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CONFORMALLY INVARIANT VARIATIONAL INTEGRALS AND THE REMOVABILITY OF ISOLATED SINGULARITIES

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We analyze the structure of two-dimensional variational integrals which are invariant under conformal mappings of the parameter domain. This allows us to prove that classical solutions of the corresponding Euler equations cannot have isolated singularities if their Dirichlet integral is finite.

1. Introduction

In the calculus of variations one typically considers variational problems of the following kind:

$$\text{Minimize } u \mapsto I(u) = \int_\Omega F(x,u(x),Du(x))\,dx!$$

Here $\Omega \subset \mathbb{R}^n$ is an open set and $u : \Omega \to \mathbb{R}^N$ $(n,N \in \mathbb{N})$ is supposed to vary in some class of functions. If F satisfies

$$m_1|p|^2 - k_o \le F(x,u,p) \le m_2|p|^2 + k_o$$

for constants $o < m_1 \le m_2$, $k_o \ge o$, almost all $x \in \Omega$ and all $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$, one calls F an *integrand of quadratic growth.*

In the following we shall restrict our attention to the case $n = 2$ and $N > 1$. This implies that the Euler equations

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lead to a *system* of partial differential equations. By a fundamental result of Morrey [MCB] minima of such problems in H_2^1 are Hölder continuous. However, if n is large enough there are examples of discontinuous minima [GM]. So one might ask the question:
Are stationary points of the variational problem, that is weak solutions to the corresponding Euler equations, regular? An example by Frehse [F] however shows that this is not true in general.
So one is led to consider a narrower class of variational integrals.
Now the geometric origin of many variational problems suggests the investigation of those functionals which are *invariant under conformal mappings* of the parameter domain. And in fact it seems to be a reasonable *conjecture* that *stationary points of such integrals are regular.* For a further discussion in this direction the reader is referred to the elegant notes by Hildebrandt [H].
In this paper we shall try to give supporting evidence to this conjecture.
In section two we analyze the structure of integrands F leading to conformally invariant variational integrals. As a corollary of Theorem 1 we show that one can introduce a new Riemannian metric on the target space $\mathbb{R}^N$ such that the variational integral may be considered as an H-surface functional. This shows that the conformal invariance forces the problem to have a geometric interpretation. In particular regularity results for surfaces of bounded mean curvature as in [GM4] may be applied.

In section three we prove the removability of isolated singularities of classical solutions to the Euler equations provided the Dirichlet integral is finite. The proof is a generalization of a method by Sacks and Uhlenbeck [SU], where they apply it to harmonic mappings on two-dimensional manifolds. An example in [HKW] shows that this result cannot be extended to dimensions $n \geq 3$.

The results presented here are taken from [GM1], [GM2], [GM3]. Although the conjecture mentioned above still is open, we think that our paper may be regarded as a small contribution to the problem of solving the regularity question.

Finally let me thank Stefan Hildebrandt for his encouragement and many fruitful discussions.

2. The structure of variational integrals which are conformally invariant

At first let us fix the notation and specify our assumptions.
Let $N \in \mathbb{N}$ and suppose that the function

$$F : \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R}$$

satisfies the inequalities

$$(2.1) \quad m_1 |p|^2 \le F(u,p) \le m_2 |p|^2$$

for every $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{2N}$ with constants $o < m_1 \le m_2$.
Concerning the regularity of F we assume

$$F(\cdot,p) \in C^1(\mathbb{R}^N), \; F(u,\cdot) \in C^2(\mathbb{R}^{2N})$$

for any $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{2N}$.

If $\Omega \subset \mathbb{R}^2$ is an open set we denote by $H_2^1(\Omega,\mathbb{R}^N)$ the well known Sobolev space of L^2-functions on Ω the distribution derivatives of which are also square integrable.

Assuming that $F(u,Du)$ is measurable for any $u \in H_2^1(\Omega,\mathbb{R}^N)$ we may define

$$I(u) := \int_\Omega F(u(x),Du(x))dx.$$

In the following we shall restrict our attention to variational integrals which are *conformally invariant*. More precisely we suppose:
If $\tilde{\Omega},\Omega \subset \mathbb{R}^2$ are open and $\Phi : \tilde{\Omega} \to \Omega$ is a diffeomorphism which

is conformal, we have

$$I(u) = I(u \circ \Phi)$$

for any $u \in H_2^1(\Omega, \mathbb{R}^N)$.

An easy calculation shows that this condition implies for any $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{2N}$ and any $y \in \tilde{\Omega}$

$$(2.2) \quad F(u, pD\Phi(y)) = \frac{1}{2}|D\Phi(y)|^2 F(u,p),$$

where $|D\Phi(y)|$ denotes the euclidean norm of $D\Phi(y)$.
Choosing $\Phi(x) = \lambda x$, $\lambda \neq o$, we get

$$F(u, \lambda p) = \lambda^2 F(u,p)$$

and using the regularity of F we conclude

$$(2.3) \quad F(u,p) = \frac{1}{2} F_{p_\alpha^i p_\beta^k}(u,o) p_\alpha^i p_\beta^k .$$

Here and in the sequel we use the summation convention.
Setting

$$A_{\alpha\beta}^{ik}(u) := F_{p_\alpha^i p_\beta^k}(u,o), \quad A^{ik} = (A_{\alpha\beta}^{ik})_{\alpha,\beta=1,2}$$

we have the symmetry relation $A_{\alpha\beta}^{ik} = A_{\beta\alpha}^{ki}$ and may rewrite the integral I as

$$I(u) = \frac{1}{2} \int {}^t\nabla u^i \, A^{ik}(u) \nabla u^k,$$

where ${}^tB$ denotes the transpose of the matrix B. We now exploit the invariance of F with respect to rotations and choose $\Phi(x) = S_\vartheta x$, where $\vartheta \in]-\pi,\pi]$
and

$$S_\vartheta = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} .$$

Using (2.2), (2.3) and the definition of A^{ik} we get

$$p^i A^{ik} \, {}^t(p^k) = p^i [S_\vartheta A^{ik} \, {}^tS_\vartheta] \, {}^t(p^k).$$

The symmetry $A^{ik} = {}^t(A^{ki})$ now implies for $i,k = 1,..,N$
and $u \in \mathbb{R}^N$

$$A^{ik}(u) = S_\vartheta \, A^{ik}(u) \, {}^tS_\vartheta .$$

Inserting $\vartheta = \pi/2$ we conclude that A^{ik} is of the form

$$A^{ik} = \begin{pmatrix} a^{ik} & -b^{ik} \\ b^{ik} & a^{ik} \end{pmatrix}$$

where the coefficients satisfy the relations $a^{ik} = a^{ki}$ and $b^{ik} = -b^{ki}$.

We now define G_{ik} and B_{ik} by

$$G_{ik}(u) = a^{ik}(u), \; B_{ik}(u) = -b^{ik}(u).$$

Then G is symmetric, B skew-symmetric and we may once again rewrite the variational integral as

$$(2.4) \quad I(u) = \frac{1}{2}\int G_{ik}(u)\nabla u^i \cdot \nabla u^k + B_{ik}(u)\det(\nabla u^i, \nabla u^k).$$

The following two special cases of (2.4) are well known.

(i) The *"Dirichlet integral"*

$$(2.5) \quad I(u) = \frac{1}{2}\int |\nabla u|^2$$

Here $G_{ik}(u) = \delta_{ik}$ and $B \equiv 0$.

(ii) The *"H-surface functional"*

$$(2.6) \quad I(u) = \frac{1}{2}\int |\nabla u|^2 + Q(u)\cdot(D_1 u \wedge D_2 u).$$

Here $N = 3$, $G_{ik}(u) = \delta_{ik}$, $2B_{12}(u) = Q_3(u)$, $2B_{13}(u) = -Q_2(u)$, $2B_{23}(u) = Q_1(u)$ and $\operatorname{div} Q\,(u) = 4H(u)$.

Let us now turn to the Euler equations for I.
From

$$D_1\{F_{p_1^i}(u,Du)\} + D_2\{F_{p_2^i}(u,Du)\} = F_{u^i}(u,Du)$$

for i=1,...,N we get

$$G_{ik}\,\Delta u^k + \frac{1}{2}\{\partial_l G_{ik} - \partial_i G_{kl} + \partial_k G_{li}\}\nabla u^k \cdot \nabla u^l =$$

$$= \frac{1}{2}\{\partial_l B_{ik} + \partial_i B_{kl} + \partial_k B_{li}\}\det(\nabla u^k, \nabla u^l);$$

and if (G^{ik}) denotes the inverse of (G_{ik}) we arrive at

$$(2.7)\quad \begin{aligned} &2\Delta u^i + G^{im}\{\partial_l G_{mk} - \partial_m G_{kl} + \partial_k G_{lm}\}\nabla u^k \cdot \nabla u^l = \\ &= G^{im}\{\partial_l B_{mk} + \partial_m B_{kl} + \partial_k B_{lm}\}\det(\nabla u^k, \nabla u^l). \end{aligned}$$

Let us remark that (2.1) implies that G is *positive definite.*

We summarize our observations in

THEOREM 1

Let $F : \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R}$ *satisfy (2.1) and the regularity assumptions made above.*

If the variational integral

$$I(u) = \int F(u, Du)$$

is conformally invariant, then F *has the form*

$$(2.8)\quad F(u,p) = G_{ik}(u)p^i \cdot p^k + B_{ik}(u)\det(p^i, p^k),$$

where G *is symmetric and positive definite, while* B *is skew-symmetric.*

In addition we have the following proposition, which is well known in the cases (2.5), (2.6) mentioned above.

PROPOSITION 1

Suppose F *has the form (2.8) and* u *is a* C^2*-solution of the Euler equations (2.7) of the associated integral* I. *If* Ψ *is defined by (identify* (x_1, x_2) *with the complex number* $z = x_1 + ix_2$)

$$(2.9)\quad \begin{aligned} \Psi(z) = &[G_{kl}(u)D_2u^kD_2u^l - G_{kl}(u)D_1u^kD_1u^l](x_1,x_2) \\ &+ i[2G_{kl}(u)D_1u^kD_2u^l](x_1,x_2), \end{aligned}$$

then Ψ *is a holomorphic function.*

This property will be used in section 3.

Let us now take a closer look at the case $N = 3$. We shall show that *solutions of the Euler equations may be considered as solutions to a suitable H-surface equation in a certain Riemannian manifold.*

We consider $\mathbb{R}^3$ equipped with the Riemannian metric (g_{ij}) defined by

$$g_{ij}(u) := G_{ij}(u) = F_{p_1^i p_1^j}(u,o) = F_{p_2^i p_2^j}(u,o).$$

As usual we set $g := \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$

Then in (2.7) the coefficient of $\nabla u^k \cdot \nabla u^l$ is nothing else but the *Christoffel symbol* Γ^i_{kl} corresponding to (g_{ij}).

Defining H by

(2.10) $H(u) := (\partial_1 B_{23}(u) + \partial_2 B_{31}(u) + \partial_3 B_{12}(u))/(4g(u))^{1/2}$

one easily checks that (2.7) is equivalent to

(2.11) $\Delta u^i + \Gamma^i_{kl}\ \nabla u^k \cdot \nabla u^l = 2H\sqrt{g}\ g^{ik}(D_1 u \wedge D_2 u)_k$.

Thus we have the

COROLLARY

Suppose F *has the form (2.8) and* N = 3. *Then one may introduce a new Riemannian metric on* $\mathbb{R}^3$ *by setting* $g_{ij} = G_{ij}$ *and define* H *by (2.10) such that the Euler equations corresponding to the integral* I *are (2.11).*

REMARK

If u is a C^2-solution of (2.11) with rank $Du = 2$ and if Ψ defined by (2.9) is identically zero - that is u is conformally parametrized - then u is a surface of mean curvature H in $(\mathbb{R}^3, (g_{ij}))$, c.f. [HK].

In the general case $N \geq 3$ one may consider the problem of prescribing the mean curvature vector at each point of a surface in a Riemannian manifold in terms of the tangent plane to the surface, c.f. [GR]. Then the mean curvature vector is to be given by a (2,1)-tensor H.

We define as above a new Riemannian metric on $\mathbb{R}^N$ by

$$g_{ij}(u) := G_{ij}(u)$$

and a differential 2-form α on $\mathbb{R}^N$ by

$$\alpha(u) := \tfrac{1}{2} B_{ik}(u)\, du^i du^k\ .$$

Now denote by H the unique (2,1)-tensor such that

$$d\alpha(U,V,W) = \langle U, 2H(V,W)\rangle$$

for all tangent vectors U,V,W, where $\langle\cdot,\cdot\rangle$ denotes the new scalar product on $\mathbb{R}^N$.
Then the functional I can be written as

$$I(u) = \frac{1}{2}\int \langle D_1u, D_1u\rangle + \langle D_2u, D_2u\rangle + 2\alpha(D_1u, D_2u)$$

and its Euler equations are

$$\nabla_{D_1u}D_1u + \nabla_{D_2u}D_2u = 2H(D_1u, D_2u).$$

Here ∇ denotes the Levi-Civita connection corresponding to $\langle\cdot,\cdot\rangle$.
From (2.7) one sees that an explicit expression for H is

$$H^i = \frac{1}{4}\, g^{ij}(\partial_l B_{jk} + \partial_j B_{kl} + \partial_k B_{lj})\, du^k du^l \ .$$

Thus the results of this section show that *considering conformally invariant variational integrals turns out to be nothing else but the study of the familiar "H-surface functional" in a Riemannian manifold.*
In particular the regularity results of [GM4] for weak solutions apply.

3. Removability of isolated singularities

We prove the following
THEOREM 2
Suppose F *has the form (2.8), satisfies (2.1), and* G,B *and their derivatives are bounded and Hölder-continuous. If* $\Omega \subset \mathbb{R}^2$ *is open,* $x \in \Omega$, $u \in C^2(\Omega\setminus\{x\})$, $\nabla u \in L^2_{loc}(\Omega)$, *and* u *is a solution of the Euler equations (2.7) corresponding to the conformally invariant variational integral*

$$I(u) = \int F(u, Du)$$

on $\Omega\setminus\{x\}$, *then* u *may be extended to* Ω *as a* C^2*-solution of the Euler equations.*

Before proving this theorem we need a number of estimates which will be established in several lemmas.

To simplify the notation we sometimes write (2.7) as

(3.1) $\Delta u^i = A^i(u,\nabla u)$

where $A^i(u,p) = A^{ikl}_{\alpha\beta}(u)p^k_\alpha p^l_\beta$ and $A^{ikl}_{\alpha\beta} \in C^\alpha(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Most of the following lemmas do not need the special structure of (2.7). Therefore it will be sufficient to consider (3.1) instead.

Note that (2.1) implies for any $\xi, u \in \mathbb{R}^N$

(3.2) $m_1|\xi|^2 \le G_{ik}(u)\xi^i\xi^k \le m_2|\xi|^2$.

We denote by A an L^∞-bound for the coefficients $A^{ikl}_{\alpha\beta}$ and by $B = B_1(o)$ the unit disc in $\mathbb{R}^2$.
The spaces H^1_p, L^p are defined in the usual way.
Different constants will sometimes be denoted by the same letter C.

In the first lemma we derive an *a-priori inequality.*
LEMMA 1
For any $o < \rho < \sigma \le 1$ *and any* $p \ge 1$ *there exists a constant* $K = K(\rho,\sigma,p,A) > o$ *such that for every solution* $u \in C^2(\overline{B},\mathbb{R}^N)$ *of (3.1) we have the estimate*

(3.3) $\|\nabla u\|_{H^1_p(B_\rho)} \le K[1 + \|\nabla u\|^3_{L^4(B_\sigma)}]\ \|\nabla u\|_{L^4(B_\sigma)}$.

PROOF
For $o < \tau_1 < \tau_2 \le 1$ we choose a cut-off function $\phi \in C^\infty_c(B_{\tau_2})$ which is identically equal to one on B_{τ_1}.
Using (3.1) we get on B

$$|\Delta(\phi u)| \le \|\Delta\phi\|_{L^\infty(B)}|u| + 2\|\nabla\phi\|_{L^\infty(B)}|\nabla u| + A|\nabla(\phi u)||\nabla u| + A\|\nabla\phi\|_{L^\infty(B)}|\nabla u||u| .$$

With a constant $K_1 = K_1(\tau_2 - \tau_1)$ this implies for $r \ge 1$

$$\|\Delta(\phi u)\|_{L^r(B_{\tau_2})} \le A\||\nabla(\phi u)||\nabla u|\|_{L^r(B_{\tau_2})} + AK_1\||u||\nabla u|\|_{L^r(B_{\tau_2})} + K_1\|u\|_{H^1_r(B_{\tau_2})} .$$

As ϕu has compact support in B_{τ_2} a well known estimate from potential theory, [MCB] Th. 3.4.2b), implies the existence of a number $C_r > o$ such that ($r > 1$!)

$$\|\phi u\|_{H^2_r(B_{\tau_2})} \le C_r \|\Delta(\phi u)\|_{L^r(B_{\tau_2})} .$$

Together with the previous inequality this yields

$$\|\phi u\|_{H^2_r(B_{\tau_2})} \le AC_r \| |\nabla(\phi u)| |\nabla u| \|_{L^r(B_{\tau_2})} +$$

(3.4)

$$+ AK_1C_r \| |u| |\nabla u| \|_{L^r(B_{\tau_2})} + K_1C_r \|u\|_{H^1_r(B_{\tau_2})} .$$

Note that (3.4) also holds for $u+c$ instead of u, $c = \text{const.}$, because $u+c$ satisfies (3.1) with $A^i_c(v,p) = A^i(v-c,p)$ instead of A^i.

Consider now $\bar{u} = u - ⨍_{B_{\tau_2}} u$, where $⨍_D u$ denotes the mean value of u over the set D.

In (3.4) we choose $\tau_1 = (\rho+\sigma)/2, \tau_2 = \sigma$ and $r = 2$. By Hölder's inequality we have ($\nabla\bar{u} = \nabla u$)

$$\|\phi\bar{u}\|_{H^2_2(B_\sigma)} \le AC_2 \|\nabla(\phi\bar{u})\|_{L^4(B_\sigma)} \|\nabla u\|_{L^4(B_\sigma)} +$$

$$+ AK_1C_2 \|\bar{u}\|_{L^4(B_\sigma)} \|\nabla u\|_{L^4(B_\sigma)} + K_1C_2 \|\bar{u}\|_{H^1_2(B_\sigma)} .$$

By Poincaré's and Hölder's inequality

$$\max\{\|\bar{u}\|_{L^4(B_\sigma)}, \|\bar{u}\|_{H^1_2(B_\sigma)}\} \le K(\sigma) \|\nabla u\|_{L^4(B_\sigma)} .$$

Using the fact that $\phi \equiv 1$ on $B_{(\rho+\sigma)/2}$ we get with a new constant $K_2 = K_2(\sigma, \rho-\sigma, A)$

$$\|\nabla u\|_{H^1_2(B_{(\rho+\sigma)/2})} \le \|\bar{u}\|_{H^2_2(B_{(\rho+\sigma)/2})} \le$$

(3.5) $$\le K_2(1 + \|\nabla u\|_{L^4(B_\sigma)}) \|\nabla u\|_{L^4(B_\sigma)} .$$

Now consider $\hat{u} = u - \fint_{B_{(\rho+\sigma)/2}} u$ and, using again Poincaré's and Sobolev's inequality, we have $(\nabla\hat{u} = \nabla u)$

$$\|\hat{u}\|_{H^1_{2p}(B_{(\rho+\sigma)/2})} \le K(p,\rho+\sigma)\,\|\nabla u\|_{H^1_2(B_{(\rho+\sigma)/2})} \le$$

$$\le K\,K_2(1+\|\nabla u\|_{L^4(B_\sigma)})\,\|\nabla u\|_{L^4(B_\sigma)} \,.$$

Using again (3.4) with $\hat{u}, r = p, \tau_1 = \rho,\ \tau_2 = (\rho+\sigma)/2$ we deduce

$$\|\phi\hat{u}\|_{H^2_p(B_{(\rho+\sigma)/2})} \le AC_p\|\nabla(\phi\hat{u})\|_{L^{2p}}\|\nabla u\|_{L^{2p}} +$$

$$+ AK_1C_p\|\hat{u}\|_{L^{2p}}\|\nabla u\|_{L^{2p}} + K_1C_p\|\hat{u}\|_{H^1_p} \le$$

$$\le K[1 + \|\nabla u\|^3_{L^4(B_\sigma)}]\,\|\nabla u\|_{L^4(B_\sigma)} \,,$$

where $K = K(\rho,\sigma,p,A)$.

As $\phi \equiv 1$ on B_ρ, this implies (3.3). □

Using the finiteness of the Dirichlet integral we now derive a fundamental estimate.

LEMMA 2

There exists $\varepsilon = \varepsilon(A) > 0$, *such that for all solutions* $u \in C^2(\bar{B},\mathbb{R}^N)$ *of (3.1) satisfying* $\int_B |\nabla u|^2 < \varepsilon$ *the following is true:*

For any $p \ge 1$ *and* $0 < \rho < 1$ *there is* $K = K(p,\rho,\varepsilon) > 0$, *such that*

$$(3.6)\quad \|\nabla u\|_{H^1_p(B_\rho)} \le K\|\nabla u\|_{L^2(B)} \,.$$

PROOF

Because of Lemma 1 it is sufficient to estimate

$\|\nabla u\|_{L^4(B_\sigma)}$ in terms of $\|\nabla u\|_{L^2(B)}$ $(\rho < \sigma < 1)$.

Let $\sigma = (\rho + 1)/2$. We may assume $\fint_B u = 0$.

We use (3.4) with $r=\frac{4}{3}$, $\tau_1=\sigma$, $\tau_2=1$. Now Poincaré's and Sobolev's inequalities yield $(4=nr/(n-r))$

$$\| |\nabla(\phi u)|\,|\nabla u| \|_{L^r(B)} \le \|\nabla u\|_{L^2(B)} \|\nabla(\phi u)\|_{L^4(B)}$$

$$\le C\|\nabla u\|_{L^2(B)} \|\phi u\|_{H^2_r(B)} \;;$$

$$\|u\|_{H^1_r(B)} \le C\|\nabla u\|_{L^2(B)} \;;$$

$$\| |u|\,|\nabla u| \|_{L^r(B)} \le \|\nabla u\|_{L^2(B)} \|u\|_{L^4(B)} \le C\|\nabla u\|^2_{L^2(B)} \;.$$

Together with (3.4) we get

$$(1-AC_{4/3}C\sqrt{\varepsilon})\,\|\phi u\|_{H^2_{4/3}(B)} \le$$

$$\le (K_1C_{4/3}C+AK_1C_{4/3}C\sqrt{\varepsilon})\,\|\nabla u\|_{L^2(B)} \;.$$

Choosing $\varepsilon<(AC_{4/3}C)^{-2}$ and, using the fact that $\phi\equiv 1$ on B_σ, we have shown

$$\|\nabla u\|_{L^4(B_\sigma)} \le C\|\phi u\|_{H^2_{4/3}(B)} \le K(A,\varepsilon)\,\|\nabla u\|_{L^2(B)} \;.$$

This implies the desired inequality. □

The previous lemma is now used to give a *pointwise estimate* of $|\nabla u|$ away from the singularity.

LEMMA 3

Suppose $u\in C^2(B_2\setminus\{o\})$ *is a solution of (3.1) satisfying* $\int_{B_2}|\nabla u|^2<\varepsilon$.

There exists a constant $C_o=C_o(K)$, *such that for any* $x_o\in B\setminus\{o\}$

$$(3.7)\quad |\nabla u(x_o)| \le C_o\Big[\fint_{B_{2|x_o|}}|\nabla u|^2\Big]^{1/2} \;.$$

REMARK

Here ε, K are the constants from Lemma 2 with $\rho = 1/2$ and $p = 4$.

PROOF.

For $x_o \in B \setminus \{o\}$ define

$$\tilde{u}(x) := u(x_o + x(|x_o|/2)), \quad x \in B.$$

Then $\tilde{u} \in C^2(\bar{B})$ is a solution of (3.1) on B. Using the conformal invariance of the Dirichlet integral we get

$$\int_B |\nabla\tilde{u}|^2 = \int_{B_{|x_o|/2}(x_o)} |\nabla u|^2 \le \int_{B_2} |\nabla u|^2 < \varepsilon ,$$

so that the assumptions of Lemma 2 are satisfied. Together with Sobolev's inequality we conclude

$$|\nabla\tilde{u}(o)| \le C \|\nabla\tilde{u}\|_{H^1_4(B_{1/2})} \le C\, K\, \|\nabla\tilde{u}\|_{L^2(B)} .$$

By the definition of $\tilde{u}$ this implies (3.7). □

We are now in a position to improve Proposition 1.

LEMMA 4

Suppose $u \in C^2(B_2 \setminus \{o\})$ *is a solution of (2.7) satisfying* $\int_{B_2} |\nabla u|^2 < \varepsilon$.

Then $\Phi(z) = z\Psi(z)$, *where* Ψ *is the function defined in Proposition 1, is holomorphic on* B_2.

PROOF

We show $\lim_{z\to o} z\,\Phi(z) = o$.

By the definition of Ψ we have for $z \neq o$

$$|\Psi(z)| \le 2m_2 |\nabla u(x)|^2.$$

Now (3.7) implies

$$|z\Phi(z)| \leq 2m_2|\nabla u(x)|^2|x|^2 \leq$$
$$\leq (m_2 c_o^2/2\pi)\int_{B_{2|x|}} |\nabla u|^2 .$$

The absolute continuity of the integral proves our claim.

□

Using Lemma 4 we now prove a relation which will be essential in the proof of Theorem 2.

PROPOSITION 2.
Assume the hypotheses of Lemma 4. For any $o < r < 2$ *we have* $(z = r\, e^{i\vartheta})$

$$(3.8)\quad \int_o^{2\pi} G_{jk}(u)u_\vartheta^j u_\vartheta^k d\vartheta = r^2 \int_o^{2\pi} G_{jk}(u)u_r^j u_r^k d\vartheta .$$

PROOF
We have for $|z| = r$

$$\mathrm{Re}[z\Phi(z)] = G_{jk}(u)u_\vartheta^j u_\vartheta^k - G_{jk}(u)r^2u_r^j u_r^k ,$$

and by Lemma 4

$$\int_{|z|=r} \Phi(z)dz = o.$$

This implies

$$o = \mathrm{Im}\left[\int_{|z|=r} \Phi(z)dz\right] = \mathrm{Im}\left[i\int_o^{2\pi} \Psi(re^{i\vartheta})(re^{i\vartheta})^2 d\vartheta\right] =$$
$$= \int_o^{2\pi} \mathrm{Re}[z\Phi(z)]d\vartheta ,$$

which proves (3.8). □

REMARK
Note that the following equation holds for $|z| = r$

$$(3.9)\quad G_{jk}(u)(r^2u_r^j u_r^k + u_\vartheta^j u_\vartheta^k) = r^2 G_{jk}(u)\nabla u^j\cdot\nabla u^k .$$

We are now able to give the

PROOF *of Theorem 2.*

Without loss of generality we may assume $\Omega = B_2(o)$, $x = o$ and

$$\int_{B_2} |\nabla u|^2 < \varepsilon,$$

where $\varepsilon > o$ will only depend on A(L^∞-bound for the coefficients) and m_1, m_2.

Let us define a *comparison function* q, which is *rotationally symmetric*.

For that purpose set $(m \in \mathbb{N} \cup \{o\})$

$$\mu_m = \frac{1}{2\pi} \int_o^{2\pi} u(2^{-m} \cos \vartheta, 2^{-m} \sin \vartheta) d\vartheta ;$$

μ_m is the mean value of u on the circle of radius 2^{-m}.

If $x \in B_1 \setminus \{o\}$ and $2^{-m} \leq |x| \leq 2^{-m+1}$ we define q by

$$q(x) := q(|x|) = \alpha_m \log|x| + \beta_m ,$$

where $\alpha_m = (\mu_{m-1} - \mu_m)/\log 2$, $\beta_m = m(\mu_{m-1} - \mu_m) + \mu_m$.

Then q is continuous on $\overline{B} \setminus \{o\}$, harmonic for $2^{-m} < |x| < 2^{-m+1}$ with boundary values μ_m resp. μ_{m-1} .

For x with $2^{-m} < |x| \leq 2^{-m+1}$ we get

$$\begin{aligned} |q(|x|) - u(x)| &\leq |q(|x|) - \mu_{m-1}| + |\mu_{m-1} - u(x)| \\ &\leq |\mu_m - \mu_{m-1}| + |\mu_{m-1} - u(x)|. \end{aligned}$$

The second term can be further estimated by Lemma 3

$$\begin{aligned} |u(x) - \mu_{m-1}| &\leq \max_{2^{-m} \leq |x'|, |x''| \leq 2^{-m+1}} |u(x') - u(x'')| \\ &\leq 2\pi 2^{-m} \max_{2^{-m} \leq |x'| \leq 2^{-m+1}} |\nabla u(x')| \\ &\leq 2\pi \max_{2^{-m} \leq |x'| \leq 2^{-m+1}} |\nabla u(x')| |x'| \\ &\leq 2\sqrt{\pi}\, C_o \sqrt{\varepsilon}. \end{aligned}$$

For the first term the same estimate is true; this implies

$$\text{(3.10)} \qquad \sup_{0 \neq |x| \leq 1} |q(|x|)-u(x)| \leq 4\sqrt{\pi}\, C_o \sqrt{\varepsilon}\ .$$

We denote by A_m the annulus $B_{2^{-m+1}} \setminus \overline{B}_{2^{-m}}$; as $q, u \in C^2(\overline{A}_m)$ Green's formula yields

$$\int_{A_m} |\nabla(q-u)|^2 = \int_{\partial A_m} (q-u)\cdot\partial_\nu(q-u)\, d\mathbb{H}^1 - \int_{A_m} (q-u)\cdot\Delta(q-u).$$

The boundary integral can be written as

$$\int_{\partial A_m} (q-u)\cdot\partial_\nu(q-u)\, d\mathbb{H}^1 =$$

$$= 2^{1-m} \int_0^{2\pi} (q(2^{1-m})-u(2^{1-m}e^{i\vartheta}))\cdot(q'(2^{1-m})-u_r(2^{1-m}e^{i\vartheta}))\,d\vartheta$$

$$- 2^{-m} \int_0^{2\pi} (q(2^{-m})-u(2^{-m}e^{i\vartheta}))\cdot(q'(2^{-m})-u_r(2^{-m}e^{i\vartheta}))\,d\vartheta\ .$$

Because of our choice of q the integrals containing q' vanish. Therefore we end up with

$$\int_{B \setminus B_{2^{-m}}} |\nabla(q-u)|^2 = 2^{-m} \int_0^{2\pi} (q(2^{-m})-u(2^{-m}e^{i\vartheta}))\cdot u_r(2^{-m}e^{i\vartheta})\,d\vartheta$$

$$- \int_0^{2\pi} (q(1)-u(e^{i\vartheta}))\cdot u_r(e^{i\vartheta})\,d\vartheta + \int_{B \setminus B_{2^{-m}}} (q-u)\cdot\Delta u\ .$$

This is true because the remaining boundary integrals cancel each other in view of the continuity of all functions involved, and because q is harmonic on A_m. By (3.7) and (3.10) we get

$$\left|2^{-m} \int_0^{2\pi} (q(2^{-m})-u(2^{-m}e^{i\vartheta}))\cdot u_r(2^{-m}e^{i\vartheta})\,d\vartheta\right| \leq$$

$$\leq C\sqrt{\varepsilon}\, \Big[\int_{B_{2^{1-m}}} |\nabla u|^2\Big]^{1/2}\ ,$$

so that this boundary integral vanishes if we let $m \to \infty$. Using (3.1) and (3.10) we may estimate the last integral by

$$(3.11)\quad \left|\int_{B\setminus B_{2^{-m}}} (q-u)\cdot\Delta u\right| \le CA\sqrt{\varepsilon}\int_B |\nabla u|^2 .$$

Letting $m\to\infty$ we get the equation

$$\int_B |\nabla(q-u)|^2 = \int_0^{2\pi} (u(e^{i\vartheta})-q(1))\cdot u_r(e^{i\vartheta})d\vartheta + $$

$$+ \int_B (q-u)\cdot\Delta u.$$

Now choose $\delta<(m_1/2m_2)$ and $\varepsilon<(\delta/CA)^2$. Then (3.11) and Hölder's inequality imply

$$(3.12)\quad \int_B |\nabla(q-u)|^2 \le \left[\int_0^{2\pi} |u(e^{i\vartheta})-\mu_0|^2 d\vartheta\right]^{1/2}\left[\int_0^{2\pi} |u_r(e^{i\vartheta})|^2 d\vartheta\right]^{\frac{1}{2}}$$

$$+ \delta\int_B |\nabla u|^2 .$$

By (3.2), (3.8) and (3.9) the left hand side can be estimated by

$$\int_B |\nabla(q-u)|^2 \ge \frac{1}{m_2}\int_B G_{jk}(u)\nabla(q-u)^j\cdot\nabla(q-u)^k =$$

$$= \frac{1}{m_2}\int_B \frac{1}{r^2}\left[G_{jk}(u)(r^2(q-u)^j_r(q-u)^k_r + u^j_\vartheta u^k_\vartheta\right] \ge$$

$$\ge \frac{1}{m_2}\int_B \frac{1}{r^2} G_{jk}(u)u^j_\vartheta u^k_\vartheta =$$

$$= \frac{1}{2}\left\{\frac{1}{m_2}\int_B \frac{1}{r^2} G_{jk}(u)u^j_\vartheta\, u^k_\vartheta + \frac{1}{m_2}\int_0^1 r\int_0^{2\pi} G_{jk}(u)u^j_r u^k_r d\vartheta\, dr\right\} =$$

$$= \frac{1}{2m_2}\int_B G_{jk}(u)\nabla u^j\cdot\nabla u^k \ge \frac{m_1}{2m_2}\int_B |\nabla u|^2 .$$

By Poincaré's inequality we get from (3.12)

$$\left(\frac{m_1}{2m_2} - \delta\right)\int_B |\nabla u|^2 \le$$

$$\leq C\Big[\int_0^{2\pi} |u_\vartheta(e^{i\vartheta})|^2 d\vartheta\Big]^{1/2}\Big[\int_0^{2\pi} |u_r(e^{i\vartheta})|^2 d\vartheta\Big]^{1/2} \leq$$

$$\leq C \int_0^{2\pi} |\nabla u(e^{i\vartheta})|^2 d\vartheta \ .$$

Introducing $\gamma = (\frac{m_1}{2m_2} - \delta)/C$ we have shown

$$\gamma \int_B |\nabla u|^2 \leq \int_0^{2\pi} |\nabla u(e^{i\vartheta})|^2 d\vartheta \ .$$

Now all the calculations made above can also be done for $u(\rho x)$ instead of u, $0 < \rho \leq 1$; the result is

$$(3.13)\quad \gamma \int_{B_\rho} |\nabla u|^2 \leq \rho^2 \int_0^{2\pi} |\nabla u(\rho e^{i\vartheta})|^2 d\vartheta \ .$$

As is well known, this differential inequality implies for $\rho \leq 1$

$$\int_{B_\rho} |\nabla u|^2 \leq \rho^\gamma \int_B |\nabla u|^2 \ .$$

Together with Lemma 3 we get for $|x| \leq \frac{1}{2}$

$$|\nabla u(x)|\,|x| \leq C|x|^{\gamma/2} [\int_B |\nabla u|^2]^{1/2} \ .$$

Thus for any x_o with $|x_o| \leq 1/4$ and any $r \leq \frac{1}{4}$ we have

$$\int_{B_r(x_o)} |\nabla u|^2 \leq C^2 \varepsilon \int_{B_r(x_o)} |w|^{\gamma-2} = \varepsilon C^2 2\pi \int_0^r \rho^{\gamma-1} d\rho = M\rho^\gamma \quad ,$$

where $M = (2\pi\,\varepsilon\,C^2)/\gamma$.

The Dirichlet-Growth-Theorem now yields the Hölder continuity of u on $B_{1/4}$. Using a result of Tomi [T], we now deduce the Hölder continuity of the first derivatives of u. Now (3.1) can be regarded as an equation with Hölder continuous right hand side and fundamental results from potential theory complete the proof of Theorem 2. □

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