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ZEROS OF ACCRETIVE OPERATORS

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In the investigation of accretive operators in Banach spaces X, the existence of zeros plays an important role, since it yields surjectivity results as well as fixed point theorems for operators S such that $I-S$ is accretive. Let $D \subset X$ and $T: D \to X$ an operator such that the initial value problems

(1) $\qquad u'(t) = -Tu(t) \quad , \quad u(0) = x \in D$

are solvable. Then T has a zero iff (1) has a constant solution for some $x \in D$. Under certain assumptions on D and T it is possible to show that (1) has a unique solution $u(t,x)$ on $[0,\infty)$, for every $x \in D$. In this case, define $U(t): D \to D$ by $U(t)x = u(t,x)$. If T is accretive it turns out that $U(t)$ is nonexpansive for every $t \geq 0$. This fact constitutes the basis for several authors concerned with this subject. They proceed with assumptions on D and X ensuring either that the $U(t)$ must have a common fixed point x_o or that $U(p)$ has a fixed point x_p for every $p \geq 0$. In the first case, $U(t)x_o$ is a constant solution of (1), whence $Tx_o = 0$. In the second case, $U(t)x_p$ is a p-periodic solution of (1). Hence, one has to impose additional conditions on T which imply that a p-periodic solution must be constant, for some $p > 0$.

The main purpose of the present paper is to show that, in certain situations, either the operators $U(t)$ are actually strict contractions or T may be approximated by operators T_n such that the corresponding $U_n(t)$ are strict contractions. Thus, we obtain several results in general Banach spaces and a unification of some results in special spaces.

1. Preliminaries

Let X be a real Banach space, X^* its dual and $F: X \to 2^{X^*}$ the duality map defined by

$$F(x) = \{x^* \in X^* : x^*(x) = |x|^2, |x^*| = |x|\} .$$

By means of F, the generalized pairings

$$(\cdot,\cdot)_- \quad , \quad (\cdot,\cdot)_+ : X \times X \to R^1$$

are defined as

$$(2)\qquad \begin{aligned}(x,y)_- &= \inf\{x^*(x) : x^* \in F(y)\} \;,\\ (x,y)_+ &= \sup\{x^*(x) : x^* \in F(y)\} \;.\end{aligned}$$

The following properties are immediate consequences of the definitions.

$$(3)\qquad \begin{aligned}(x+\alpha y,y)_{\pm} &= \alpha|y|^2 + (x,y)_{\pm}\\ (x+y,z)_{\pm} &\leq (x,z)_{\pm} + |y||z|\end{aligned}$$

(4) $(x,y)_- \leq (x,y)_+$, with equality holding everywhere if X^* is strictly convex

(5) If $x\colon (a,b] \to X$ is weakly differentiable at t_0 , $\phi(t) = |x(t)|$ and $D^-\phi(t) = \limsup_{h\to 0+} h^{-1}\{\phi(t)-\phi(t-h)\}$ then $\phi(t_0)D^-\phi(t_0) \leq (x'(t_0),x(t_0))_-$.

Definition 1. *Let* $D \subset X$. *We call* $T\colon D \to X$ *accretive if* $(Tx-Ty,x-y)_+ \geq 0$ *for every* $x,y \in D$, *and strongly accretive if* $(Tx-Ty,x-y)_+ \geq \alpha(|x-y|)|x-y|$ *for every* $x,y \in D$, *where* $\alpha\colon R^1_+ \to R^1_+$ *is continuous with* $\alpha(0) = 0$ *and* $\alpha(r) > 0$ *for* $r > 0$.

Remark 1. The usual definition af accretiveness is "$\langle Tx-Ty,x^*\rangle \geq 0$ for some $x^* \in F(x-y)$" . It coincides with our weaker assumption if X^* is strictly convex.

If $D \subset X$, $\rho(x,D)$ denotes the distance from $x \in X$ to D .

2. Zeros of certain operators on arbitrary closed subsets

Theorem 1. *Let* $D \subset X$ *be closed*, $T\colon D \to X$ *continuous and* $(Tx-Ty,x-y)_- \geq \alpha(|x-y|)|x-y|$, *with* α *as in Definition* 1 *satisfying* $\alpha(r) \to \infty$ *as* $r \to \infty$ *and* $\liminf_{r\to 0} \alpha(r)/r > 0$. *Suppose in addition that*

(6) $\quad \rho(x-\lambda Tx,D) = o(\lambda)$ *as* $\lambda \to 0+$, *for every* $x \in D$

holds. Then $0 \in T(D)$.

Proof. (i) We may assume $0 \in D$. Let $f(u) = -Tu$ for $u \in D$. Since $\rho(u+\lambda f(u),D) = o(\lambda)$ and

$$(f(u)-f(v),u-v)_+ = -(Tu-Tv,u-v)_- \leq 0$$

for $u,v \in D$, the initial value problem (1) has a unique solution $u(t,x)$ on $[0,\infty)$, by Theorem 4 in [6].

(ii) There is a "ball" $K = \{x \in D : |x| \leq R\}$ such that $u(t,x) \in K$ for each $x \in K$ and each $t \geq 0$. To see this, let $\phi(t) = |u(t,x)|$. By (5) and (3), we have

$$\phi(t)D^-\phi(t) \leq -(Tu-T(0),u-0)_+ + |T(0)|\phi(t) \leq -\alpha(\phi(t))\phi(t) + |T(0)|\phi(t).$$

Hence, with $\beta = |T(0)|$,

(7) $\quad \phi(t)D^-\phi(t) \leq [\beta-\alpha(\phi(t))]\phi(t)$ in $t > 0$, $\phi(0) = |x|$.

Let $R = \inf\{r > 0 : \alpha(\rho) > \beta$ in $(r,\infty)\}$. If $\beta = 0$ we are done. Therefore, we may assume $\beta > 0$ and we have $0 < R < \infty$. If $|x| \leq R$ then (7) implies $\phi(t) \leq R$ in $t \geq 0$.

(iii) Let $U(t)x = u(t,x)$. By (ii), $U(t): K \to K$ for every $t \geq 0$. We claim that there exists $p > 0$ such that $U(p)$ is a strict contraction. Let $\phi(t) = |U(t)x-U(t)y|$. We have $\phi(0) = |x-y|$ and $\phi(t)D^-\phi(t) \leq -\alpha(\phi(t))\phi(t)$ in $t > 0$, hence $\phi(t) \leq \rho(t,|x-y|)$, where $\rho(t,r)$ denotes the solution of $\rho' = -\alpha(\rho)$, with $\rho(0) = r$. Therefore, we need only show that $\rho(p,r) \leq \frac{1}{2}r$ for some $p > 0$ and each $r \in [0,2R]$. Since $\liminf_{r\to 0} \alpha(r)/r > 0$, there exist $c > 0$ and $r_o > 0$ such that $\alpha(r) \geq cr$ in $[0,r_o]$. Now, if $r \leq r_o$ then $\rho' \leq -c\rho$, and therefore $\rho(t,r) \leq r/2$ for $t \geq c^{-1}\log 2$. If, however, $r_o < r \leq 2R$ then let t_r be the first time with $\rho(t,r) = r_o$. Since $\gamma = \inf\{\alpha(r) : r \geq r_o\} > 0$, we have $\rho' \leq -\gamma$ in $[0,t_r]$, hence $t_r \leq \gamma^{-1}(2R-r_o) = \bar{t}$, and thus

$$\rho(t,r) \leq r_o \exp[-c(t-\bar{t})] \quad \text{for } t \geq \bar{t} .$$

This implies $\rho(t,r) \leq \frac{1}{2}r$ for all $t \geq \bar{t} + c^{-1}\log(2r_o/r)$ and $r \in (r_o,2R]$. Hence, $\rho(p,r) \leq \frac{1}{2}r$ for $p = \bar{t} + c^{-1}\log 2$ and every $r \in [0,2R]$.

(iv) By (iii) there is a unique fixed point $x_p \in K$ of $U(p)$. Since $u(0,x_p) = u(p,x_p)$ and (1) is uniquely solvable, $u(t,x_p)$ is p-periodic. By (iii), we have

$$|u(t,x_p)-x_p| = |u(t+p,x_p)-x_p| = |U(p)u(t,x_p)-U(p)x_p| \leq$$

$$\leq \frac{1}{2}|u(t,x_p)-x_p| \quad ,$$

hence $u(t,x_p) = x_p$ for every $t \geq 0$, and therefore $Tx_p = 0$.

q.e.d.

Remark 2. Theorem 1 has been proved in [9,Theorem 3] in case $\alpha(r) = cr$ and T satisfies in addition a global Lipschitz condition. Obviously, we may replace "$\alpha(r) \to \infty$ as $r \to \infty$" by the weaker condition "$\liminf_{r\to\infty} \alpha(r) > |Tx_o|$ for some $x_o \in D$" . If $\liminf_{r\to 0} \frac{\alpha(r)}{r} = 0$ then the conclusion in (iii) may be wrong, as is shown by the example $\alpha(r) = r^q$ with $q > 1$.

Corollary 1. Let $D \subset X$ be closed, $T: D \to X$ continuous, $(Tx-Ty,x-y)_+ \leq k|x-y|^2$ for some $k < 1$, and

$$\rho((1-\lambda)x + \lambda Tx,D) = o(\lambda) \quad \text{as } \lambda \to 0+ ,$$

for each $x \in D$. Then T has a unique fixed point.

Corollary 1 is Proposition 3 from [6] . It follows immediately from Theorem 1 applied to $S = I-T$.

3. Zeros of strongly accretive operators on convex sets.

In case D is also convex, Theorem 1 can be improved considerably. This depends on two facts. At first, the existence theorem mentioned in step (i) holds if

$$(f(u) - f(v),u-v)_- \leq 0 ,$$

and secondly the boundary condition (6) is equivalent (in this case) to

(8) "If $x \in D$, $x^* \in X^* \setminus \{0\}$ and $x^*(x) = \sup_D x^*(y)$ then $x^*(-Tx) \leq 0$" ,

as follows immediately from the duality formula

$$\rho(z,D) = \max\{x^*(z)-\sup_D x^*(y):|x^*|=1\} \quad \text{for } z \in X.$$

Theorem 2. Let $D \subset X$ be closed and convex, $T: D \to X$ continuous and strongly accretive, and condition (6) hold. If

<u>either</u> "$(Tx,x)_+ \geq 0$ <u>for</u> $|x| \geq R$" <u>or</u> "$|Tx| \to \infty$ <u>as</u> $|x| \to \infty$" <u>then</u> $0 \in T(D)$.

<u>Proof</u>. Since everything is invariant under translation of D, except "$(Tx,x)_+ \geq 0$" , we may assume $0 \in D$, but we have to change "$(Tx,x)_+ \geq 0$" into "$(Tx,x+x_o)_+ \geq 0$ for $|x+x_o| \geq R$" (some $x_o \in X$ fixed) . Let $T_n = T + \frac{1}{n}I$.
If $x \in \partial D$, $x^* \in X^* \setminus \{0\}$ and $x^*(x) = \sup_D x^*(y)$ then

$$x^*(-T_n x) = x^*(-Tx) - \frac{1}{n}x^*(x) \leq 0 \quad ,$$

since $x^*(-Tx) \leq 0$ by (6) and (8) , and $x^*(x) \geq 0$ (since $0 \in D$) . Hence, (6) is also true for T_n . In addition, T_n is strongly accretive with $\alpha_n(r) = \frac{1}{n}r$. Now, in the proof of Theorem 1 applied to T_n we only have to change step (i): since D is convex, $(-(T_nu-T_nv),u-v)_- = -(T_nu-T_nv,u-v)_+ \leq 0$ is sufficient for (1) to have a unique global solution. Hence, T_n has a zero $x_n \in D$, i.e. $Tx_n = -\frac{1}{n}x_n$ for every n. Suppose first that "$|Tx| \to \infty$ as $|x| \to \infty$" holds. Since T is accretive, we obtain $|Tx_n| = |\frac{1}{n}x_n| \leq |T(0)|$. Hence, (x_n) must be bounded too.

If, however, $(Tx,x+x_o)_+ \geq 0$ for $|x+x_o| \geq R$, then $|x_n+x_o| \geq R$ implies $(x_n,x_n+x_o)_- \leq 0$. Let $x^* \in F(x_n+x_o)$. Then $|x^*| = |x_n+x_o|$ and $|x_n+x_o|^2 = x^*(x_n)+x^*(x_o) \leq x^*(x_n)+|x_n+x_o||x_o|$. This implies $|x_n+x_o|^2 \leq (x_n,x_n+x_o)_- + |x_n+x_o||x_o|$, and therefore $|x_n| \leq \max\{R+|x_o| , 2|x_o|\}$ for every n .

Since in both cases $|x_n| \leq c$ for some $c > 0$ and every n , we obtain

$$\alpha(|x_n-x_m|)|x_n-x_m| \leq (Tx_n-Tx_m,x_n-x_m)_+ \leq c(\frac{1}{n} + \frac{1}{m})|x_n-x_m|,$$

hence $\alpha(|x_n-x_m|) \to 0$ as $n,m \to \infty$. Therfore, (x_n) is a Cauchy sequence and thus convergent to some $x \in D$. Since T is continuous and $Tx_n = -\frac{1}{n}x_n \to 0$ as $n \to \infty$, $Tx = 0$.

q.e.d.

Notice that both "conditions at infinity" in Theorem 2 are weaker than "$\alpha(r) \to \infty$ as $r \to \infty$" . The following fixed point theorem is an immediate consequence of Theorem 2 .

Corollary 2. *Let $D \subset X$ be closed and convex ; $T: D \to X$ continuous and $(Tx-Ty,x-y)_- \le \alpha(|x-y|)|x-y|$ with $\alpha: R^1_+ \to R^1$ continuous and $\alpha(r) < r$ for $r > 0$; $\rho((1-\lambda)x+\lambda Tx,D) = o(\lambda)$ as $\lambda \to 0+$, for each $x \in D$. If D is unbounded , assume either "$|x-Tx| \to \infty$ as $|x| \to \infty$" or "$(Tx,x)_- \le |x|^2$ for $|x| \ge R$" . Then T has exactly one fixed point.*

Theorem 3. *Let $D \subset X$ be open , $T: D \to X$ continuous and strongly accretive with α satisfying in addition $\liminf_{r\to\infty} \alpha(r) > 0$. Then $T(D)$ is open .*

Proof. Let $x_o \in D$ and $K_{r_o}(x_o) = \{x : |x-x_o| \le r_o\} \subset D$. We have to show that there is some $\delta > 0$ such that $K_\delta(Tx_o) \subset T(D)$. Without loss of generality, we assume $x_o = 0$. Let $\delta > 0$ be such that $R_\delta = \inf\{r : \alpha(\rho) > \delta \text{ in } (r,\infty)\} < r_o$. Let $y \in K_\delta(T(0))$ and $T_n = T + \frac{1}{n}I$. Then, the initial value problem

(9) $\quad u' = -T_n u + y \quad , \quad u(0) = x \in K_{R_\delta}(0)$

has a unique local solution $u(t,x)$. Let $\phi(t) = |u(t,x)|$. As in the proof of Theorem 1 we obtain

$$\phi(t)D^-\phi(t) \le [|y-T(0)|-\alpha(\phi(t)]\phi(t) \ , \ \phi(0) = |x| \ .$$

Hence, $\phi(t) \le R_\delta$. This implies that $u(t,x)$ can be extended to a unique solution on $[0,\infty)$ with $|u(t,x)| \le R_\delta$ for $t \ge 0$. Since the operators $U(t)$, corresponding to (9) , are strict contractions from $K_{R_\delta}(0)$ into itself (for $t > 0$) , there exists x_n such that $T_n x_n - y = 0$ and $|x_n| \le R_\delta$. Hence, $Tx_n = -\frac{1}{n}x_n + y \to y$ as $n \to \infty$. Since

$$\alpha(|x_n-x_m|) \le |Tx_n-Tx_m| \to 0 \quad \text{as} \quad n,m \to \infty \ ,$$

we have $x_n \to x$ for some $x \in K_{R_\delta}(0)$ and $Tx = y$.

q.e.d.

Corollary 3. *Let $T: X \to X$ be continuous and strongly accretive. Assume either "$\liminf_{r\to\infty} \alpha(r) > 0$" or "$|Tx| \to \infty$ as $|x| \to \infty$" . Then T is a homeomorphism of X onto X .*

Proof. In case $\liminf_{r\to\infty} \alpha(r) > 0$, $T(X)$ is open by Theorem 3, and closed since $\alpha(|x-y|) \leq |Tx-Ty|$. Hence, $T(X) = X$, T is one to one, and $\alpha(|T^{-1}x - T^{-1}y|) \leq |x-y|$ implies the continuity of T^{-1} . Now, assume $|Tx| \to \infty$ as $|x| \to \infty$. Since $T-y$ (for fixed y) has the same properties as T , Theorem 2 implies $T(X) = X$, and T is one to one. If $y_n \to y$ then $T(T^{-1}y_n) = y_n$ and therefore $(T^{-1}y_n)$ is bounded. Together with $\alpha(|T^{-1}y_n - T^{-1}y|) \leq |y_n - y| \to 0$, this implies $T^{-1}y_n \to T^{-1}y$.

q.e.d.

A result similar to Corollary 3 is Theorem 4 in [3] , where $\alpha(r) = cr$ and T satisfies in addition a global Lipschitz condition.

4. Projectional solvability of equations involving strongly accretive operators.

In this section, we consider a real Banach space with some projection scheme $\{(X_n),(P_n)\}$, where X_n is a finite dimensional subspace of X , P_n a linear projection from X into X_n with $|P_n| = 1$ for every n and $P_n x \to x$ for each $x \in X$. Recall that the equation $Tx = y$ is said to be projectionally solvable if $P_n Tx = P_n y$ has exactly one solution $x_n \in X_n$, $x_n \to x$ as $n \to \infty$ and $Tx = y$.

Theorem 4. _Let_ $T: X \to X$ _be continuous_ , $(Tx-Ty, x-y)_- \geq \alpha(|x-y|)|x-y|$ _with_ α _as in Definition_ 1 _and_ $\liminf_{r\to\infty} \alpha(r) > 0$. _Then_ $Tx = y$ _is projectionally solvable for every_ $y \in X$.

Proof. Since $|P_n| = 1$ it is easy to see that $P_n^* F(x) \subset F(x)$ for $x \in X_n$. Hence, if $x \in X_n$ and $y \in X$ then $(P_n y, x)_- \geq (y,x)_-$. This implies that $P_n T$ has the same properties on X_n as T on X . Since $\liminf_{r\to\infty} \alpha(r) > 0$ there is exactly one $x_n \in X_n$ with $P_n T x_n = P_n y$ and exactly one $x_o \in X$ with $Tx_o = y$, by Corollary 3 . Since $P_n T x_o = P_n T x_n$ and $TP_n x_o \to Tx_o = y$ as $n \to \infty$, we obtain $\alpha(|x_n - P_n x_o|) \leq |y - TP_n x_o| \to 0$ as $n \to \infty$, hence $x_n \to x_o$ as $n \to \infty$.

q.e.d.

Remark 3. Theorem 4 has been proved in [7, Corollary 11] under the additional condition "α strictly increasing, $\alpha(r) \to \infty$ as $r \to \infty$, X reflexive, X^* strictly convex and F weakly continuous", and in [1, Theorem 8] under the additional condition "X^* strictly convex and F uniformly continuous on bounded sets" which is equivalent to "X^* is uniformly convex".

5. Some consequences for accretive operators.

Proposition 1. *Let X, D and T be as in Theorem 2, but instead of "T strongly accretive" assume "T accretive and T(D) closed". Then* $0 \in T(D)$.

Proof. In the proof of Theorem 2, we have obtained $Tx_n = -\frac{1}{n}x_n$ and (x_n) bounded, hence, $Tx_n \to 0$ and therefore $0 \in T(D)$.

q.e.d.

Recall that $T: D \to X$ is called pseudo-contractive if

$$|x-y| \leq |(1+\lambda)(x-y) - \lambda(Tx-Ty)|$$

for every $\lambda \geq 0$ and all $x,x \in D$. It is known that I-T is accretive if T is pseudo-contractive. Hence, Proposition 1 yields a fixed point theorem for such mappings. A similar result has been proved in [4, Theorem 1] : Let D be open and bounded, $0 \in D$, $T: \overline{D} \to X$ Lipschitzian and pseudo-contractive, $(I-T)(\overline{D})$ closed and the Leray-Schauder condition "$Tx \neq \lambda x$ for $x \in \partial D$ and $\lambda > 1$" satisfied, then T has a fixed point. In case D is also convex, we need only assume that T is continuous, but it is easy to see that our boundary condition "$\rho((1-\lambda)x + \lambda Tx,D) = o(\lambda)$" is stronger than that of Leray-Schauder, in general. In particular, if D is a ball with center 0 then our condition is equivalent to "$(Tx,x)_+ \leq |x|^2$ on ∂D".

The next proposition, proved in [10, Theorem 5] and [8, Theorem 2], is a typical result of the kind mentioned in the introduction.

Proposition 2. _Let_ X _be_ _reflexive_, $D \subset X$ _closed_ _bounded_ _convex_ _and_ _of_ _normal_ _structure_, $T\colon D \to X$ _accretive_ _and_ _Lipschitz_ _continuous_, _with_ _condition_ (6) _satisfied_. _Then_ $0 \in T(D)$.

Proof (following [10]). Since X and D are as indicated and $U(t)\colon D \to D$ is nonexpansive, U(p) has a fixed point x_p for every $p \geq 0$, by a well known fixed point theorem of Kirk. Hence, $U(t)x_p$ is p-periodic. But a theorem of Lasota/Yorke [5] says that T satisfying $|Tx-Ty| \leq L|x-y|$ has nonconstant p-periodic solutions for $p \geq 4/L$ only. Hence, $U(t)x_p$ is constant for $p < 4/L$ and therefore $Tx_p = 0$.

q.e.d.

If X in Proposition 2 is uniformly convex then "of normal structure" is automatically satisfied and "Lipschitz" is unnecessary, since by a fixed point theorem of Browder the U(t) have a fixed point x_o in common, whence $Tx_o = 0$ [9, Theorem 2]. In this case, it is even possible to replace "D bounded" by "$|Tx| \to \infty$ as $|x| \to \infty$", since the theorem on common fixed points remains true [2, p. 873]; see also [6, Propos. 4]. This observation implies $T(X) = X$ if "X uniformly convex, $T\colon X \to X$ accretive and continuous and $|Tx| \to \infty$ as $|x| \to \infty$" holds, a result that has been announced in [3, Theorem 5] under the additional condition "T uniformly continuous on bounded sets".

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