatorname{Aut} G$ *is finite and* $\operatorname{Inn} G$ *is an abelian p-group. Then $G \simeq G_1 \times H$ where G_1 has no elements of order prime to p in its center, G and G_1 have isomorphic central quotient groups, and for $p > 2$, if* $\operatorname{Aut} G$ *is of odd order, then* $\operatorname{Aut} G \simeq \operatorname{Aut} G_1$.

Proof. Let H be the subgroup of T consisting of those elements of order prime to p. Since G/F is a finite nilpotent group, it is the direct product of its sylow p-subgroups. Thus

$$G/F = (G_1/F) \times (HF/F)$$

where G_1/F is the unique sylow p-subgroup of G/F. Since $G_1 \cap H = 1$, it follows that $G = G_1 \times H$. Clearly the center of G_1 has no elements of order prime to p, and G and G_1 have isomorphic central quotient groups.

Now suppose that $p > 2$ and $\operatorname{Aut} G$ is of odd order. Since the automorphisms of the finite abelian group H all extend to G, they must be of odd order. Hence,

H is trivial or cyclic of order 2. Noting that $G/H \simeq G_1$ and considering the exact sequence

$$\mathrm{Hom}((G_1)_{\mathrm{ab}}, H) \rightarrowtail \mathrm{Aut}\, G \twoheadrightarrow \mathrm{Aut}(G/H),$$

we see that the left-hand term is a 2-group, and hence trivial. Therefore, $\mathrm{Aut}\, G \simeq \mathrm{Aut}\, G_1$ and the proof is complete.

This lemma will be used extensively to simplify the groups under consideration. Another useful result is the following:

Lemma 3.2. *An automorphism α of (an arbitrary) group G with center C commutes with every inner automorphism of G if and only if α induces the identity on $G/C = Q$. Moreover, if this holds, α leaves every element of G' fixed.*

The proof for the first part may be found in Zassenhaus [13, p. 52] where such automorphisms are called "normal." The second part follows easily. It may also be found as a special case of a result proven by P. Hall [6, Lemma 8.4(i)].

4. Elementary Abelian 2-groups as Automorphism Groups

We will now concentrate on elementary abelian 2-groups, which are the easiest automorphism groups with which we will deal.

Lemma 4.1. *If* $\mathrm{Aut}\, G$ *is an elementary abelian* 2*-group, then G has an outer automorphism.*

Proof. We may generate G by C and elements $a_1, \ldots, a_m$ where a_i^2 lies in C for each i. Recall that if G has a presentation $\langle X | R \rangle$, that is, if there is an exact sequence

$$R^{F(X)} \rightarrowtail F(X) \twoheadrightarrow G$$

where $F(X)$ is the free group on X, then a function $\gamma: X \to F(X)$ induces an endomorphism of G if and only if γ maps elements of R to $R^{F(X)}$. In our case we define γ to be the function inverting each a_i and each element of C. Since $[x^{-1}, y^{-1}] = [x, y]$ in G, it follows that γ extends to an automorphism of G. Since γ transforms the center of G non-trivially, it must be outer. This completes the proof.

Corollary 4.2. *Suppose G is an infinite non-abelian group and* $\mathrm{Aut}\, G$ *is an elementary abelian* 2*-group. Then the rank of* $\mathrm{Aut}\, G$ *is at least* 3.

Proof. If $\mathrm{Aut}\, G$ is an elementary abelian 2-group, then so is $Q = G/C \simeq \mathrm{Inn}\, G$. Since Q is not cyclic, it must have rank at least 2. By Lemma 4.1, G has an outer automorphism and thus $\mathrm{rank}(\mathrm{Aut}\, G) \geqq 3$. This complete the proof.

5. Elementary Abelian p-groups as Automorphism Groups, p Odd

In this section we will prove that if Aut G is an elementary abelian p-group for some odd p, then G has an outer automorphism. For $p>3$, we will also prove that the rank of Aut G must be a composite number. The proofs of these results need several preparatory lemmas.

Lemma 5.1. *Suppose that* Aut G *is finite group of odd order and* $Q=G/C$ *is an elementary abelian p-group for some odd prime p. Then G may be generated by C together with elements* $a_1, \ldots, a_m$ *that are independent modulo C, such that for some* s, $0<s<m$,

$$\begin{aligned} a_1^p &= f_1, \\ &\vdots \\ a_s^p &= f_s, \\ a_{s+1}^p &= e_{s+1}, \\ &\vdots \\ a_m^p &= e_m, \end{aligned}$$

where $f_1, \ldots, f_s$ *are linearly independent modulo* pF, *and* $e_{s+1}, \ldots, e_m$ *lie in T.*

Proof. By (3.3) the mapping $xC \mapsto x^p + (pF+T)$ is a homomorphism of Q onto $C/(pF+T)$. We choose $a_1, \ldots, a_s$ to be preimages of a basis of $C/(pF+T)$. Then $a_i^p = f_i + e_i$ where the f_i's are linearly independent modulo pF and each e_i lies in T. We enlarge $\{a_1, \ldots, a_s\}$ to a basis $\{a_1, \ldots, a_m\}$ of Q. Then $a_i^p = e_i$ lies in T for all $i>s$. Now we enlarge $\{f_1+e_1, \ldots, f_s+e_s\}$ to a basis B of $C/(pF+T)$. Replacing F by $\langle B, pF\rangle$ we may assume that $C = F \oplus T$ and that:

$$a_i^p = f_i, \quad i \leq s,$$
$$a_i^p = e_i, \quad i > s,$$

where the f_i's are linearly independent in F/pF.

Now suppose that $s=0$. Then F is a direct factor of G and the automorphism inverting F extends to an automorphism of G of even order which is a contradiction.

If $s=m$, the mapping that inverts each a_i and each element of F, and fixes each element of T extends to an automorphism of G of order 2. This contradiction completes the proof.

Lemma 5.2. *Suppose that* Aut G *is an elementary abelian p-group. If T is a p-group, then it is itself an elementary abelian p-group.*

Proof. We choose generators for G according to the previous lemma. Suppose that t is an element of order p^2 in T. We define an automorphism γ on G by letting

$$a_1\gamma = a_1 t,$$
$$a_i\gamma = a_i, \quad i \neq 1,$$

$$x\gamma = x, \quad x \in T,$$
$$f_1\gamma = f_1 t^p,$$
$$f_i\gamma = f_i, \quad i \neq 1.$$

This may be done since the f_i's are linearly independent modulo pF. Since $a_1\gamma^p = a_1 t^p \neq a_1$, this automorphism is not of order p and this contradiction completes the proof.

Now suppose that Aut G is an elementary abelian p-group of rank n for some odd prime p. We will eliminate various possibilities for n by constructing automorphisms of G that violate Lemma 3.2, thus making Aut G non-abelian. By Lemmas 3.1 and 5.2 we may assume in this construction that T is an elementary abelian p-group.

Lemma 5.3. *Suppose that* Aut G *is an elementary abelian p-group of rank n, for some prime* $p>3$. *If T is an elementary abelian p-group, then the rank of T divides n.*

Proof. Again we select generators $a_1, \ldots, a_m$ according to Lemma 5.1, and enlarge $\{f_1, \ldots, f_s\}$ to a basis $\{f_1, \ldots, f_s, \ldots, f_r\}$ of F modulo pF. Let γ be any automorphism of G. Since G is nilpotent, Corollary 5.4 of Robinson [10] implies that F has a finite automorphism group. By the Theorem of Hallett and Hirsch, the automorphisms of F of prime order have order 2 or 3. Since $(\gamma|_C)^p = 1$, the automorphism induced by γ on $C/T = F$ must be the identity. Hence, $f_i\gamma = f_i + f_i\theta$ where $\theta\colon F \to T$ is a homomorphism. Also, by Lemma 3.2 we know that γ must induce the identity on $Q = G/C$. Therefore, for each g in G there are elements f in F and t in T such that $g\gamma = gft$. Since $g(\gamma^p) = g \equiv gf^p$ (modulo T) it follows that $f = 1$. Consequently, $g\gamma = gt$. Thus we may write $a_i\gamma = a_i t_i$, for some t_i in T.

Now suppose that every automorphism of G fixes T pointwise. Applying γ to the power relations of G we see that

$$(a_i^p)\gamma = f_i\gamma = f_i + f_i\theta = (a_i\gamma)^p = a_i^p t_i^p = f_i, \quad \text{for } i \leqq s.$$

Therefore, $f_i\theta = 0$ for $i \leqq s$. For $i > s$, $f_i\theta$ may be chosen arbitrarily in T since $\{f_1, \ldots, f_r\}$ is a basis for F modulo p and since there are no more relations involving the f_i that could obstruct the construction of such homomorphisms. This means that Aut G can be generated by automorphisms of the following type:

$$\begin{aligned} &a_i \mapsto a_i t_i, \quad \text{some } t_i \text{ in } T,\\ &c \mapsto c, \quad \text{for all } c \text{ in } C, \end{aligned} \tag{5.1}$$

$$\begin{aligned} &a_i \mapsto a_i,\\ &t \mapsto t, \quad t \text{ in } T,\\ &f_i \mapsto f_i, \quad i \leqq s,\\ &f_i \mapsto f_i + f_i\theta, \quad i > s. \end{aligned} \tag{5.2}$$

Automorphisms of type (5.1) form a group of automorphisms of G isomorphic with $\mathrm{Hom}(Q, T)$. The automorphisms of type (5.2) form a complement in Aut G.

Therefore, the rank of Aut G is (rank T) (rank $Q+r-s$) and the result is proven in this case.

We will now show that every automorphism of G leaves T pointwise fixed. For suppose there is an element x which lies in T but not in $T\cap(G'G^p)$. Then $\langle x\rangle$ is a summand of G and therefore G has an automorphism of order 2. Thus we may assume $T\subseteq G'G^p$. By Lemma 3.2, elements of G' are fixed by every automorphism of G. Also, for any g in G and any automorphism γ, it follows that $g^p\gamma=(g\gamma)^p=(g t)^p=g^p t^p=g^p$. Thus, G^p is fixed elementwise by γ, and hence so is T. This completes the proof.

This Lemma would eliminate many possibilities for the rank of Aut G if we could prove that the rank of T is not 1. We do this in the next result.

Lemma 5.4. *If* Aut G *is a finite abelian group of odd order and both Q and T are elementary abelian p-groups, then T and G' are not cyclic.*

Proof. Suppose to the contrary that such a G exists with cyclic T. We select generators for G according to Lemma 5.1, that is, we select generators $a_1,\ldots,a_m$ for G modulo C such that

$$\begin{aligned}
&a_1^p=f_1,\\
&\vdots\\
&a_s^p=f_s,\\
&a_{s+1}^p=e_{s+1},\\
&\vdots\\
&a_m^p=e_m,
\end{aligned}$$

where $f_1,\ldots,f_s$ are linearly independent elements in F modulo pF, and e_i lies in T. Note that $0<s<m$.

Suppose first that a_m commutes with $a_{s+1},\ldots,a_{m-1}$. Since a_m does not lie in the center of G, we may assume that a_1 and a_m do not commute. Thus, $c=[a_1,a_m]\neq 1$. Since T is cyclic, it must be cyclic of order p and hence $G'=T$. Thus, c generates G' and consequently for each i, $[a_i,a_m]=c^{r(i)}$ for some $r(i)$. For $i=2,3,\ldots,s$ we replace a_i by $a_i a_1^{-r(i)}$. Since $f_1,\ldots,f_s$ are independent modulo pF, so are $f_1,f_2-r(2)f_1,\ldots,f_s-r(s)f_1$. Also,

$$[a_i a_1^{-r(i)},a_m]=[a_i,a_m][a_1,a_m]^{-r(i)}=1.$$

Hence, we may assume in addition to the power relations that a_m commutes with all a_i except a_1. We define a map on G as follows:

$$\begin{aligned}
&a_1\gamma=a_1 a_m,\\
&a_i\gamma=a_i, \quad i\neq 1,\\
&f_1\gamma=f_1+e_m,\\
&f_i\gamma=f_i, \quad i\neq 1,\\
&t\gamma=t, \quad \text{for all } t \text{ in } T.
\end{aligned}$$

It may easily be checked that this gives rise to an automorphism of G. By Lemma 3.2, Aut G is not abelian, which is a contradiction.

If $e_i=0$ for all $i>s$, then the mapping inverting each a_i and each element of F, and fixing each element of T extends to an automorphism of even order. Thus, we may assume that $a_m^p=e_m\neq 0$ and that a_m does not commute with one of $a_{s+1},\ldots,a_{m-1}$, let us say with a_{m-1}. Now if $e_{m-1}=s\,e_m$ we replace a_{m-1} by $a_{m-1}a_m^{-s}$ and assume without loss of generality that $e_{m-1}=0$. Since $c=[a_{m-1},a_m]$ generates G', we have $[a_{m-1},a_i]=r(i)\,c$, for $i<m-1$. Replacing a_i by $a_i a_m^{-r(i)}$ we may assume that a_{m-1} commutes with all a_i except a_m. We define an automorphism on G as follows:

$$a_i\gamma=a_i,\qquad i\neq m,$$

$$a_m\gamma=a_m a_{m-1},$$

$$c\gamma=c,\qquad \text{for all } c \text{ in } C.$$

Again, Lemma 3.2 implies that Aut G is not abelian and the proof is complete.

Corollary 5.5. *Suppose that* Aut G *is an elementary abelian p-group for some prime* $p>3$. *Then the rank of* Aut G *is not prime.*

Proof. Suppose that Aut G is an elementary abelian p-group of rank n. By Lemmas 3.1 and 5.2 we may assume that T is an elementary abelian p-group also. From the proof of Lemma 5.3 we have

$$\operatorname{rank}(\operatorname{Aut} G)=(\operatorname{rank} T)(\operatorname{rank} Q+r-s)$$

where $r-s>0$. Since T and Q are not cyclic, Aut G is not of prime rank and the proof is complete.

This Corollary places restrictions on the rank of an automorphism group that is an elementary abelian p-group, $p>3$. In the next section we will discover two further restrictions ($n>7$, $n\neq 9$). For $p=3$ we show in the next section that $n>7$. It will be shown elsewhere that there are no other restrictions.

Corollary 5.6. *Suppose that* Aut G *is an elementary abelian p-group for any prime p. Then G has an outer automorphism.*

Proof. If $p=2$, then this result is Lemma 4.1. If p is odd, then we may assume that T is an elementary abelian p-group of rank at least 2. Since the rank of $Q\simeq\operatorname{Inn} G$ is at least 2, this means $|Q|=|\operatorname{Inn} G|<|\operatorname{Hom}(Q,T)|$. The embedding (3.1) now implies that G has an outer automorphism and the proof is complete.

6. Inn G and T Elementary Abelian p-groups of Small Rank, p Odd

If Inn G and T are elementary abelian p-groups of small rank, then it is possible to construct automorphisms that do not commute with Inn G. Here "small rank" means no larger than 3. If Aut G were an elementary abelian p-group of rank $n<8$, then embedding (3.1) implies that Q and T are of small rank, and hence Aut G is not abelian. For $n=9$ and $p>3$, Lemmas 5.3, 5.4, and 6.1 imply that both $Q\simeq\operatorname{Inn} G$ and T have rank 3 and again Aut G is not abelian. The task of proving these results is divided into several lemmas.

Lemma 6.1. *Suppose that* Aut G *is a finite group of odd order and that Q and T are elementary abelian p-groups. Then the rank of Q is not* 2.

Proof. Suppose to the contrary that the rank of Q is 2. By Lemma 5.1 we may assume that G is generated by C, a_1, and a_2 such that

$$a_1^p = f_1,$$
$$a_2^p = e_2,$$

where f_1 is not trivial modulo pF. We define an automorphism of G as follows:

$$a_1\gamma = a_1,$$
$$a_2\gamma = a_2^{-1},$$
$$t\gamma = -t, \qquad \text{for } t \text{ in } T,$$
$$f_1\gamma = f_1.$$

However, γ is of even order and this contradiction completes the proof.

Corollary 6.2. *Suppose that* Aut G *is an elementary abelian p-group for some odd prime p. Then the rank of* Aut G *is not* 1, 2, 3, 4, *or* 5.

Proof. We may assume that T is an elementary abelian p-group of rank n. By Lemma 5.4 we know that $n \geqq 2$. By the previous Lemma we know that the rank m of Q is at least 3. From embedding (3.1), we have that $6 \leqq mn \leqq \text{rank}(\text{Aut}\, G)$. This completes the proof.

If Aut G is an elementary abelian p-group, the remaining ranks to be excluded for Aut G are 6, 7, and (for $p > 3$) 9. To do this we consider the case in which $Q \simeq \text{Inn}\, G$ and T are elementary abelian p-groups, and Q is of rank 3. In addition we assume that Aut G is a finite abelian group of odd order. By Lemma 5.1, G may be generated by C, a_1, a_2, and a_3 such that one of the following holds:

$$a_1^p = f_1,$$
$$a_2^p = f_2, \tag{6.1}$$
$$a_3^p = e_3,$$

where f_1 and f_2 are linearly independent in F, and e_i lies in T, or

$$a_1^p = f_1,$$
$$a_2^p = e_2, \tag{6.2}$$
$$a_3^p = e_3,$$

where f_1 is not trivial modulo pF, and e_i lies in T. The following result is very useful.

Lemma 6.3. *Suppose that Q and T are elementary abelian p-groups of rank less than or equal to* 3. *Then for all x, y, z, w in G there is some g such that $[x, y]$ and $[z, w]$ both lie in $[g, G]$.*

This follows immediately from the relations (3.2) and from the fact that G' has rank at most 3. It also follows that every element of G' is a commutator.

Suppose now that the rank of G' is 3. If we are in the situation of (6.1), that is, if a_3 is of finite order but a_1 and a_2 are not, then it follows that $a_3^p=[r,s]$ for some r,s in G. If $\langle r,s,a_3,C\rangle=G$, then we may choose $a_1=r$ and $a_2=s$. Then the mapping inverting a_1 and a_2 and fixing a_3 (which we will briefly denote as $(-1, -1, 1)$) extends to an automorphism of G. By Lemma 3.2, Aut G is not abelian. Now suppose that $\langle r,s,a_3,C\rangle\neq G$. If r and s are both of finite order, then $a_3^p=[r,s]=1$ since the torsion subgroup of G is abelian. In this case the same mapping as above extends to an automorphism of G. Thus, we may assume that at least one of r and s has infinite order and may be chosen as a_1. We then choose a_2 accordingly. Thus, $a_3^p=[a_3,g]$ for some g. The mapping $(1, 1, -1)$ extends to an automorphism of G.

Now suppose that (6.2) holds, that is, suppose that a_2 and a_3 both are of finite order. Then $G^p\cap T$ is of rank at most 2 and is contained in some $[x,G]$. If x is of infinite order we choose it to be a_1, and then $(1,-1,-1)$ extends to an automorphism of G. If x is of finite order we choose it to be a_2. If $a_2^p=1$, then the mapping $(-1,1,-1)$ extends to an automorphism of G. If $a_2^p\neq 1$, then $a_2^p=[a_2,y]$ for some y in G. Suppose that y is of finite order. Then we may choose it as a_3. If $a_3^p=[a_2,z]$ and z is of finite order, it follows that $z=a_3^r$ for some r. Replacing a_3 by $a_3a_2^{-r}$, we may assume without loss of generality that $a_3^p=1$. If z is of infinite order we choose it as a_1. In either case the mapping $(-1, -1, 1)$ extends to an automorphism of G. The only case left to consider is that in which y is of infinite order. For this we choose y to be a_1. It is then possible to choose a_3 so that $a_3^p=[a_2,a_3]^r$ for some r. The mapping which sends a_1 to a_1a_2 and fixes a_2 and a_3 now extends to an automorphism of G.

Let us now consider the case in which G' has rank 2. In this case it is always possible to choose a basis for G modulo C such that one of the commutators $[a_1,a_2]$, $[a_2,a_3]$, or $[a_1,a_3]$ vanishes. Let us first consider the case in which $[a_1,a_3]=1$ and both a_1 and a_2 are of infinite order. Then the mapping fixing a_1 and a_3 and sending a_2 to a_2a_3 extends to an automorphism of G. If $[a_1,a_2]=1$ and a_1 and a_2 are of infinite order, then $a_3^p=[a_3,a_1^ra_2^s]\,t$ for some r and s where t is in some complement of G' in T. In this case $(1,1,-1)$ extends to an automorphism of G.

Now suppose that only a_1 is of infinite order and that $[a_1,a_2]=1$. Now $T=G'\oplus C_p$ and $a_2^p=[a_3,a_1^ra_2^s]\,h$, $a_3^p=[a_3,a_1^ta_2^u]\,k$ where h and k lie in C_p. Either $h=1$ or we may replace a_3 by $a_3a_2^w$ for an appropriately chosen w and assume that $k=1$. Suppose that p divides r. If p also divides t, then the mapping $(-1,1,1)$ extends to an automorphism of G. If p does not divide t, then we replace a_1 by $a_1^ta_2^u$. The mapping $(1,-1,1)$ extends to an automorphism. If p does not divide r, we take $a_1^ra_2^s$ as a_1. Replacing a_3 by $a_3a_2^{-t}$ we may assume that $a_2^p=[a_3,a_1]\,h$ and $a_3^p=[a_3,a_2^u]\,k$. If $h=1$, the mapping $(-1, 1, -1)$ extends to an automorphism of G. If $h\neq 1$, this mapping also extends (on $T=G'\oplus C_p$ we send $[a_3,a_1]$ to $[a_3,a_1]\,h^2$, $[a_3,a_2]$ to $[a_3,a_2]$, and invert each element of C_p). The only case remaining is that in which $[a_2,a_3]=1$. Then $(-1, 1, 1)$ extends to an automorphism of G. This covers all possible cases. We have proven the following:

Lemma 6.4. *If $Q = G/C$ and T are elementary abelian p-groups of rank at most* 3 *for some odd prime p, then* Aut G *is not abelian.*

7. Summary

We will now prove our main result.

Proposition 7.1. *Suppose that G is an infinite non-abelian group and that* Aut G *is an elementary abelian p-group of rank n.*

(i) *If* $p=2$, *then* $n \geqq 3$.
(ii) *If* $p=3$, *then* $n \geqq 8$.
(iii) *If* $p>3$, *then* $n=8$, *or* $n \geqq 10$ *and n is composite.*

Proof. Suppose $p=2$. Then this result is Corollary 4.2.

Suppose that p is odd. By Lemmas 3.1 and 5.2, we may assume that T, the torsion subgroup of the center of G, is an elementary abelian p-group. By Lemmas 5.4 and 6.1 we may assume that $\operatorname{rank}(T)>1$ and $\operatorname{rank}(Q)>2$. By Corollary 6.2 we know that the rank of Aut G is at least 6. If Aut G has rank 6 or 7, then the embedding (3.1) implies that $\operatorname{rank}(T)=2$ and $\operatorname{rank}(Q)=3$. This contradicts Lemma 6.4.

Now suppose that $p>3$. Then Corollary 5.5 implies that the rank of Aut G is composite. Suppose that Aut G is of rank 9. Then Q and T must both be of rank 3. This contradicts Lemma 6.4 and completes the proof.

Remark. The conditions on p and n above are also sufficient to ensure the existence of an infinite non-abelian group whose automorphism group is an elementary abelian p-group of rank n.

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