On bounded polynomials in several variables.

By

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1. Introduction.

The properties of bounded polynomials have been of interest ever since the appearance of the celebrated memoir of Tschebychev entitled Theorie des méchanismes connus sous le nom de parallelogrammes[1]). Questions concerning them arise not only in problems of polynomial approximation[2]), but also in connection with the theory of analytic functions. For polynomials in one variable, there is a considerable literature[3]). For polynomials in several variables, less has been done[4]).

The present notes had their origin in a study of the question as to the region of convergence of the power-series obtained by dropping the parentheses in series of harmonic polynomials known to converge in a certain circle. For this raised the question, given the maximum absolute value of a homogeneous polynomial of degree n in a circle about the

[1]) Mémoirs présentés à l'Academie Impérial des Sciences de St.-Petersbourg par divers savants, 7 (1852), pp. 539—568; Oeuvres 1 (Petrograd 1899), pp. 111—143.

[2]) For an extensive bibliography on the approximation problem, see the report of Dunham Jackson on The general theory of approximation by polynomials and trigonometric sums, Bull. Amer. Math. Soc. 27 (1921), pp. 415—431.

[3]) See the articles of I. Schur and G. Szegö in various numbers of the Mathematische Zeitschrift, in particular, I. Schur, Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall, 4 (1919), pp. 271—287; G. Szegö, Über einen Satz von A. Markoff, 23 (1925), pp. 45—61, also the literature cited by these authors. In addition may be cited Szegö in Acta Lit. ac. Sci. Reg. Univ. Hungaricae Francisco-Josephinae 3 (1923), Féjer in Journ. für Math. u. Phys. 146 (1916), and Math. Ann. 85 (1922), Egerváry in Archiv f. Math. u. Phys. 27 (1918), Dulac in Acta Math. 31 (1907).

[4]) But see Tonelli, I polynomi d'approssimazione de Tschebichev, Annali di Mat. 15 (1908), esp. p. 73; J. Chokhatte, Sur une formule générale dans la théorie des polynomes de Tschebycheff et ses applications, Comptes Rendus 181 (1925), pp. 329—331.

origin, what can be said of its coefficients? Results on this problem are attained in the following pages for polynomials in two and in m variables. The first derivatives of bounded homogeneous and non-homogeneous polynomials are then studied, and extensions of theorems due to S. Bernstein and A. Markoff are obtained. It is shown that all results hold for polynomials with complex coefficients.

2. The coefficients of bounded homogeneous polynomials.

In what follows, we shall be concerned with polynomials $P_n(x_1, x_2, \ldots, x_m)$, whose degrees do not exceed the index n, and whose coefficients may be real or imaginary. The variables will be restricted to the real region $r^2 = x_1^2 + x_2^2 + \ldots + x_m^2 \leqq 1$, and in this region the modulus of P_n will be required not to exceed 1. The proofs will be given on the assumption that the coefficients are real; this restriction will be removed at the end (p. 64). In the present section, P_n will be supposed to be homogeneous, and to depend on two variables only.

Theorem I. *Let $P_n(x, y)$ be a homogeneous polynomial of degree n, such that $|P_n| \leqq 1$ for $r^2 = x^2 + y^2 = 1$. Then the modulus of the coefficient a_k, of the term in $x^k y^{n-k}$ cannot exceed the binomial coefficient ${}^nC_k$. For values of k other than 0 and n, this bound is attained only when P_n is a constant of unit modulus times the real or imaginary part of $(x + iy)^n$.* [5]

To determine the maximum of $|a_k|$, we may restrict ourselves to polynomials which are either even in y or odd in y, since all the coefficients of P_n occur in one or the other of the properly bounded polynomials $\frac{1}{2}[P_n(x, y) + P_n(x, -y)]$ or $\frac{1}{2}[P_n(x, y) - P_n(x, -y)]$. Taking first the case in which $n - k$ is even, we consider the polynomial

$$P_n(x, y) = a_n x^n - a_{n-2} x^{n-2} y^2 + a_{n-4} x^{n-4} y^4 - \ldots,$$

and compare it with the polynomial

$$r^n \cos n\theta = x^n - {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 - \ldots .$$

If $-1 < \lambda < 1$, the difference

$$D(x, y, \lambda) = r^n \cos n\theta - \lambda P_n(x, y)$$
$$= d_n x^n - d_{n-2} x^{n-2} y^2 + d_{n-4} x^{n-4} y^4 - \ldots$$

is positive at all points of $r = 1$ where $r^n \cos n\theta = 1$, and negative where

[5]) The bound for a_0 and a_n is attained also for the polynomials x^n and y^n. Since the submission of this paper for publication, there has appeared the book of S. Bernstein, Leçons sur les propriétés extrémales des fonctions analytiques d'une variable réelle, Paris (1926). Professor Szegö has kindly pointed out that the bounds on the coefficients in theorem I, above, are an immediat result of inequalities given on page 56 of this book.

this function is -1. Hence $D(x, y, \lambda)$ has exactly one root[6]) between each pair of successive extremes of $\cos n\theta$, i. e. all its roots are real and distinct. Now it follows from Descartes rule of signs[7]) that if all the roots of a polynomial in x are real, and if any coefficient vanishes which is between two non-vanishing coefficients, the coefficients adjacent to the vanishing one must be different from zero and have opposite signs. This result is applicable to $D(x, y, \lambda)$, and the numbers d_n, d_{n-2}, $d_{n-4}, \ldots$ are all different from zero and have the same signs, for if either the first or last vanished, the polynomials would have a multiple root, and we have seen that this is not the case. Now $|a_n| = |P_n(1, 0)|$ cannot exceed 1, so that $d_n = 1 - \lambda a_n$ is positive, and therefore also d_{n-2}, d_{n-4}, $d_{n-6} \ldots$. The limits of these functions of λ, as $\lambda \to 1$ or as $\lambda \to -1$ are therefore $\geqq 0$, and so $|a_n| \leqq {}^nC_k$.

The second part of the theorem for $n-k$ even emerges when we consider the limiting form of $D(x, y, \lambda)$ as $\lambda \to 1$ or -1. The roots remain real, and can coalesce only in pairs, unless the limit is an identically vanishing polynomial. If $|a_k|$ attains its maximum for k other than 0 or n, then for $\lambda = 1$ or $\lambda = -1$, an intermediate one of the numbers d_n, d_{n-2}, $d_{n-4}, \ldots$ would vanish, and hence all preceding or all following numbers of this sequence would have to vanish. But this would require too high a multiplicity for a root. The only alternative is $D(x, y, 1) \equiv 0$ or $D(x, y, -1) \equiv 0$, i. e. $P_n \equiv \pm r^n \cos n\theta$.

For the case in which $n-k$ is odd, the same reasoning may be applied with the substitution of $r^n \sin n\theta$ for $r^n \cos n\theta$, until we arrive at the point where it is necessary to show that the numbers d_{n-1}, d_{n-3}, $d_{n-5}, \ldots$ are all positive. This fact will follow from a lemma, which will also find later application:

Lemma 1. *If* $P_n(x, y) = \alpha_n x^n + n\alpha_{n-1} x^{n-1} y + \ldots$ *is a homogeneous polynomial with real coefficients, such that* $|P_n| \leqq 1$ *for* $r = 1$, *then* $\alpha_n^2 + \alpha_{n-1}^2 \leqq 1$.

If ϱ and ε are determined so that $\alpha_n = \varrho \cos\varepsilon$ and $\alpha_{n-1} = \varrho \sin\varepsilon$, then $P_n(x, y) - \varrho r^n \cos(n\theta + \varepsilon) = R(x, y)$ will be a homogeneous polynomial containing y^2 as a factor. If, now, ϱ were greater than 1, $R(x, y)$ would have signs opposite to those of $\cos(n\theta + \varepsilon)$ at every extreme of this function, and hence it would have n distinct roots. But

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[6]) We mean by root of $D(x, y, \lambda)$, a value of the ratio $x : y$ for which $D(x, y, \lambda)$ vanishes; or also one of the values of θ which corresponds to such a value by the equations $x = r\cos\theta$, $y = r\sin\theta$, say in the interval $0 \leqq \theta < \pi$.

[7]) See, for instance, Weber, Lehrbuch der Algebra 1 (Braunschweig 1898), p. 350.

this contradicts the fact that it has a double root for $y = 0$. Hence $\varrho^2 = \alpha_n^2 + \alpha_{n-1}^2 \leqq 1$, as was to be proved.

From the lemma it follows that in a polynomial containing only odd powers of y, $|a_{n-1}| \leqq n$, and hence that $d_{n-1} > 0$ for $-1 < \lambda < 1$. Theorem I is thus etablished.

It may be remarked that the elementary method employed above leads to a very simple proof of the theorems of W. Markoff[8]) with respect to the coefficients of polynomials $P_n(x)$ of degree not exceeding n and maximum absolute value not exceeding 1 for $-1 \leqq x \leqq 1$. As comparison functions are used

$$T_n(x) = \cos(n \arccos x) \quad \text{and} \quad T_{n-1}(x) = \cos((n-1)\arccos x).$$

3. An application to convergence of power-series.

Before proceeding with the generalization of Theorem I to polynomials in more variables, let us consider an application. In the theory of the logarithmic potential, the potential, $U(x, y)$, due to a certain distribution of masses, is shown to be susceptible of expansion in a series

$$U = \sum_{n=0}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta),$$

which converges uniformly and absolutely in any closed region interior to the circle $r = R$ which passes through that point, or limit point, of the distribution, nearest the origin. It is also shown that the power-series in x and y,

$$S = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \dots$$

obtained by expanding $\cos n\theta$ and $\sin n\theta$ in the series in U, and replacing $r\cos\theta$ by x, and $r\sin\theta$ by y, converges in a neighborhood of the origin, i. e. that $U(x, y)$ is analytic in x and y. The extent of the region of convergence is not usually considered.

Theorem I enables us to give a region within which S is certainly convergent. For, from the uniform convergence of U, it follows that to any $\lambda > 0$ there corresponds an N such that for $n > N$,

$$|r^n(a_n \cos n\theta + b_n \sin n\theta)| \leqq 1 \quad \text{for} \quad r \leqq R - \lambda,$$

i. e. that $|r^n(R-\lambda)^n(a_n \cos n\theta + b_n \sin n\theta)| \leqq 1$ for $r \leqq 1$. Hence, by the theorem, $(R-\lambda)^n |a_{ij}| \leqq {}^nC_k$, where $i + j = n$. Accordingly S is dominated by the series obtained by expanding the terms of

[8]) Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen, Math. Ann. 77 (1916), pp. 213—258.

$\sum\left[\frac{|x|+|y|}{R-\lambda}\right]^n$. It therefore converges at the points of the square $|x|+|y|<R$. [9])

The result may be extended by means of the theorem of the next section. A harmonic function in space, whose representation by a series of spherical harmonics converges uniformly in any closed region interior to the sphere $r=R$ will be represented by a power-series in x, y and z, which converges at the points of the octahedron $|x|+|y|+|z|<R$, and similarly for higher space.

In two dimensions, the only possible additional points of convergence of the power series lie on the sides of the square or on the coördinate axes, as Bôcher showed. The situation is different in three or more dimensions, for the three dimensional region in which the power-series converges may go beyond the octahedron, and have various shapes, but exact results, as far as I know, have not been obtained.

4. Generalization to polynomials in *m* variables.

Theorem II. *Let $P_n(x_1, x_2, \ldots, x_n)$ be a homogeneous polynomial of degree n, such that $|P_n| \leqq 1$ for $r^2 = x_1^2 + x_2^2 + \ldots + x_m^2 = 1$. Then the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$ cannot exceed in absolute value the polynomial coefficient* $\frac{n!}{k_1!\, k_2! \ldots k_m!}$. It should be remarked, however, that this formula does not, in general, give actual maxima[10]).

The theorem reduces to the preceding one when $m=2$. Let us suppose it has been established when there are $m-1$ variables. We make the substitution $x_1 = \varrho\, y_1, x_2 = \varrho\, y_2, \ldots, x_{m-1} = \varrho\, y_{m-1}, x_m = x_m$, and impose the restrictions $y_1^2 + y_2^2 + \ldots + y_{m-1}^2 = 1,\ \ \varrho^2 + x_m^2 = 1$. The polynomial P_n then takes the form $\sum_0^n A_{k_m} \varrho^{n-k_m} x_m^{k_m}$, where A_{k_m} is a homogeneous polynomial in $y_1, y_2, \ldots, y_{m-1}$, of degree $n-k_m$, and subject, by Theorem I to the inequality $|A_{k_m}| \leqq \frac{n!}{(n-k_m)!\, k_m!}$. Now Theorem II holds, by hypothesis, if m is replaced by $m-1$, and so permits

[9]) This is included in a result obtained by Bôcher, by a different method; On the regions of convergence of power-series which represent two-dimensional harmonic functions, Trans. Amer. Math. Soc. **15** (1909), pp. 271—278.

[10]) For instance, the coefficient of the term in $x\,y\,z$ in $P_3(x, y, z)$ has as maximum absolute value $3\sqrt{3}$, whereas the theorem gives the bound 6.

us to infer that the coefficient of $y_1^{k_1} y_2^{k_2} \dots y_{m-1}^{k_{m-1}}$ in A_{k_m}, i. e. the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$ in P_n, cannot exceed

$$\frac{n!}{(n-k_m)!\,k_m!} \times \frac{(n-k_m)!}{k_1!\,k_2!\dots k_{m-1}!} = \frac{n!}{k_1!\,k_2!\dots k_m!},$$

as was to be shown.

5. The derivatives of bounded homogeneous polynomials in x and y.

We are here concerned with the maximum absolute value of the derivate DP_n of a polynomial P_n, in any direction. In the case of polynomials with real coefficients, this will also be the maximum of the magnitude of the gradient of the polynomial, i. e. of the vector whose components along the coördinate axes are the corresponding partial derivatives of P_n. We start with the case of two variables.

Theorem III. *Let $P_n(x, y)$ be a homogeneous polynomial of degree n, such that $|P_n| \leqq 1$ for $r = 1$. Then $|DP_n|$ never exceeds n for $r \leqq 1$.*

It is obvious that $|\nabla P_n|$, the magnitude of the gradient of P_n, attains its maximum for points of the circle $r \leqq 1$ on the circumference $r = 1$, because of the homogeneity of P_n. Since the gradient is invariant under a rotation, we are at liberty to assume that the point at which its magnitude is greatest is the point $(1, 0)$. But at this point, the polynomial $P_n(x, y) = \alpha_n x^n + n\alpha_{n-1} x^{n-1} y + \dots$ has the partial derivatives $\frac{\partial P_n}{\partial x} = n\alpha_n$ and $\frac{\partial P_n}{\partial y} = n\alpha_{n-1}$. Hence, by Lemma 1, the magnitude of the gradient cannot exceed n, as was to be proved.

It is of interest to compare this result with the theorem of Bernstein[11]), which, although usually enunciated for trigonometric polynomials, holds for the boundary values of polynomials in x and y. We infer from Bernstein's theorem that the *tangential* derivatives of P_n cannot exceed n in absolute value. Obviously, the normal derivatives of the homogeneous function P_n cannot exceed n, so that the gradient cannot exceed $\sqrt{2}\,n$ in magnitude.

Our theorem is thus sharper than what we should infer in this way. On the other hand, it is apparently less general than Bernstein's theorem

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[11]) Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné, Mémoir couronné, Brussels, 1912. (Included in the Mémoirs publiés par la Classe des Sciences de l'Académie Royale de Belgique (2) 4, p. 20.) For a more complete statement and simple proof, see de la Vallee Poussin, Sur le maximum du module de la derivée d'une expression trigonométrique d'ordre et du module bornes, Comptes Rendus 166 (1918), pp. 843—846; or Leçons sur l'approximation des fonctions d'une variable réelle, Paris 1919, pp. 39—42.

in the information it gives with respect to tangential derivatives, for the latter applies also to the values on $r = 1$ of non-homogeneous polynomials. Bernstein's theorem can be inferred from Theorem III, and for reasons that will appear later, it is worth while to give the reasoning.

Let $P_n(x, y)$ be any polynomial of degree n, with real coefficients, and not greater than 1 in absolute value for $r = 1$. Suppose the point at which the tangential derivative is greatest has been brought by a rotation to $(1, 0)$. By multiplying the terms of P_n by appropriate powers of $r = \sqrt{x^2 + y^2}$, it can be brought, without altering its values on $r = 1$, to the form $P_n(x, y) = Q_n(x, y) + rR_{n-1}(x, y)$, where Q_n and R_{n-1} are homogeneous polynomials of degrees indicated by the subscripts, one degree being even, and one odd. Hence, as both $P_n(x, y)$ and $P_n(-x, -y)$ are not greater than 1 in absolute value on $r = 1$, it follows that $|Q_n| + |R_{n-1}| \leqq 1$ on $r = 1$, and therefore the homogeneous polynomial, $S_n(x, y) = Q_n(x, y) + xR_{n-1}(x, y)$ is similarly bounded. Hence, its tangential derivative at $(1, 0)$, which coincides with that of P_n, cannot exceed n, and this gives the desired result.

Incidentally, there has been proved the following lemma, which will be useful in the next section.

Lemma 2. *Given the function* $F_n(x, y) = Q_n(x, y) + rR_{n-1}(x, y)$, *where* $r = \sqrt{x^2 + y^2}$, *and* Q_n *and* R_{n-1} *are homogeneous polynomials of degrees indicated by the subscripts; if* $|F_n| \leqq 1$ *for* $r = 1$, *then no tangential derivative of* F_n *on* $r = 1$ *can exceed* n *in absolute value.*

An obvious consequence of theorem III and the property of homogeneity is the following:

Let S *denote a region of the* x, y*-plane contained in a circle of radius* R *about the origin, and containing a concentric circle of radius* ϱ. *Then if the homogeneous polynomial* $P_n(x, y)$ *is not greater than* M *in absolute value in* S, $|DP_n| \leqq MnR^{n-1}/\varrho^n$ *in* S. It will presently appear that the same is true of homogeneous polynomials in m variables. Evidently, this corellary may be used in the discussion of the termwise differentiability of series of homogeneous polynomials.

6. Derivatives of homogeneous polynomials in m variables.

Theorem IV. *Let* $P_n(x_1, x_2, \ldots, x_m)$ *be a homogeneous polynomial of degree* n *whose absolute value does not exceed* 1 *for* $r = 1$. *Then* $|DP_n| \leqq n$ *for* $r \leqq 1$.

Suppose the gradient of P_n attains its maximum magnitude at $(a_1, a_2, \ldots, a_m)$, and that it has at this point the components $(b_1, b_2, \ldots, b_m)$.

We rotate the axes about the origin so that the plane of the origin and the points $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_m)$ becomes the plane of x_1, x_2. P_n then becomes a homogeneous polynomial in x_1 and x_2, when the other variables are equated to 0, and the gradient of maximum magnitude is unaltered by the process. Thus the desired result follows immediately from Theorem III.

By the use of reasoning previously employed, we may establish a generalization of Bernstein's theorem to m dimensions:

Theorem V. *Let $P_n(x_1, x_2, \ldots, x_m)$ be any polynomial of degree not greater than n, whose absolute value does not exceed 1 for $r = 1$. Then the tangential derivatives of P_n on the hypersphere $r = 1$ never exceed n in absolute value.*

In particular, if $S_n(\theta, \varphi)$ is a linear function of spherical harmonics of orders not exceeding n, such that $|S_n| \leqq 1$, then

$$\sqrt{\left(\frac{\partial S_n}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S_n}{\partial \varphi}\right)^2} \leqq n.$$

7. A. Markoff's theorem and its extension to m-space.

Markoff's[12]) theorem is to the effect that if $P_n(x)$ is a polynomial of degree not exceeding n and absolute value on $-1 \leqq x \leqq 1$ not exceeding 1, then the absolute value of its derivative on the same interval does not exceed n^2.

This theorem, in a more precise form, may be obtained as follows. $P_n(x)$ is a particular case of a polynomial in x and y, and as such, its tangential derivatives on the circle $r = 1$ cannot exceed n in absolute value, by Bernstein's theorem. That is, $\left| y \frac{\partial P_n}{\partial x} - x \frac{\partial P_n}{\partial y} \right| \leqq n$, and therefore $|P_n'(x)| \leqq n / |y| = n / \sqrt{1 - x^2}$. The rest is a result of the following emma:

l If a polynomial of degree $n - 1$ is dominated by the function $n / \sqrt{1 - x^2}$ on $-1 \leqq x \leqq 1$, it is dominated on the same interval by n^2. Suppose this were not so, and that the polynomial $P_{n-1}(x)$, dominated by $n / \sqrt{1 - x^2}$ on $-1 \leqq x \leqq 1$, exceeded n^2 at a point $x = a$ of this interval. By legitimate sign changes we are free to assume that $0 \leqq a \leqq 1$, and that $P_{n-1}(a) > 0$. We consider the polynomial of degree $n - 1$, $D(x) = P_{n-1}(x) - \lambda \frac{n \sin n\varphi}{\sin \varphi}$, where $\cos\varphi = x$, and λ is a number sufficiently slightly greater than 1 so that $P_{n-1}(a) > \lambda n^2$. The function $D(x)$

[12]) On a problem of Mendeleieff (in Russian), Memoirs of the St. Petersburg Academy 62 (1889). See also I. Schur, loc. cit., esp. p. 272–277.

must alternate in sign at the interior extremes of $\sin n\varphi$, i. e. for $x = \cos\frac{(2n-1)\pi}{2n}$, $\cos\frac{(2n-3)\pi}{2n}, \ldots, \cos\frac{\pi}{2n}$, and hence have $n-1$ roots in the intervals between these points. Now the point a cannot lie between 0 and $\cos\frac{\pi}{2n}$, for in this interval, $n/\sqrt{1-x^2} = n/\sin\varphi \leqq n/\sin\frac{\pi}{2n}$, a number easily shown to be less than n^2 for $n \geqq 2$ (for $n=1$ the theorem is trivial). Hence $D(x)$, negative at $\cos\frac{\pi}{2n}$, would be positive at a point between this one and 1, and so would have n roots. This is impossible, since $D(x)$ is not identically 0. We thus arrive at the result contained in the article of Schur (loc. cit.):

If $P_n(x)$ is a polynomial of degree not greater than n, whose absolute value does not exceed 1 for $-1 \leqq x \leqq 1$, then $P_n'(x)$ is dominated in this interval by the smaller of the two numbers $n/\sqrt{1-x^2}$ and n^2.[13])

The result is readily generalized to polynomials in m variables:

Theorem VI. *Let $P_n(x_1, x_2, \ldots, x_m)$ be a polynomial of degree not greater than n, and of absolute value not greater than 1 for $r^2 = x_1^2 + x_2^2 + \ldots + x_m^2 \leqq 1$. Then for $r \leqq 1$, DP_n is dominated by the smaller of the two numbers $n/\sqrt{1-r^2}$ and n^2.*

By a rotation of axes such as was used in the last section, we may reduce the general case to that of two variables. We may think of the polynomial $P_n(x, y)$ as one in x, y, z[14]), and as such its tangential derivatives on the sphere $r=1$ cannot exceed n in absolute value, by theorem V. If α, β, γ, be the direction cosines of a (non-vertical) tangent to the sphere, we infer that $\left|\alpha\frac{\partial P_n}{\partial x} + \beta\frac{\partial P_n}{\partial y}\right| \leqq n$, and hence that the derivative of $P_n(x, y)$ in the direction $\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$, $\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, at any interior point $r<1$, is not greater in absolute value than $n/\sqrt{\alpha^2+\beta^2}$, a function of α and β which, under the conditions on α, β, γ, and x, y, z, cannot exceed $n/|z| = n/\sqrt{1-r^2}$.

Now the derivative of $P_n(x, y)$ in a fixed direction at the points of a line through the origin and containing the point at which this derivative

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[13]) Bernstein gives the following theorem: If the polynomial $P_n(x)$ of degree not greater than n satisfies the inequality $|P_n(x)| \leqq L$ on (a, b), then $|P_n'(x)|\sqrt{(a-x)(x-b)} \leqq nL$ on (a, b). On the best polynomial approximation to continuous functions, Thesis, 1912 (in Russian). I am indebted to Prof. T. A. Shohat for this note.

[14]) The coefficients of the terms in z being 0.

is greatest in absolute value, is a polynomial in r (r being allowed negative values), and so, by the reasoning employed above, cannot exceed n^2. Theorem VI is thus established.

8. Removal of the restriction to real coefficients.

In multiplying a polynomial by a complex number of unit modulus, we change neither the bounds of the absolute value of the polynomial, nor of its coefficients, nor of its derivatives. Such a multiplier may always be so chosen as to reduce to a real number any given coefficient, or derivative at a point. Thus the given coefficient, or derivative at a point will be the corresponding coefficient or derivative of the real part of the new polynomial, and this real polynomial will be bounded in absolute value by unity in any region in which the given polynomial is so bounded. The results attained for polynomials with real coefficients therefore hold for polynomials with complex coefficients.

Oct. 15, 1925.

(Eingegangen am 2. November 1925.)