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A Relation Between Growth and the Spectrum of the Laplacian

Robert Brooks*

Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742, U.S.A.

1. Introduction

Let M be a smooth, complete, non-compact Riemannian manifold, and denote by Δ the Laplace-Beltrami operator on $L^2(M)$, with sign chosen so that it is a positive operator.

We denote by λ_0 the greatest lower bound of the spectrum of Δ. It is given by the variational formula

$$\lambda_0 = \inf_f \frac{\int_M f \cdot \Delta f}{\int_M f^2}$$

where f runs over smooth functions with compact support on M.

Actually, writing $\int_M f \cdot \Delta f = \int_M \|\operatorname{grad} f\|^2$, we may weaken the smoothness assumption on f. It is standard that we may take f to be uniformly Lipschitz.

A somewhat more interesting invariant of M, which we denote by λ_0^{ess}, is the greatest lower bound of the essential spectrum of Δ, where the essential spectrum consists of points of the spectrum of Δ which are either accumulation points of points on the spectrum or which correspond to discrete eigenvalues of Δ with infinite multiplicity. It is classical that if M is compact, the essential spectrum is empty. We also clearly have that $\lambda_0 \leqq \lambda_0^{\text{ess}}$, and that $\lambda_0^{\text{ess}}(M) = \lim_K \lambda_0(M-K)$, where K runs through all compact subsets of M, see [7] or [8].

The object of this paper is to present estimates for λ_0^{ess}. To state our main result, we pick a point $x_0 \in M$, and for each $r>0$, we denote by $B(r)$ the ball of radius r, and $V(r)$ the volume if this ball. Then we set

$$\mu = \lim_{r\to\infty} \sup \frac{1}{r} \log(V(r));$$

μ is the exponential growth of M.

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One checks by a simple use of the triangle inequality that μ is independent of the choice of x_0. Furthermore, under simple geometric assumptions (for instance, if the Ricci curvature is bounded below by some negative number), one has from standard comparison arguments [2] that μ is finite.

Theorem 1. *If the volume of M is infinite, then $\lambda_0^{\text{ess}} \leqq \frac{1}{4}\mu^2$.*

Theorem 1 is a rather simple consequence of Agmon's work on exponential decay of harmonic functions, a simplified and somewhat disguised account of which we present as Theorem 2 below. He related his results to us in connection with Corollary 1 below.

We remark that if M has finite volume, then trivially $\lambda_0 = 0$. Theorem 1 leaves open the possibility that, for finite volume, M may have discrete spectrum. Such M have been constructed in [8].

We also remark that μ is a rather simple invariant of the metric, and is readily computable in many cases. Furthermore, it is easily estimated in terms of the curvature of M, by the standard comparison theorems [2]. In addition, if M arises as the Riemannian universal cover of a compact manifold M', then a lemma of Milnor [10] allows one to compute μ in terms of group theory and simple geometric properties of M' (see Corollary 5 below).

Theorem 1 gives a sharp estimate when M is a simply connected rank 1 symmetric space of non-compact type. This can be verified by looking at standard tables computing these constants – see, for example, [1]. A conceptual explanation of why this inequality is sharp is given in Corollary 2 below, where we interpret Theorem 1 as an "isoperimetric inequality" of a rather special sort.

We also present some estimates for λ_0^{ess} from below, under some geometrically restrictive assumptions. The basic techniques for these lower estimates are, first of all, the decomposition principle of Donnelly-Li [8], and, secondly, the lower estimate of Cheeger [5]. Taken together with Theorem 1, these allow one to compute λ_0^{ess} under quite general circumstances.

On the other hand, Theorem 1 may easily fail to be sharp. For instance, let π be an amenable (for instance, solvable) group with exponential growth. Such groups are easily constructed, and one such group is given in [10]. Then if M_0 is any compact manifold with $\pi_1(M_0) = \pi$, and M is the Riemannian universal cover of M_0, a lemma of Milnor [10] insures that $\mu > 0$, while the theorem of [4] provides that $\lambda_0^{\text{ess}} = 0$. The obstruction to the inequality of Theorem 1 being sharp is that in general the "distance spheres" of M need not be the most efficient candidates for isoperimetric inequalities in M.

The problem of estimating λ_0 and λ_0^{ess} for non-compact manifolds has been considered by Pinsky [11, 12] for surfaces, and by Cheng [6] and Donnelly and Li ([7] and [8]) for general manifolds. We recover their estimates for λ_0^{ess} here by using the Comparison Theorem [2] to estimates for λ_0^{ess} here by using the Comparison Theorem [2] to estimate μ, with the exception that, when M is a surface, [11] and [12] contain estimates of λ_0^{ess} in terms of the behavior of the metric in a sector of M. We do not pursue this question here.

See also [14] for related results.

See [4] for an interpretation of λ_0 in terms of random walks.

In § 2 below we present the proof of Theorem 1. § 3 is then devoted to an assortment of consequences of Theorem 1, some of which have been indicated above.

It is a great pleasure to acknowledge the help of Shmuel Agmon, who called my attention to what is here Theorem 2, and to thank Carlos Berenstein for his help and encouragement.

2. Proof of Theorem 1

Theorem 1 will follow from:

Theorem 2. *Let K be a compact (possibly empty) subset of M, and $\lambda_0(M-K)=$ the greatest lower bound of the spectrum of Δ on $L^2(M-K)$, with Dirichlet boundary conditions on ∂K.*

Let $\rho(x)=\rho(x,x_0)$ denote the distance from a fixed point $x_0 \in M$. If

$$\int_{M-K} e^{-2\alpha\rho(x)}dx<\infty, \quad \text{for some } \alpha \text{ satisfying } 0<\alpha<\sqrt{\lambda_0(M-K)}$$

then

$$\int_{M-K} e^{2\alpha\rho(x)}dx<\infty.$$

Proof of Theorem 1 from Theorem 2. We first remark that if $2\alpha>\mu$, then

$$\begin{aligned}\int_M e^{-2\alpha\rho(x)}dx &\leqq \sum_{r=1}^{\infty}[V(r)-V(r-1)]e^{-2\alpha(r-1)}\\ &=\sum_{r=1}^{\infty}V(r)e^{-2\alpha r}[e^{2\alpha}-1]\end{aligned}$$

where the last expression is easily seen to be finite, by comparison with a geometric series, using the fact that $2\alpha>\mu$.

We now conclude from Theorem 2 that if $2\alpha>\mu$ and $\alpha<\sqrt{\lambda_0(M-K)}$, we have $\int_{M-K} e^{2\alpha\rho(x)}dx<\infty$. But this is clearly impossible, since $M-K$ has infinite volume.

Therefore, there is no such α, and we have shown $\lambda_0(M-K)\leqq\frac{1}{4}\mu^2$. Taking the limit over arbitrarily large K, we see that $\lambda_0^{\mathrm{ess}}\leqq\frac{1}{4}\mu^2$, proving Theorem 1.

Proof of Theorem 2. For convenience, we may multiply the metric by a constant to assume $\lambda_0(M-K)=1$, since if $\lambda_0(M-K)=0$, there is nothing to prove.

We consider a test function $f(x)=e^{h(x)}\cdot\chi(x)$, where we assume $\chi(x)$ has compact support in $M-K$. Then

$$\int_{M-K}\|\operatorname{grad} f\|^2=\int_{M-K} e^{2h(x)}[\|dh\cdot\chi+d\chi\|^2]\geqq\int_{M-K} f^2.$$

Thus

$$\begin{aligned}\int_{M-K} f^2[1-\|\operatorname{grad} h\|^2]&\leqq\int_{M-K} e^{2h}[2\chi\cdot\langle dh,d\chi\rangle+\|\operatorname{grad}\chi\|^2]\\ &\leqq\int_{M-K} e^{2h}[2\chi\cdot\|\operatorname{grad} h\|\,\|\operatorname{grad}\chi\|+\|\operatorname{grad}\chi\|^2].\end{aligned}$$

We now suppose $\|\operatorname{grad} h\| \leqq \alpha < 1$, and, for some increasing sequence K_i of compact subsets of $M-K$ such that $\bigcup K_i = M-K$, we set

$$\begin{aligned} \chi_i(x) &= 0 \quad \text{if } x \in (M-K)-K_i \\ &= \frac{1}{d}\rho(x, M-K_i) \quad \text{if } 0 \leqq \rho(x, M-K_i) \leqq d \\ &= 1 \quad \text{if } \rho(x, M-K_i) \geqq d \end{aligned}$$

where d is fixed for the discussion.

Then $\|\operatorname{grad} \chi_i\| \leqq 1/d$, and $\operatorname{grad}(\chi_i)$ is supported in a neighborhood $B_d(\partial K_i)$ of radius d about ∂K_i. Thus

$$\int_{M-K} f^2[1-\alpha^2] \leqq \left(\frac{2}{d}+\frac{1}{d^2}\right) \int_{B_d(\partial K_i)} e^{2h}$$

Taking limits as $i \to \infty$, and assuming that $\int_{M-K} e^{2h}$ is integrable, we get

$$\int_{M-K} e^{2h}[1-\alpha^2] \leqq \left(\frac{2}{d}+\frac{1}{d^2}\right) \int_{B_d(\partial K)} e^{2h}$$

Under the assumption $\int_{M-K} e^{-2\alpha\rho(x)} < \infty$, we may set

$$h_j(x) = \min[\alpha\rho(x), -\alpha\rho(x)+j].$$

Note that, for each j, $\|\operatorname{grad} h_j\| \leqq \alpha$, and e^{2h_j} is integrable for all j. Note also that h_j increases pointwise to $h = \alpha\rho(x)$. Thus, for j sufficiently large, we have

$$\int_{M-K} e^{2h_j}[1-\alpha^2] \leqq \left(\frac{2}{d}+\frac{1}{d^2}\right) \int_{B_d(\partial K)} e^{2\alpha\rho(x)}$$

so

$$\int_{M-K} e^{2h_j} \leqq (\text{const})$$

where (const) is a finite constant independent of j. Taking the limit as $j \to \infty$ gives

$$\int_{M-K} e^{2\alpha\rho(x)} \leqq (\text{const})$$

which is the conclusion of Theorem 2.

3. Examples and Consequences

We now turn to some consequences of Theorem 1.

Corollary 1. *If M has subexponential growth, then $\lambda_0^{\text{ess}}(M) = 0$.*

Proof. The condition that M has subexponential growth is precisely the condition that $\mu = 0$.

We remark that this result was proved in [3] under some technical assumptions on M.

To give the next corollary, we introduce the isoperimetric constant h, following Cheeger [5]:

$$h = \inf_N \frac{\text{area}(N)}{\text{vol}(\text{int} N)},$$

where N runs over all compact $(n-1)$-dimensional submanifolds of M dividing M into two components, and $\text{int}(N)$ denotes the bounded component.

Corollary 2. $h \leqq \mu$. *Furthermore, equality holds if the ratio* $\frac{S(r)}{V(r)}$ *tends to* h *as* $r \to \infty$, *where* $S(r)$ *denotes the surface area of the distance sphere of radius* r. *In this case,* $\lambda_0 = \lambda_0^{\text{ess}} = \frac{1}{4}\mu^2$.

Proof. The inequality holds since, according to Cheeger's inequality [5], we have $\lambda_0 \geqq \frac{1}{4}h^2$, and $\frac{1}{4}\mu^2$ is an upper bound for λ_0 by Theorem 1.

An alternate proof of this inequality is obtained by noting that $S(r) = V'(r)$. Thus the differential inequality $\frac{V'(r)}{V(r)} \geqq h$ integrates to give $V(r) \geqq (\text{const}) \cdot e^{hr}$, establishing the inequality.

On the other hand, an inequality of the form $\frac{S(r)}{V(r)} < h + \varepsilon$ for r sufficiently large gives $\mu \leqq h + \varepsilon$, establishing the second statement.

The third statement is then obvious.

We may extend this notion in the following way: if K is any smooth, compact submanifold of M, of the same dimension as M, we set

$$h^K = \inf_N \frac{\text{area}(N)}{\text{vol}(\text{int}(N))}$$

where N runs over compact submanifolds of $M-K$ dividing $M-K$ into a compact component, $\text{int}(N)$, which does not meet ∂K, and other components containing ∂K and the unbounded part of M.

According to Cheeger [5] again, $\frac{1}{4}(h^K)^2$ represents a lower bound to the spectrum of the Laplacian on $M-K$ with Dirichlet boundary conditions on ∂K, while according to the Decomposition Principle of Donnelly and Li [8], the essential spectrum of the Laplacian of M agrees with the essential spectrum of the Laplacian of $M-K$ with Dirichlet boundary conditions on ∂K. Setting h^{ess} to be the supremum of h^K for all K, we get

Corollary 3. $\frac{1}{4}(h^{\text{ess}})^2 \leqq \lambda_0^{\text{ess}} \leqq \frac{1}{4}\mu^2$.

In general, it is about as hard to estimate h and h^{ess} as it is to estimate λ_0 and λ_0^{ess}. The following device provides an important exception:

Assume that for some point $x_0 \in M$, the exponential map $\exp\colon T_{x_0}(M) \to M$ is a diffeomorphism; in particular, M is diffeomorphic to $\mathbb{R}^n$. Denote by θ the function on $T_{x_0}(M) \cong \mathbb{R}^n$ such that $\theta \cdot dx_1 \wedge \ldots \wedge dx_n$ is the induced volume form on $T_{x_0}(M)$.

Lemma. *If* $\frac{\partial\theta}{\partial r} \geqq 0$, *then* $h^{\mathrm{ess}} \geqq \lim_{r\to\infty} \inf_{d(x,x_0)>r} \frac{1}{\theta}\frac{\partial\theta}{\partial r}(x)$.

Proof (see Yau's proof [13] of McKean's Theorem [9]). It is well-known that

$$-\Delta r = \frac{1}{\theta}\frac{\partial\theta}{\partial r} + \frac{(n-1)}{r}.$$

For any K, if $\frac{1}{\theta}\frac{\partial\theta}{\partial r} \geqq c$ on $M-K$, we may integrate the above expression over $\operatorname{int}(N)$ to get

$$\int_{\operatorname{int}(N)} \Delta r \geqq c \operatorname{vol}(\operatorname{int}(N)).$$

On the other hand, by Stokes' Theorem,

$$\int_{\operatorname{int}(N)} \Delta r = \int_N {*dr} \leqq \operatorname{area}(N)$$

where the inequality comes from $|{*dr}| \leqq 1$. Letting K be arbitrarily large establishes the lemma.

Corollary 4. *Under the assumptions of the Lemma, we have*

$$\frac{1}{4}\left(\lim_{r\to\infty} \inf_{d(x,x_0)\geqq r} \frac{1}{\theta}\frac{\partial\theta}{\partial r}(x)\right)^2 \leq \lambda_0^{\mathrm{ess}} \leqq \frac{1}{4}\left(\lim_{r\to\infty} \sup_{d(x,x_0)\geqq r} \frac{1}{\theta}\frac{\partial\theta}{\partial r}(x)\right)^2.$$

Proof. The left-hand inequality follows from the Lemma, together with Corollary 3. The right-hand side follows from the obvious inequality

$$\lim_{r\to\infty} \sup_{d(x,x_0)\geqq r} \frac{1}{\theta}\frac{\partial\theta}{\partial r}(x) \geqq \mu.$$

It is an easy matter to construct examples where both sides agree, giving a sharp computation of λ_0^{ess}. We note also that it is standard to deduce the conditions of the Lemma as well as bounds for $\frac{1}{\theta}\frac{\partial\theta}{\partial r}$ from curvature conditions on M via the Comparison Theorem [2].

We remark that the lower bound of Corollary 4 may be proved equally well from the integration-by-parts argument of [11].

To state our final corollary, let M_0 be a compact manifold, and M its universal cover. We pick a fundamental domain F for the action of $\pi_1(M_0)$ on M – we may think of F as arising from "cutting M_0 open", and then lifting this cut-open M_0 to M.

Then $\{g \cdot F : g \in \pi_1(M_0)\}$ is a tiling of M. Let $g_1, \ldots, g_k$ denote the finitely many elements of $\pi_1(M_0)$ such that $g_i \cdot F$ adjoins F. It is clear that $g_1, \ldots, g_k$ generate $\pi_1(M_0)$, and that if $g_i \in \{g_1, \ldots, g_k\}$, then $g_i^{-1} \in \{g_1, \ldots, g_k\}$.

For each ε, let $N(\varepsilon)$ denote the maximal number of translates of F which meet a ball of radius ε.

Corollary 5. $\lambda_0^{\mathrm{ess}} \leqq \frac{1}{4}\left(\frac{N(\varepsilon)}{2\varepsilon} \cdot \log(k-1)\right)^2$.

Proof. According to Theorem 1, it suffices to show that

$$\mu \leqq \frac{N(\varepsilon)}{2\varepsilon} \cdot \log(k-1).$$

We first remark that the exponential growth of $\pi_1(M_0)$, computed with respect to the generators $g_1, \ldots, g_k$, is at most $\log(k-1)$, as can be seen by noting that there are at most $(k)\cdot(k-1)^{n-1}$ distinct words in $g_1, \ldots, g_k$ of length n.

The remainder of the proof of Corollary 5 now mimics the argument of Milnor [10]. We claim that if $x_0 \in F$ is fixed, and $g \in \pi_1(M)$ is given, then

$$\ell(g) \leqq N(\varepsilon)\left(\left[\frac{d(x_0, g(x_0))}{2\varepsilon}\right]+1\right)$$

where $\ell(g)$ is the length of g in the word metric given by $g_1, \ldots, g_k$.

But joining x_0 to $g(x_0)$ by a minimal geodesic γ, we cover γ by $\left[\frac{d(x_0, g(x_0))}{2\varepsilon}\right]+1$ balls of radius ε. Each one of these balls meets at most $N(\varepsilon)$ copies of F. We may write $g=\prod g_i$, where we introduce one g_i whenever we cross from one copy of F to another, which establishes our claim.

The estimate $\mu \leqq \left(\frac{N(\varepsilon)}{2\varepsilon}\right)\cdot\log(k-1)$ now follows from the definitions, since given $\delta>0$, we may find r sufficiently large so that

$$\begin{aligned} V(r) &\leqq \operatorname{vol}(F)\cdot \#(g: d(x_0, g(x_0))<r+D) \\ &\leqq \operatorname{vol}(F)\cdot \#\left(g: \ell(g) \leqq \left(\frac{N(\varepsilon)}{2\varepsilon}\right)\left[(1+\delta)(r+D)+\frac{1}{N(\varepsilon)}\right]\right. \\ &\leqq \operatorname{vol}(F)\cdot(\text{const})\, e^{\frac{N(\varepsilon)}{2\varepsilon}\log(k-1)(1+\delta)r}, \end{aligned}$$

where D is the diameter of M, so that letting $\delta \to 0$ and $r \to \infty$ gives $\mu \leqq \frac{N(\varepsilon)}{2\varepsilon} \cdot \log(k-1)$, as desired.

We remark that as $\varepsilon \to 0$, $N(\varepsilon)$ remains bounded, and as $\varepsilon \to \infty$, $N(\varepsilon)$ grows exponentially in ε if M has exponential growth. It appears that an optimal value of ε would seem to be at about half the injectivity radius of M_0. For this value of ε, $N(\varepsilon)$ measures how many fundamental domains meet at a vertex of F, together with how small the various faces of F are.

The formula of Corollary 5 gives qualitative expression to the following principle: if M is a fixed space with $\lambda_0(M)>0$, and if M_0 is a compact quotient of M by a groups of isometries, and if the injectivity radius of M_0 is large, then M_0 must be "complicated", e.g., the fundamental group of M_0 is large.

We will make this principle more precise at a later opportunity.

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