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Determination of the optimal initial temperature distribution in a porous bed

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Summary. This paper suggests the application of the minimum principle of Pontryagin to the solution of an optimal control problem for a porous packed bed cooled by a flow of incompressible fluid. The procedure for determination of the optimal initial temperature distribution in a one-dimensional packed bed is developed. The amount of heat transferred to the fluid phase is utilized as the optimization criterion. It is necessary to maximize this amount under the following constraints: (a) a given amount of heat is initially stored in the packed bed and (b) a given duration of the process. As the control the initial temperature of the packed bed is considered. Qualitative changes in the behavior of the optimal initial temperature distribution take place as the duration of the process is increased.

Notation

a_{sf}	specific surface area common to solid and fluid phases, m^2/m^3
c_p	specific heat at constant pressure, J $kg^{-1}K^{-1}$
h_{sf}	fluid-to-solid phase heat transfer coefficient, W $m^{-2}K^{-1}$
I_ν	modified Bessel function of the order ν
L	dimensionless length of the porous slab
L'	length of the porous slab, m
t	dimensionless time
t'	time, s
t_f	dimensionless duration of the process
T	temperature, K
T_1 and T_2	reference temperatures, K
$u_{\min}$	the lower boundary for admissible controls
$u_{\max}$	the upper boundary for admissible controls
v	velocity of the fluid phase, m s^{-1}
z	dimensionless Cartesian coordinate
z'	Cartesian coordinate, m

Greek symbols

ε	porosity
Φ	dimensionless temperature of the fluid phase
Φ_{in}	dimensionless inlet temperature of the fluid phase
λ_1	the Lagrange multiplier
θ	dimensionless temperature of the solid phase
θ_0	dimensionless initial temperature of the solid phase
ρ	density, kg m^{-3}

Subscripts

f	fluid
s	solid

1 Introduction

Optimization problems in heat and mass transfer have recently drawn considerable attention [1] – [4]. This is because these problems are interesting from the fundamental point of view and also are relevant to many practical applications.

In this paper we suggest the application of the minimum principle of Pontryagin to the solution of an optimal control problem for a porous packed bed cooled by a flow of incompressible fluid. Our task is to determine the optimal initial temperature distribution in a packed bed to maximize its performance. More precisely, we maximize the amount of heat transferred by the packed bed to the fluid for a given duration of the process and for the given amount of heat energy initially stored in the packed bed. To the best of the author's knowledge, this is the first attempt to calculate the optimal initial temperature distribution in a packed bed.

Investigation of the transient response of packed beds and their performance has been a subject of permanent interest for scientific investigations. This is because of the important applications of porous beds, such as the storage of heat energy. Recent works [5]–[9] present numerical solutions for non-thermal equilibrium, condensing, forced fluid flow through porous packed beds. In these references the two energy equation model is utilized in which the temperature difference between the fluid and solid phases is taken into account. Proceeding from this model, [10] analysed the temperature difference between the fluid and solid phases and found them to exhibit wave properties.

2 Statement of the problem

Most of the analytical studies of these phenomena are concentrated on the Schumann model of a packed bed suggested in [11]. In the Schumann model a flow of incompressible fluid through a packed bed is considered, and the thermal conduction terms in both the fluid and solid phase energy equations are neglected. In the present paper we follow this model and employ the following assumptions:

- The fluid phase is incompressible and the mass flow rate at every cross section of the packed bed is constant
- Thermal, physical, and transport properties are constant
- The conduction heat transfer is negligible in both the fluid and solid phases
- Heat transfer and fluid flow are one-dimensional.

As follows from [12], under these assumptions the equations governing the solid and fluid temperature distributions can be presented in the following nondimensional form:

$$\frac{\partial \theta}{\partial t} = \phi - \theta \qquad \Lambda \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial z} = \theta - \phi \tag{1}$$

where

$$\Lambda = \frac{\varepsilon \rho_f c_{pf}}{(1-\varepsilon)\, \rho_s c_s}.$$

Here the dimensionless temperature of the solid phase is defined as

$$\theta(z,\, t) = \frac{T_s - T_1}{T_2 - T_1}, \tag{2}$$

and the dimensionless temperature of the fluid phase is defined as

$$\phi(z, t) = \frac{T_f - T_1}{T_2 - T_1} \tag{3}$$

where T_1 and T_2 are reference temperatures chosen to suitably normalize the initial and boundary conditions.

The dimensionless time and coordinate in Eqs. (1) are defined as follows:

$$t = \frac{h_{sf} a_{sf} t'}{(1-\varepsilon)\, \rho_s c_s},$$

and

$$z = \frac{h_{sf} a_{sf} z'}{\varepsilon \rho_f c_{pf} v}.$$

The analytical solutions for Eqs. (1) for different boundary conditions are obtained in [13]–[17]. In [18], [19] the solution for the case when the inlet fluid temperature is a function of time and the initial temperature of the packed bed is a function of the space variable is obtained. In [18], the following initial and boundary conditions are utilized:

$$\theta(z, 0) = \theta_0(0), \tag{4.1}$$

$$\phi(0, t) = \phi_{\mathrm{in}}(t). \tag{4.2}$$

Upon simple rearrangement, the solution obtained in [18] can be put into the following form:

$$\theta(z, t) = \exp(-z) \int_0^{t-\Lambda z} \phi_{\mathrm{in}}(t - \Lambda z - \tau) \exp(-\tau)\, I_0[(4\tau z)^{1/2}]\, d\tau + \exp(\Lambda z - t) \left[\theta_0(z) + \int_0^z \theta_0(z - \xi) \exp(-\xi) \left(\frac{t - \Lambda z}{\xi} \right)^{1/2} I_1[\{4\xi(t - \Lambda z)\}^{1/2}]\, d\xi \right] \tag{5}$$

$$\phi(z, t) = \exp(\Lambda z - t) \int_0^z \theta_0(z - \xi) \exp(-\xi)\, I_0[\{4\xi(t - \Lambda z)\}^{1/2}]\, d\xi + \exp(-z) \left[\phi_{\mathrm{in}}(t - \Lambda z) + \int_0^{t-\Lambda z} \phi_{\mathrm{in}}(t - \Lambda z - \tau) \exp(-\tau) \left(\frac{z}{\tau} \right)^{1/2} I_1[(4\tau z)^{1/2}]\, d\tau \right]. \tag{6}$$

Equations (5), (6) determine the temperatures of the solid and fluid phases at a particular point in the porous bed with the position z' (or corresponding dimensionless coordinate z) after this point is reached by the temperature front moving from the fluid inlet boundary with a velocity v, i.e. when $t' \geqq z'/v$. In the dimensionless coordinates this condition is $t \geqq \Lambda z$. Because the thermal conductivities of both the solid and fluid phases are neglected, for $t < \Lambda z$ the temperature of the solid phase at this point equals the initial temperature determined by the function $\theta_0(z)$ in Eq. (4.1).

We consider a one-dimensional porous slab of the length L'. The dimensionless length of the slab is then defined as

$$L = \frac{h_{sf} a_{sf} L'}{\varepsilon \rho_f c_{pf} v}.$$

It is assumed that the inlet fluid phase temperature is constant. In this case it is always possible to select the reference temperatures, T_1 and T_2, in Eqs. (2) and (3) so that the dimensionless inlet temperature equals zero. This essentially simplifies Eqs. (5) and (6), because in this case the first term on the right-hand side of Eq. (5) and the second term on the right-hand side of Eq. (6) are equal to zero.

It is assumed that the initial packed bed temperature is given by some function of the coordinate z, $\theta_0(z)$. Consider the following optimal problem. The function $\theta_0(z)$ is considered as a control. It is assumed that this function is a bounded, piecewise continuous function with a minimum value $u_{\min}$ and a maximum value $u_{\max}$. As the optimization criterion the amount of heat energy transferred to the fluid phase is used. It is necessary to maximize this amount of heat under the following constraints: (a) a given amount of heat energy is stored initially in the packed bed and (b) a given duration of the process.

The mathematical formulation of this problem is as follows. It is necessary to determine the optimal control $\hat{\theta}_0(z)$ that maximizes the following functional:

$$\Phi(\theta_0) = \int_{AL}^{t_f} \phi(L, t)\, dt \to \max \tag{7}$$

where the function $\phi(L, t)$ is determined by Eq. (6), under the following constraints:

$$\int_0^L \theta_0(\xi)\, d\xi = E = \text{const} \tag{8}$$

and

$$u_{\min} \leqq \theta_0(z) \leqq u_{\max}. \tag{9}$$

The lower integration limit in Eq. (7) corresponds to the moment of time when the fluid front reaches the outlet boundary, and the upper limit corresponds to the duration of cooling. We consider the case when $t_f > AL$.

3 Solution and analysis

To bring the problem (7)–(9) to the form of an optimal control problem it is necessary to rearrange the functional (7). To accomplish this, Eq. (6) for the function $\phi(z, t)$ is first rearranged by the following change of the integration variable:

$$\varsigma = z - \xi. \tag{10}$$

Then, accounting for the assumption that the inlet temperature of the fluid phase is zero, Eq. (6) at the outlet boundary $z = L$ can be written as

$$\phi(L, t) = \int_0^z \theta_0(\varsigma) \exp(AL + \varsigma - L) \exp(-t)\, I_0[\{4(L - \varsigma)(t - AL)\}^{1/2}]\, d\varsigma. \tag{11}$$

Then, utilizing Eq. (11) and changing the integration order, Eq. (7) can be recast as

$$\Phi(\theta_0) = \int_{AL}^{t_f} \phi(L, t)\, dt = \int_0^L \theta_0(\varsigma)\Psi(\varsigma)\, d\varsigma \to \max \tag{12}$$

where

$$\Psi(\varsigma) = \exp(\Lambda L + \varsigma - L) \int_{\Lambda L}^{t_f} \exp(-t)\, I_0[\{4(L-\varsigma)(t-\Lambda L)\}^{1/2}]\, dt\,.$$

The functional $\Phi(\theta_0)$ is the performance functional for our problem.

The problem given by Eqs. (8), (9), and (12) is an optimal control problem. It can be solved by the minimum principle of Pontryagin considered, for example, in [20]–[21]. Application of this principle leads to the following requirement:

$$\hat{\theta}_0(z)\,[\lambda_1 - \Psi(z)] \rightarrow \min \tag{13}$$

where λ_1 is the Lagrange multiplier.

Equation (13), when applied accounting for the constraint (9), makes it possible to determine the optimal control, $\hat{\theta}_0(z)$, as

$$\hat{\theta}_0(z) = u_{\min} \text{ if } \lambda_1 - \Psi(z) > 0$$

$$\hat{\theta}_0(z) = u_{\max} \text{ if } \lambda_1 - \Psi(z) < 0\,. \tag{14}$$

To make use of Eqs. (14) it is necessary to calculate the value of the Lagrange multiplier, λ_1. To do this, transcendental equation (8) needs to be solved accounting for Eq. (14). To solve this problem, first a segment that unequivocally contains the desired value of λ_1 was selected. Then an algorithm for finding a root of a transcendental equation on a given segment was applied to Eq. (8).

Figure 1 depicts the optimal controls, $\hat{\theta}_0(z)$, for different durations of the process, t_f, for the following data: $u_{\min} = 0$, $u_{\max} = 1$, $E = L/2$, $L = 1$, $\Lambda = 0.05$. As can be seen in Fig. 1, for a small

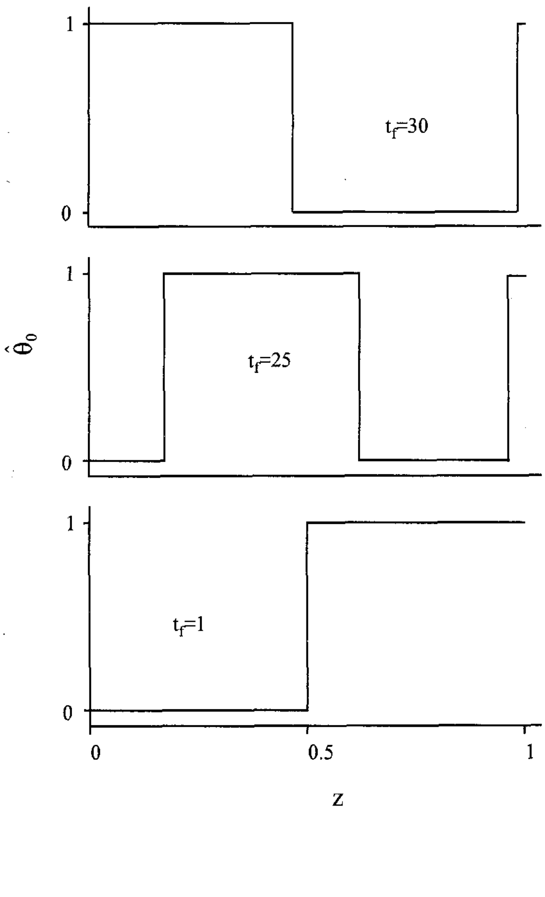

Fig. 1. The optimal initial temperature distributions in the porous slab

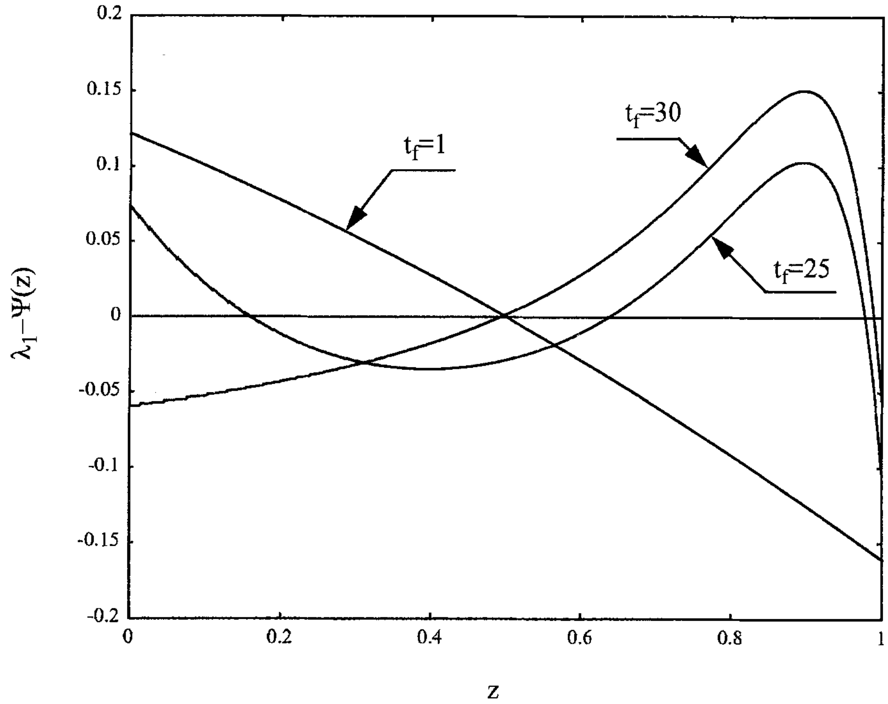

Fig. 2. The functions $\lambda_1 - \Psi(z)$. For $t_f = 25$ and for $t_f = 30$ the value of these functions is multiplied by 10^7

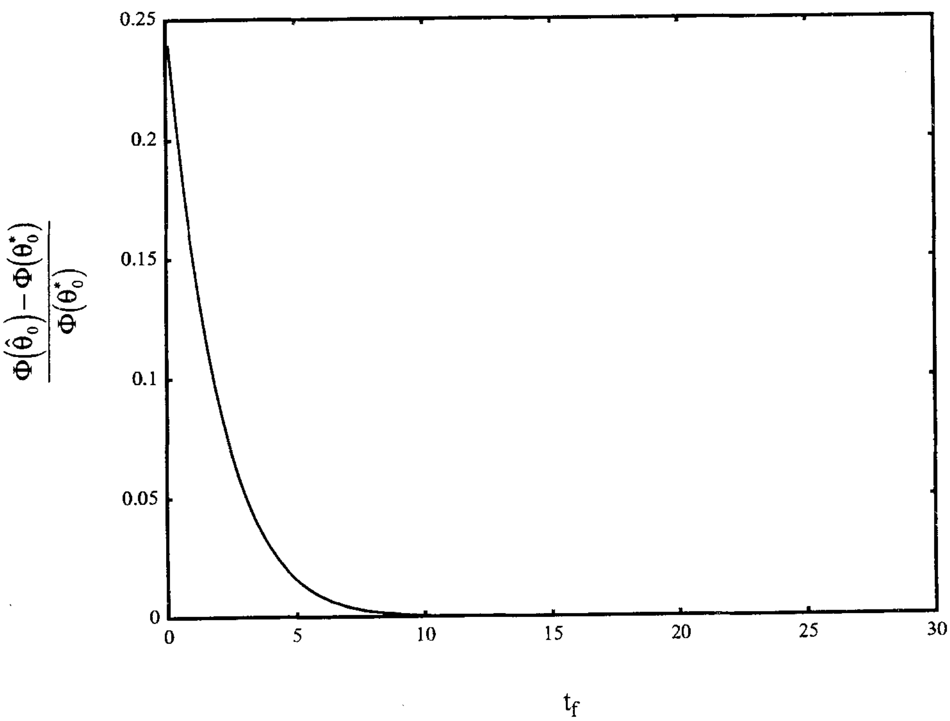

Fig. 3. The dependence on the gain in the amount of the heat energy transferred to the fluid phase when the optimal initial temperature distribution is utilized instead of the constant initial temperature of the duration of the process

duration ($t_1 = 1$) the optimal initial temperature $\hat{\theta}_0(z)$ takes its minimum value, u_{min}, in the first half of the slab and its maximum value, u_{max}, in the second half of the slab. With an increase in the duration ($t_f = 25$) the distribution of the optimal initial temperature becomes more complicated. Now, the optimal initial temperature takes its minimum value in the beginning of the slab, then its maximum value, then again its minimum value and then again its maximum value. With a further increase in the duration ($t_f = 30$) a new change in the distribution occurs: the maximum value occurs in the beginning of the slab, then the minimum value and then again the maximum value. The corresponding functions $\lambda_1 - \Psi(z)$, which determine these distributions of the optimal temperature, are depicted in Fig. 2.

Thus Fig. 1 shows that with an increase in the duration of cooling the distribution of the optimal initial temperature changes not only quantitatively but also qualitatively. To understand the reason for these qualitative changes we consider two extreme cases, namely, a very short and a very long duration of the process. For a very short duration the temperature front has just reached the outlet boundary of the porous slab and the minima–maxima behavior is obviously beneficial. Indeed, otherwise after contacting with the hot part of the packed bed the fluid will be cooled in the cold part, and the resulting amount of heat transferred to the fluid will be small. For a very long duration of the process its duration becomes sufficient to cause a redistribution of the temperature in the packed bed and to heat the part of the packed bed which was initially cold. This results in moving the part which should be initially at a maximum temperature to the beginning of the slab.

It is interesting to compare the value that the performance functional $\Phi(\theta_0)$ takes on the optimal functions, calculated according to Eqs. (14), and on the functions $\theta_0{}^*(z) \equiv 1/2$. These functions, $\theta_0{}^*(z)$, correspond to a constant initial temperature distribution in the slab. For $E = L/2$ (this was used to calculate Fig. 1) the functions $\theta_0{}^*(z) \equiv 1/2$ apparently satisfy constraint (8). The ratio $[\Phi(\hat{\theta}_0) - \Phi(\theta_0{}^*)]/\Phi(\theta_0{}^*)$ characterizes the gain in the amount of the heat energy transferred to the fluid phase when the optimal initial temperature distribution is utilized instead of the constant initial temperature. The dependence on this ratio of the duration of the process is depicted in Fig. 3. It can be seen that the optimal initial temperature distribution makes it possible to considerably improve the discharging characteristics of the packed bed (up to approximately 25%) for short durations of the process. The influence of the initial temperature on the discharging efficiency quickly decreases with an increase of the duration (for $t_f > 10$ it becomes so small that it is hardly visible in Fig. 3). This conclusion agrees with the well known property of heat transfer processes to forget initial conditions. The rate at which the curve depicted in Fig. 3 tends to zero characterizes the rate at which the process forgets its initial conditions.

4 Conclusions

(i) A method for the optimization of the initial temperature distribution in a one-dimensional porous slab is suggested. As the optimization criterion the amount of heat transferred to the fluid phase is utilized.

(ii) It is shown that with an increase in the duration of the process qualitative changes in the optimal temperature distribution take place. For a small duration the optimal initial temperature takes its minimum value in the first half of the slab and its maximum value in the second half of the slab. With an increase in the duration the distribution of the optimal initial temperature becomes more complicated. The optimal initial temperature takes its minimum value in the beginning of the slab, then its maximum value, then again its minimum value and then again its maximum value. With a further increase in the duration a new change in the distribution occurs: the maximum value in the beginning of the slab, then the minimum value and then again the maximum value.

(iii) The optimal initial temperature distribution makes it possible to considerably improve the discharging characteristics of the packed bed for short durations of the process. The influence of the initial temperature on the discharging efficiency quickly decreases with an increase of the duration.

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