Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions.

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This paper reproduces a paper which was published in 1917 under the same title[1]), with considerable abbreviations and generalizations. A detailed discussion of some questions which are explained here but briefly, as well as extended references, may be found in the above mentioned paper.

§ 1.

Asymptotic expressions of solutions of differential equations containing a parameter.

1. We begin with a discussion of the system of n differential equations

$$\text{(I)} \qquad \frac{dy_i}{dx} = \sum_{k=1}^{n} a_{ik}(x, \varrho)\, y_k,$$

whose coefficients are functions of a real variable x in the interval $a \leqq x \leqq b$ and of a complex parameter ϱ. Suppose the functions $a_{ik}(x, \varrho)$ admit of expansions

$$\text{(1)} \qquad a_{ik}(x, \varrho) = \sum_{\nu=0}^{\infty} \varrho^{\varkappa-\nu}\, a_{ik}^{(\varkappa-\nu)}(x) \quad (i, k = 1, 2, \ldots, n)$$

for $|\varrho|$ sufficiently large, i. e. for

$$\text{(2)} \qquad |\varrho| \geqq R_0,$$

R_0 being a given positive constant.

If the functions $a_{ik}^{(\varkappa-\nu)}(x)$ possess derivatives of all orders in (a, b) and if the "characteristic equation"

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[1]) Petrograd 1917 (in Russian).

$$(3)\qquad \Phi(\theta)\equiv\begin{vmatrix} a_{11}^{(\varkappa)}(x)-\theta & a_{12}^{(\varkappa)}(x)\ldots & a_{1n}^{(\varkappa)}(x)\\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\\ a_{n1}^{(\varkappa)}(x) & a_{n2}^{(\varkappa)}(x)\ldots & a_{nn}^{(\varkappa)}(x)-\theta\end{vmatrix}=0$$

has simple roots

$$(4)\qquad \varphi_1^{(\varkappa)}(x),\ldots,\varphi_n^{(\varkappa)}(x)$$

for every value of x, then a matrix of formal solutions of (I) can be found of the form

$$(5)\qquad e^{\int\omega_j(x,\varrho)\,dx}\sum_{\nu=0}^{\infty}y_{ij}^{(\nu)}(x)\,\varrho^{-\nu},$$

where

$$\omega_j(x,\varrho)=\varrho^{\varkappa}\varphi_j^{(\varkappa)}(x)+\ldots+\varrho\,\varphi_j^{(1)}(x)$$

and the coefficients

$$\varphi_j^{(\nu)}(x),\qquad y_{ij}^{(\nu)}(x)$$

are determined by immediate substitution of (5) in the system (I). The series (5) are divergent in the general case, but they can be used for approximate representation of certain solutions of (I), as it is shown by the following theorem:

Theorem 1. *Suppose the coefficients* $a_{ik}(x,\varrho)$ *of the system* (I) *to satisfy the following conditions:*

1°. *Series* (1) *are convergent on the region* (2) *and the functions*

$$a_{ik}^{(\varkappa-\nu)}(x)\qquad (i,k=1,2,\ldots,n;\ \nu=0,1,2,\ldots)$$

are continuous and uniformly bounded on (a,b).

2°. *If* m *denotes a given positive integer, and integers* s *and* r *are determined by the condition*

$$m=s\varkappa+r+1\qquad\qquad (0\leqq r\leqq\varkappa-1),$$

it is assumed that the functions

$$\frac{d^{s+1}a_{ik}^{(\varkappa)}(x)}{dx^{s+1}},\ \ldots,\ \frac{d^{s+1}a_{ik}^{(\varkappa-r)}(x)}{dx^{s+1}}$$

$$\frac{d^{s}a_{ik}^{(\varkappa-r-1)}(x)}{dx^{s}},\ \ldots,\ \frac{d^{s}a_{ik}^{(-r)}(x)}{dx^{s}}$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$a_{ik}^{(-m+\varkappa)}(x),\ \ldots,\ a_{ik}^{(-m+1)}(x)$$

possess continuous derivatives of the first order on (a,b).

3°. *The roots* (4) *of the characteristic equation* (3) *are distinct for every value of* x *in* (a,b).

4°. *There exists such an infinite part* $(\mathfrak{D})$ *of the region* $|\varrho|\geqq R_0$, *in which the inequalities*

$$(6)\qquad \Re\omega_1(x,\varrho) \leqq \Re\omega_2(x,\varrho) \leqq \ldots \leqq \Re\omega_n(x,\varrho)$$

are satisfied for every value of x *in* (a, b).

Then there exists a matrix of solutions of the system (I), *of the form*

$$(7)\qquad y_{ij}^{(m)}(x,\varrho) = e^{\int_a^x \omega_j(x,\varrho)\,dx}\left\{\sum_{\nu=0}^{m-1} y_{ij}^{(\nu)}(x)\,\varrho^{-\nu} + \frac{E(x,\varrho)}{\varrho^m}\right\},$$

where $E(x, \varrho)$ *are bounded and continuous functions of* x *in* (a, b) *and of* ϱ *in* $(\mathfrak{D})$.[2])

The theorem 1 was proved in the paper[1]) using a generalization of the classic Dini's method[3]).

2. In the most important case, when the system (I) is equivalent to a single differential equation of order n, and $\varkappa = 1$, we obtain:

Theorem 2. *Suppose the coefficients of the differential equation*

$$(\mathrm{II})\qquad \frac{d^n y}{dx^n} + P_1(x,\varrho)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_n(x,\varrho)\,y = 0$$

to satisfy the following conditions:

1°. *The functions* $P_i(x, \varrho)$ *can be expanded in descending powers of* ϱ *on the region* (2):

$$(8)\qquad P_i(x,\varrho) \equiv \varrho^i \sum_{j=0}^{\infty} p_{ij}(x)\,\varrho^{-j} \equiv \varrho^i p_i(x,\varrho) \qquad (i = 1, 2, \ldots, n),$$

the coefficients $p_{ij}(x)$ *being continuous and uniformly bounded on* (a, b).

2°. *The functions*

$$p_{i0}(x) \qquad (i = 1, 2, \ldots, n)$$

possess continuous derivatives of the second order, the functions

$$p_{i1}(x) \qquad (i = 1, 2, \ldots, n)$$

possess continuous derivatives of the first order on (a, b).

3°. *The characteristic equation*

$$(9)\qquad \Phi(\theta) \equiv \theta^n + p_{10}(x)\theta^{n-1} + \ldots + p_{n-1\,0}(x)\theta + p_{n0}(x) = 0$$

has roots $\varphi_1, \ldots, \varphi_n$ *which are distinct for all values of* x *in* (a, b).

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[2]) We shall use the symbol $E(\varrho, \ldots)$ in order to denote functions of ϱ and of other variables, bounded on $(\mathfrak{D})$ or, more generally, bounded for large values of $|\varrho|$.

[1]) Chapters I and II.

[3]) The particular case of a system with coefficients linear in ϱ was discussed also by G. D. Birkhoff and R. E. Langer, The boundary problems and developements etc. Proc. of Am. Ac. of Arts and Sc. **58** (1923), n° 2. The theorem holds true when the series (1) are not convergent, but only asymptotic in $(\mathfrak{D})$, and when the inequalities (6) are replaced by more general ones: $\Re\omega_1 \leqq \Re(\omega_2 + \alpha) \leqq \ldots \leqq \Re(\omega_n + (n-1)\alpha)$, where α is a constant. In the case, when the functions $a_{ik}^{(\nu)}$ have derivatives of all orders on (a, b) the matrix (7) can be chosen as to be independent of m. Cfr. our paper (in collaboration with A. Besicovich), Math. Zeitschr. **21** (1924), pp. 119—125.

4°. *There exists such an infinite part* $(\mathfrak{D})$ *of the region* (2) *on which*

$$\Re(\varrho\varphi_1) \leqq \Re(\varrho\varphi_2) \leqq \ldots \leqq \Re(\varrho\varphi_n) \tag{10}$$

for every value of x *in* (a, b).

Then there exists a fundamental system of solutions of (II), *which on* $(\mathfrak{D})$ *can be represented by*

$$y_\lambda(x, \varrho) = e^{\varrho\int_a^x \varphi_\lambda(x)\,dx}\left\{\eta_\lambda(x) + \frac{E_\lambda(x, \varrho)}{\varrho}\right\}, \tag{11}$$

where the functions

$$\begin{gathered}\eta_\lambda(x) = \frac{1}{\sqrt{\Phi'(\varphi_\lambda)}}\,e^{-\int_a^x \frac{\Phi_1(\varphi_\lambda)}{\Phi'(\varphi_\lambda)}dx} \qquad (\lambda = 1, 2, \ldots, n),\\ \Phi'(\theta) \equiv \frac{d\Phi(\theta)}{d\theta};\quad \Phi_1(\theta) \equiv p_{11}(x)\theta^{n-1} + \ldots + p_{n-1\,1}(x)\theta + p_{n1}(x)\end{gathered} \tag{12}$$

possess continuous derivatives of the second order, and the functions $E_\lambda(x, \varrho)$ *are continuous and bounded.*

The formulas (11) *can be differentiated* $(n-1)$ *times with respect to* x, *conserving each time the highest term in* ϱ *only, so that*

$$\frac{d^s y_\lambda(x, \varrho)}{dx^s} = e^{\varrho\int_a^x \varphi_\lambda(x)\,dx}\,\varrho^s[\varphi_\lambda(x)]^s\left\{\eta_\lambda(x) + \frac{E}{\varrho}\right\}\text{[4]} \quad (s = 0, 1, \ldots, n-1). \tag{13}$$

3. A detailed discussion shows that *if the condition* 2° *of Theorem* 2 *is replaced by a more restrictive one:*

2°_1. *The functions*

$$\frac{d^2 p_{i0}(x)}{dx^2},\quad \frac{dp_{i1}(x)}{dx},\quad p_{i2}(x) \qquad (i = 1, 2, \ldots, n)$$

are continuous and of bounded variation on (a, b), *and the arguments of all the differences*

$$\varphi_i(x) - \varphi_j(x) \qquad (i, j = 1, 2, \ldots, n)$$

satisfy Dirichlet condition on (a, b), *then*

$$y_\lambda(x, \varrho) = e^{\varrho\int_a^x \varphi_\lambda(x)\,dx}\left\{\eta_\lambda(x) + \frac{\zeta_\lambda(x)}{\varrho} + \frac{E(x, \varrho)}{\varrho^2}\right\} \quad (\lambda = 1, 2, \ldots, n) \tag{14}$$

and the functions

$$\frac{d^2\eta_\lambda(x)}{dx^2},\quad \frac{d\zeta_\lambda(x)}{dx} \qquad (\lambda = 1, 2, \ldots, n)$$

are continuous and of bounded variation on (a, b).[1])

[4]) An analogous theorem was proved by G. D. Birkhoff, On the asymptotic character of solutions of certain linear diff. equat., Trans. Am. Math. Soc. **9** (1908)

[1]) pp. 75—79.

4. If the conditions of Theorems 1 and 2 are not satisfied, the discussion of the asymptotic character of solutions of (I) and (II) presents considerable difficulties, and even the formal character of the series (5) may change. For instance, if the characteristic equation has multiple roots [of the same multiplicity throughout the whole interval (a, b)], then instead of formal series (5) we obtain series, which contain, besides integral, also fractional powers of ϱ.

In the special case of an equation of the second order

$$\frac{d^2 y}{dx^2} + \varrho\, p_1(x, \varrho) \frac{dy}{dx} + \varrho^2 p_2(x, \varrho)\, y = 0,$$

$$p_1(x, \varrho) = \sum_{j=0}^{\infty} \varrho^{-j} p_{1j}(x), \quad p_2(x, \varrho) = \sum_{j=0}^{\infty} \varrho^{-j} p_{2j}(x),$$

suppose that the characteristic equation has a multiple root $\varphi(x) = -\frac{1}{2} p_{10}(x)$ for all values of x in (a, b), so that

$$p_{10}(x)^2 - 4\, p_{20}(x) \equiv 0 \quad \text{on} \quad (a, b).$$

Then it can be proved that *either*:

$$\psi(x) \equiv -\frac{1}{2} \frac{d p_{10}(x)}{dx} - \frac{1}{2} p_{10}(x)\, p_{11}(x) + p_{21}(x) \equiv 0,$$

in which case the equation has two solutions of the form

$$e^{\varrho \int_a^x \varphi(x)\, dx} \sum_{\nu=0}^{\infty} y_\lambda^{(\nu)}(x)\, \varrho^{-\nu} \qquad (\lambda = 1, 2),$$

the infinite series being *convergent* on (2), *or*: $\psi(x) \neq 0$, in which case the equation has two *formal* solutions

$$y_\lambda(x, \varrho) = e^{\varrho \int_a^x \varphi(x)\, dx \pm \sqrt{\varrho} \int_a^x \sqrt{\psi(x)}\, dx} \sum_{\nu=0}^{\infty} y_\lambda^{(\nu)}(x)\, \varrho^{-\frac{\nu}{2}},$$

where the sign $(+)$ corresponds to $\lambda = 1$, the sign $(-)$ corresponds to $\lambda = 2$.

It is easy to prove the asymptotic character of these formal series.

§ 2.

The Green's function in general.

5. The "boundary problem (L)" consists in the determination of a function $y(x)$, which satisfies the differential equation

$$\text{(I)} \qquad L(y) \equiv y^{(n)} + P_1(x)\, y^{(n-1)} + \ldots + P_n(x)\, y = f(x)$$

and n supplementary conditions

$$\text{(II)} \qquad L_i(y) \equiv \sum_{k=1}^{n} \int_a^b y^{(k-1)}(t)\, d\alpha_{ik}(t) \qquad (i = 1, 2, \ldots, n).$$

We suppose that

1°. *The functions*

(1) $$P_1(x), \ldots, P_n(x)$$

are continuous on the interval (a, b) *and the functions*

(2) $$\alpha_{ik}(x) \qquad (i, k = 1, 2, \ldots, n)$$

are of bounded variation, the integrals being taken in the sense of Stieltjes.

2°. *The linear operators*

$$L_1(y),\ L_2(y),\ \ldots,\ L_n(y)$$

are linearly independent.

If we replace the operators (II) by any n linearly independent linear combinations with constant coefficients, we obtain a problem which is equivalent to the problem (L), and we shall make no distinction between all these problems.

If the function $f(x)$ is not equal to zero identically on (a, b), the problem is *non-homogeneous*; in the contrary case the problem is called *homogeneous.*

If the homogeneous problem admits of at least one solution, which is not identically zero on (a, b), such a problem is called *compatible*; if only $y \equiv 0$ can satisfy the homogeneous problem (L), it is *incompatible.*

Denoting by

(3) $$t_{ik}^{(\nu)} \quad (i, k = 1, 2, \ldots, n;\ \nu = 1, 2, 3, \ldots)$$

the points of discontinuity of the functions $\alpha_{ik}(t)$, the end-points a, b, inclusive, we can write the operators $L_i(y)$ in the form:

(4) $$L_i(y) \equiv \sum_{k=1}^{n} \sum_{\nu=1}^{\infty} \beta_{ik}^{(\nu)} y^{(k-1)}(t_{ik}^{(\nu)}) + \sum_{k=1}^{n} \int_a^b y^{(k-1)}(t)\, d\beta_{ik}(t)$$

where the constants $\beta_{ik}^{(\nu)}$ are determined by the jumps of the functions $\alpha_{ik}(t)$, and the functions $\beta_{ik}(t)$ are continuous and of bounded variation. Finally, integrating by parts, we reduce $L_i(y)$ to the form:

(5) $$L_i(y) \equiv \sum_{k=1}^{n} \sum_{\nu=1}^{\infty} \gamma_{ik}^{(\nu)} y^{(k-1)}(t_{ik}^{(\nu)}) + \int_a^b y^{(n-1)}(t)\, d\gamma_i(t)$$

where $\gamma_{ik}^{(\nu)}$ are constants and $\gamma_i(t)$ are continuous functions of bounded variation on (a, b).[5]

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[5]) A special case of the operators (II), where the number of the points of discontinuity was finite and the functions $\beta_{ik}(t)$ were absolutely continuous on (a, b), was discussed in my paper [1]). A more special case, where all $\beta_{ik}(t) \equiv 0$, was considered in two papers by Ch. E. Wilder, Trans. Am. Math. Soc. **18** (1917), pp. 415–442, and **19** (1918), pp. 157–186.

6. Definition. *The Green's function of the problem (L) is a function G (x, t) of two variables x, t, which is determined and continuous with respect to each variable on (a, b), except for*

$$t = t_{ik}^{(\nu)} \quad (i, k = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots)$$

and for

$$x = t, \quad \text{when} \quad n = 1,$$

and which enables to represent the solution of the non-homogeneous problem (L) in the form of a definite integral

$$y(x) = \int_a^b G(x, t)\, f(t)\, dt \tag{6}$$

for an arbitrary choice of the function f(x).

The existence and the properties of the Green's function have been discussed by many authors in special cases. It is not difficult to make this discussion in our general case. Denote by

$$u_1(x),\ u_2(x),\ \ldots,\ u_n(x) \tag{7}$$

a fundamental system of solutions of the homogeneous differential equation:

$$L(y) = 0, \tag{8}$$

and let

$$\delta(x) = \begin{vmatrix} u_1^{(n-1)}(x) \ldots u_n^{(n-1)}(x) \\ u_1^{(n-2)}(x) \ldots u_n^{(n-2)}(x) \\ \cdots\cdots\cdots \\ u_1(x) \quad \ldots u_n(x) \end{vmatrix} = \delta(a)\, e^{-\int_a^x P_1(t)\, dt} \tag{9}$$

$$g(x, t) = \pm \frac{1}{2\delta(t)} \begin{vmatrix} u_1(x) \quad \ldots u_n(x) \\ u_1^{(n-2)}(t) \ldots u_n^{(n-2)}(t) \\ \cdots\cdots\cdots \\ u_1(t) \quad \ldots u_n(t) \end{vmatrix}, \qquad \begin{matrix} + \text{ if } x > t, \\ - \text{ if } x < t, \end{matrix} \tag{10}$$

$$u_{ik} = L_i(u_k) \qquad (i, k = 1, 2, \ldots, n), \tag{11}$$

$$\Delta = |u_{ik}|. \tag{12}$$

7. Theorem 3. *Only two cases are possible: either*

1°. *The determinant Δ is equal to zero, in which case the homogeneous problem (L) is compatible, the Green's function does not exist and the non-homogeneous problem (L) is impossible for an arbitrary function f(x). Or*

2°. *The determinant $\Delta \neq 0$, in which case the homogeneous problem (L) is incompatible, the Green's function exists and is determined uniquely (at the points of continuity) by the formula*

$$(13)\qquad G(x,t)=\frac{(-1)^n}{\Delta}\begin{vmatrix} u_1(x) \dots u_n(x) & g(x,t) \\ u_{11} \quad \dots u_{1n} & L_1(g)_x \\ \dots\dots\dots & \dots \\ u_{n1} \quad \dots u_{nn} & L_n(g)_x \end{vmatrix},$$

where the subscript x indicates, that the operation L_i is performed on $g(x,t)$ as on a function of x. The non-homogeneous problem (L) for an arbitrary choice of $f(x)$ has a unique solution

$$(6)\qquad y(x)=\int_a^b G(x,t)f(t)\,dt.$$

In order to prove this theorem we write the general solution of the non-homogeneous equation (I) in the form

$$(14)\qquad y(x)=\sum_{i=1}^{n} c_i u_i(x)+u_0(x),$$

where c_i are constants and

$$(15)\qquad u_0(x)=\int_a^b g(x,t)f(t)\,dt$$

is a particular solution of (I). Substitution in (II) gives a system of equations for c_i:

$$(16)\qquad \sum_{k=1}^{n} u_{ik}c_k=-L_i(u_0)\qquad (i=1,2,\dots,n).$$

If the determinant Δ of this system is different from zero, we can determine c_i uniquely. Substituting the values of c_i in (14) and using the relation

$$L_i\Big[\int_a^b f(t)g(x,t)\,dt\Big]=\int_a^b f(t)L_i(g)_x\,dt,\ ^{6)}$$

we obtain formulas (6) and (13). Using the fact that $f(x)$ is arbitrary, we easily obtain the proof of the statement 2° of Theorem 3. The statement 1° can be proved, by considering the case $f(x)\equiv 0$. and the corresponding homogeneous system:

$$(17)\qquad \sum_{k=1}^{n} u_{ik}c_k=0;\qquad \Delta=|u_{ik}|=0.$$

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[6]) This relation is a corollary of the following lemma: If $\alpha(\lambda)$ is of bounded variation on $\lambda_1\leqq\lambda\leqq\lambda_2$, $f(s)$ is an integrable function on $a\leqq s\leqq b$, and $\omega(s,\lambda)$ is bounded on the rectangle $\lambda_1\leqq\lambda\leqq\lambda_2$, $a\leqq s\leqq b$ and continuous with respect to λ for almost all values of s, then $\int_{\lambda_1}^{\lambda_2} d\alpha(\lambda)\int_a^b f(s)\,\omega(s,\lambda)\,ds=\int_a^b f(s)\,ds\int_{\lambda_1}^{\lambda_2}\omega(s,\lambda)\,d\alpha(\lambda)$. This lemma can be proved like an analogous (but non equivalent) lemma of T. Carleman, Sur les équations integrales singulières, Uppsala Universitets Årsskr. 1923, n° 3, p. 8.

8. Theorem 4. *If the conditions 1° and 2° of 5 are satisfied, the Green's function $G(x, t)$, as function of x, has the following properties:*

1°. *$G(x, t)$ is determined for all values of $t \neq t_{ik}^{(\nu)}$. It is determined for $t = a$, $t = b$ also, if we agree to replace the terms:*

$$\left.\frac{\partial^{n-1} g(x, a)}{\partial x^{n-1}}\right|_{x=a}, \quad \left.\frac{\partial^{n-1} g(x, b)}{\partial x^{n-1}}\right|_{x=b}$$

in the expressions of the operators $L_i(g)_x$, respectively by

$$\lim_{x \to a+0} \frac{\partial^{n-1} g(x, a)}{\partial x^{n-1}}, \quad \lim_{x \to b-0} \frac{\partial^{n-1} g(x, b)}{\partial x^{n-1}}.$$ [7]

2°. *For all non-singular values of t the function $G(x, t)$ is continuous ($n \geqq 2$) and has continuous derivatives up to the order $(n-2)$ inclusive, with respect to x. The derivatives of orders $(n-1)$ and n are continuous for $x \neq t$. When $x = t$, the limits*

$$\lim_{x \to t+0} \frac{\partial^{n-1} G(x, t)}{\partial x^{n-1}}, \quad \lim_{x \to t-0} \frac{\partial^{n-1} G(x, t)}{\partial x^{n-1}} \tag{18}$$

exist, and their difference is:

$$\lim_{x \to t+0} \frac{\partial^{n-1} G(x, t)}{\partial x^{n-1}} - \lim_{x \to t-0} \frac{\partial^{n-1} G(x, t)}{\partial x^{n-1}} = 1. \tag{19}$$

3°. *For all non-singular values of t and for $x \neq t$ the function $G(x, t)$ satisfies the equations:*

$$L(G)_x = 0, \quad L_i(G)_x = 0 \qquad (i = 1, 2, \ldots, n). \tag{20}$$

4°. *The Green's function $G(x, t)$ is invariant with respect to all linear transformations (with constant coefficients and non-vanishing determinant) of the fundamental system of solutions (7) and of the operators (II).*

5°. *If the Green's function $G(x, t)$ exists, and $F(x)$ is an arbitrary function, which has an absolutely continuous derivative of the order $(n-1)$ on (a, b) and satisfies the conditions*

$$L_i(F) = 0 \qquad (i = 1, 2, \ldots, n), \tag{21}$$

the function $F(x)$ can be represented in the form of a definite integral:

$$F(x) = \int_a^b G(x, t)\, L(F)_t\, dt. \tag{22}$$

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[7] Thus the Green's function remains undetermined only for the values $t = t_{ik}^{(\nu)}$ *interior* to the interval (a, b). We shall call these values of t singular, and all others as non-singular.

The statements 1°—4° follow immediately from 7 and 6. The statement 5° follows from the definition of the Green's function $G(x, t)$, because the function $F(x)$ is a solution of the non-homogeneous problem (L), where

$$f(x) = L(F).$$

9. A more detailed study of the properties of the Green's function is based upon the notion of the adjoint problem, which was introduced in the general case by G. D. Birkhoff[8]). Suppose that

1°. *The functions*

$$P_i(x) \qquad (i = 1, 2, \ldots, n)$$

possess continuous derivatives up to the order $(n - i)$ *inclusive.*

2°. *The operators* (II) *contain no terms corresponding to singular values of* t *and no integrals, so that*

$$L_i(y) \equiv A_i(y) + B_i(y) \qquad (i = 1, 2, \ldots, n), \tag{23}$$

where

$$A_i(y) \equiv \sum_{k=1}^{n} a_{ik}\, y^{(k-1)}(a), \qquad B_i(y) \equiv \sum_{k=1}^{n} b_{ik}\, y^{(k-1)}(b) \tag{24}$$

and a_{ik}, b_{ik} *are given constants.*

Integration by parts gives the so called „*Green's identity*“:

$$\int_a^b \{vL(u) - uL'(v)\}\, dx = Q_L(u, v), \tag{25}$$

where

$$L'(y) \equiv (-1)^n y^{(n)} + (-1)^{(n-1)} (P_1 y)^{(n-1)} + \ldots + P_n y \tag{26}$$

is the operator, adjoint[9]) to the operator $L(y)$, and

$$Q_L(u, v) \equiv \sum_{i=1}^{n} u^{(i-1)} \sum_{j=0}^{n-i} (-1)^j (P_{n-i-j}\, v)^{(j)} \Big|_a^b \qquad (P_0 \equiv 1) \tag{27}$$

is a bilinear form in two sets of arguments

$$u(a), u'(a), \ldots, u^{(n-1)}(a),\ u(b), u'(b), \ldots, u^{(n-1)}(b), \tag{28}$$

$$v(a), v'(a), \ldots, v^{(n-1)}(a),\ v(b), v'(b), \ldots, v^{(n-1)}(b). \tag{29}$$

The identity (25) gives us at once

$$Q_{L'}(u, v) \equiv -Q_L(v, u). \tag{30}$$

Definition. *The problem* (L'):

$$L'(y) = f(x); \quad L_i'(y) = 0 \qquad (i = 1, 2, \ldots, n) \tag{31}$$

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[8]) Boundary value and expansion problems etc. Trans. Am. Math. Soc. 9 (1908), p. 375.

[9]) Frobenius, Crelle's Journ. 85 (1878).

is adjoint to the problem (L), *if the operators* $L_i'(y)$ *are of the form:*

$$L_i'(y) \equiv A_i'(y) + B_i'(y) \qquad (i = 1, 2, \ldots, n) \tag{32}$$

$$A_i'(y) \equiv \sum_{k=1}^{n} a_{ik}' y^{(k-1)}(a); \quad B_i(y) \equiv \sum_{k=1}^{n} b_{ik}' y^{(k-1)}(b), \tag{33}$$

and are such that the bilinear form $Q_L(u, v)$ *is zero for every pair of functions* $u(x)$, $v(x)$, *whose derivatives of the order* $(n-1)$ *are absolutely continuous on* (a, b), *and which satisfy the conditions*

$$L_i(u) = 0; \quad L_i'(v) = 0 \qquad (i = 1, 2, \ldots, n). \tag{34}$$

In order to obtain the operators $L_i'(y)$ of the adjoint problem (L'), we have only to introduce n forms

$$L_{n+i}(u) \qquad (i = 1, 2, \ldots, n), \tag{35}$$

in $2n$ variables (28) which, taken together with the forms

$$L_i(u) \qquad (i = 1, 2, \ldots, n),$$

constitute a complete system of $2n$ linearly independent linear forms in $2n$ variables. Setting:

$$U_i = L_i(u) \qquad (i = 1, 2, \ldots, 2n),$$

we can write $Q_L(u, v)$ in the form:

$$Q_L(u, v) \equiv \sum_{i=1}^{2n} U_i \overset{*}{L}_{2n-i+1}(v) \equiv \sum_{i=1}^{n} \{L_i(u) L_{n+i}'(v) + L_{n+i}(u) L_i'(v)\}. \tag{36}$$

The forms

$$L_i'(v) \qquad (i = 1, 2, \ldots, 2n) \tag{37}$$

in $2n$ variables (29) constitute a complete system of $2n$ linearly independent forms, and the n first of them:

$$L_i'(y) \qquad (i = 1, 2, \ldots, n) \tag{38}$$

represent the n linear operators of the adjoint problem (L') in question.

The adjoint problem (L') is uniquely determined (in the sense of 5), because every linear transformation of the forms (35) with non vanishing determinant merely replaces the forms (38) by linear combinations of them, which are linearly independent.

10. Theorem 5. *Under the conditions of* 5 *and* 9 *the adjoint problem* (L') *exists and is uniquely determined, and we have moreover:*

1°. *The problem adjoint to* (L') *coincides with the problem* (L).

2°. *The homogeneous problems* (L) *and* (L') *are compatible or incompatible simultaneously and they both have the same number of linearly independent solutions.*

3°. *If the homogeneous problem* (L) *is compatible, the non-homogeneous problem* (L) *has a solution, when and only when the function* $f(x)$ *satisfies the condition*:

$$\int_a^b f(x)\, v(x)\, dx = 0, \tag{39}$$

where $v(x)$ *denotes any solution of the homogeneous problem* (L').

4°. *If the Green's function* $G(x, t)$ *of the problem* (L) *exists, then the Green's function* $G'(x, t)$ *of the problem* (L') *exists also, and these two functions are connected by the relation*:

$$G'(x, t) \equiv (-1)^n G(t, x) \text{ [10]}. \tag{40}$$

The problem (L) is *selfadjoint*, if the adjoint problem (L') coincides with (L). In this case the operators $L(y)$ and $L'(y)$ are identical and the operators $L_i'(y)$ are linear combinations of the operators $L_i(y)$ and vice versa. In what follows, if we say that the adjoint problem (L') exists, we suppose implicitly that the conditions of 5 and 9 are satisfied.

11. The adjoint problem was defined only under the assumptions 1° and 2° of 9. The assumption 1° is, of course, essential for the existence of the adjoint problem, but the assumption 2° is not essential, and in spite of the fact that the adjoint problem in the sense of Birkhoff can not exist, if the operators $L_i(y)$ contain singular values of t or integrals[1]), the notion of the adjoint problem can be generalized as to be adapted to the most general case of the operators $L_i(y)$, and even to the case where, instead of a single equation of the order n, we have to deal with a system of n equations of the first order. This generalization is discussed in a paper which is to be published elsewhere.

Here we may state only the

Addition to Theorem 4. *If the conditions* 1°—2° *of* 5. *and* 1° *of* 9. *are satisfied, the Green's function* $G(x, t)$, *as function of* t, *possesses at all non-singular points a continuous derivative of the* $(n-2)^{th}$ *order, the derivatives of the* $(n-1)^{th}$ *and* n^{th} *orders being continuous, except for* $t = x$ *(and singular points)*.

This follows easily from (13), if we observe that, in our case, the adjoint equation $L'(z) = 0$ admits of a fundamental system of solutions

$$z_i(t) = \frac{Y_i(t)}{\delta(t)} \qquad (i = 1, 2 \ldots n),$$

---

[10]) Cfr. Bôcher, Application and generalization of the adjoint systems, Trans. Am. Math. Soc. 14 (1913); Birkhoff[8]); Westfall, Zur Theorie der Integralgleichungen, Göttingen 1905.

[1]) pp. 108—111.

where $Y_i(t)$ denotes the cofactor of the element $u_i^{(n-1)}(t)$ in the determinant $\delta(t)$. The expression (10) for the function $g(x, t)$, then, can be rewritten as follows

$$g(x, t) = \pm \frac{1}{2} \sum_{i=1}^{n} u_i(x) z_i(t), \qquad \begin{matrix} + & \text{if} & x > t, \\ - & \text{if} & x < t. \end{matrix}$$

After that, our assertion becomes almost obvious.

§ 3.

The structure of the principal parts of the Green's function at its poles.

12. In this section we suppose that the coefficients of the operators $L(y)$ and $L_i(y)$ of the problem (L) depend on a complex parameter ϱ, and that they are analytic on a closed region $(\mathfrak{D}_0)$ of ϱ-plane. The region $(\mathfrak{D}_0)$ may coincide with the whole ϱ-plane, in which case the above mentioned coefficients are entire transcendental functions or polynomials in ϱ[11].

We suppose also that the conditions 1° and 2° of 5 are satisfied for all values of ϱ.

All the functions considered in the § 2, except the function $f(x)$, are now functions of ϱ, which fact will be indicated by a slight modification in the notation of § 2. For instance, we shall write:

$$P_i(x, \varrho), \quad u_i(x, \varrho), \quad L(y, \varrho) \quad \text{etc.}$$

instead of

$$P_i(x), \quad u_i(x), \quad L(y) \quad \text{etc.}$$

Under these conditions the functions

$$u_i(x, \varrho), \quad \delta(x, \varrho), \quad g(x, t, \varrho), \quad u_{ik}(\varrho), \quad \Delta(\varrho)$$

are analytic in ϱ, and the Green's function

$$G(x, t, \varrho)$$

is meromorphic in ϱ, except for two possible cases:

1°. $\Delta(\varrho) \equiv 0$, when the Green's function does not exist for any value of ϱ, and the homogeneous problem (L) is always compatible.

2°. $\Delta(\varrho)$ has no roots at all, the homogeneous problem (L) is always incompatible, and the Green's function always exists and is analytic in ϱ.

In all other cases the Green's function has poles in ϱ, which are called *characteristic values* of the problem (L). The Green's function does not exist for such values of ϱ, the non-homogeneous problem is

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[11]) In what follows it is assumed that ϱ always remains in $(\mathfrak{D}_0)$.

impossible for an arbitrary $f(x)$, and the homogeneous problem has solutions which are not identically zero. These solutions are called *fundamental functions* of the problem (L), corresponding to the characteristic value of ϱ in question.

13. If ϱ_0 denotes a characteristic value, we have:

$$(1) \qquad G(x, t, \varrho) = \frac{\Gamma_m(x, t)}{(\varrho - \varrho_0)^m} + \ldots + \frac{\Gamma_1(x, t)}{\varrho - \varrho_0} + \Gamma(x, t, \varrho)$$

where $\Gamma(x, t, \varrho)$ is analytic in the vicinity of $\varrho = \varrho_0$. The rational function of ϱ:

$$(2) \qquad G^{(0)}(x, t, \varrho) = \frac{\Gamma_m(x, t)}{(\varrho - \varrho_0)^m} + \ldots + \frac{\Gamma_1(x, t)}{\varrho - \varrho_0}$$

is called the *principal part* of $G(x, t, \varrho)$ for $\varrho = \varrho_0$.

We have obviously:

$$\Delta(\varrho_0) = 0,$$

and we denote by

$$(3) \qquad (\varrho - \varrho_0)^{e_1}, \ldots, (\varrho - \varrho_0)^{e_\mu}, \ldots, (\varrho - \varrho_0)^{e_n}$$
$$(e_1 \geqq e_2 \geqq \ldots \geqq e_\mu > e_{\mu+1} = \ldots = e_n = 0)$$

the elementary divisors of the matrix

$$(4) \qquad (u_{ik}(\varrho)),$$

corresponding to the factor $(\varrho - \varrho_0)$.

If we replace the functions $u_i(x, \varrho)$ and the operators $L_i(y)$ by suitably chosen linear combinations, we always can reduce the matrix (4) to the canonical form

$$(5) \qquad u_{ik}(\varrho) = \begin{cases} 0 & \text{if } k \neq i, \\ (\varrho - \varrho_0)^{e_i} & \text{if } k = i. \end{cases}$$

Since the problem (L) and the Green's function $G(x, t, \varrho)$ remain invariant under all these transformations, we can assume without loss of generality, that the fundamental system

$$(6) \qquad u_i(x, \varrho)$$

and the operators $L_i(y)$ are chosen so as to reduce (4) to the canonical form (5)[12].

$$(7) \qquad c_i(\varrho - \varrho_0)^{e_i} = - L_i\left\{\int_a^b g(x, t, \varrho) f(t)\, dt\right\} \qquad (i = 1, 2, \ldots, n),$$

---

[12]) It is worth noting, that the choice of these linear combinations depends on ϱ_0, and that the transformed system (6) might be modified, when we go over to another characteristic value.

which gives the expression for the Green's function:

$$(8)\qquad G(x,t,\varrho) = -\sum_{i=1}^{n} \frac{u_i(x,\varrho)\, L_i\{g(x,t,\varrho)\}_x}{(\varrho-\varrho_0)^{e_i}} + g(x,t,\varrho).$$

On setting in (7)

$$f(x) \equiv 0, \qquad \varrho = \varrho_0,$$

we see at once that the homogeneous problem (L) has μ linearly independent solutions:

$$(9)\qquad u_1(x,\varrho_0), \ldots, u_\mu(x,\varrho),$$

when $\varrho = \varrho_0$, and that every fundamental function of the problem (L) corresponding to the characteristic value ϱ_0 must be a linear combination of the functions (9).

We can also prove that the μ first fractions of the sum in (8) are reduced to lowest terms, so that the numerators and the denominators of each of them have no common factors in $(\varrho - \varrho_0)$. For this purpose it is enough to show that

$$L_i\{g(x,t,\varrho_0)\}_x \not\equiv 0 \qquad (i = 1, 2, \ldots, \mu).$$

It is evident that

$$u_i(x,\varrho_0) \not\equiv 0.$$

Now, supposing that

$$L_i\{g(x,t,\varrho_0)\}_x \equiv 0 \qquad (i = 1, 2, \ldots, \mu)$$

and using the equalities

$$L_i\{u_k(x,\varrho)\}/_{\varrho=\varrho_0} = u_{ik}(\varrho_0) = 0 \quad (i = 1, 2, \ldots, \mu,\ k = 1, 2, \ldots, n),$$
$$L_i\{u_0(x,\varrho)\}/_{\varrho=\varrho_0} = 0,$$

we see that the general solution of $L(y,\varrho_0) = f(x)$ satisfies the conditions

$$L_i(y) = 0 \qquad (i = 1, 2, \ldots, \mu),$$

for an arbitrary $f(x)$, which is impossible.

14. Theorem 6. *If the conditions* 1°, 2° *of* 5 *are satisfied and, in the neighbourhood of any root* $\varrho = \varrho_0$ *of the equation* $\Delta(\varrho) = 0$, *the matrix*

$$(4)\qquad (u_{ik}(\varrho))$$

has elementary divisors

$$(3)\quad (\varrho-\varrho_0)^{e_i} \quad (i = 1, 2, \ldots, n;\ e_1 \geqq e_2 \geqq \cdots \geqq e_\mu > e_{\mu+1} = \ldots = e_n = 0),$$

then

1°. *The characteristic value* ϱ_0 *is a pole of the Green's function, of the multiplicity* e_1.

2°. *The homogeneous problem* (L) *has* μ *linearly independent solutions for* $\varrho = \varrho_0$ *(fundamental functions).*

3°. *The principal part of the Green's function for $\varrho = \varrho_0$ is of the form:*

$$G^{(0)}(x, t, \varrho) = \sum_{s=0}^{e_1-1} \frac{\Gamma_{e_1-s}(x, t)}{(\varrho - \varrho_0)^{e_1-s}},$$

where

$$\Gamma_k(x, t) = \sum_{i, s} \Phi_{is}(x)\, \Psi_{is}^{(k)}(t)$$

and the summation is taken over all values of i and s satisfying the conditions

$$1 \leqq i \leqq \mu; \quad s + k \leqq e_i.$$

The functions $\Phi_{is}(x)$ („principal functions of the problem (L) corresponding to $\varrho = \varrho_0$") are well determined functions of x and have continuous derivatives of the order n on (a, b). The functions $\Psi_{is}^{(k)}(t)$ are well determined for all non-singular values of t in (a, b).

The functions

(10) $$\Phi_{i0}(x) \qquad (i = 1, 2, \ldots, \mu)$$

represent a complete set of μ linearly independent fundamental functions of the problem (L) for $\varrho = \varrho_0$.

The set of principal functions coincides with the set of fundamental functions, when and only when all the elementary divisors are simple, that is when $e_1 = e_2 = \ldots = e_\mu = 1$.

4°. *If the matrix (4) is reduced to its canonical form (5), the conditions*

(11) $$\int_a^b f(t)\, L_i\{g(x, t, \varrho)\}_x /_{\varrho=\varrho_0} dt = 0 \qquad (i = 1, 2, \ldots, \mu)$$

are necessary and sufficient for the existence of a solution of the nonhomogeneous problem (L) for $\varrho = \varrho_0$.

5°. *If the adjoint problem (L') exists and if the matrix (4) is reduced to the canonical form (5), the functions*

(12) $$L_i\{g(x, t, \varrho)\}_x /_{\varrho=\varrho_0} \qquad (i = 1, 2, \ldots, \mu)$$

represent a complete set of μ linearly independent solutions of the homogeneous adjoint problem (L') for $\varrho = \varrho_0$ [1]*.*

15. The case when the adjoint problem exists and $e_1 = 1$ presents a particular interest. In this case we have

(13) $$G^{(0)}(x, t, \varrho) = \frac{1}{\varrho - \varrho_0} \sum_{i=1}^{\mu} \Phi_i(x)\, \Psi_i(t).$$

---

[1]) pp. 117–124. On the p. 124 there is given an example, in which the Green's function has multiple poles and the principal functions appear simultaneously with the fundamental functions.

The functions

$$(14)\qquad \Phi_1(x), \ldots, \Phi_\mu(x)$$

and

$$(15)\qquad \Psi_1(x), \ldots, \Psi_\mu(x)$$

are certain special complete sets of fundamental functions of the problems (L) and (L') respectively.

In the applications very often we know a priory two sets of fundamental functions of the problems (L) and (L'), and then the question arises, how is it possible to determine the principal part of the Green's function, using only these known sets and without reducing the matrix (4) to the canonical form, which requires complicated calculations. Let

$$(16)\qquad \varphi_1(x), \ldots, \varphi_\mu(x)$$

and

$$(17)\qquad \psi_1(t), \ldots, \psi_\mu(t)$$

be given complete sets of fundamental functions of the problems (L) and (L'). The functions (14) are linear combinations of (16) and the functions (15) are linear combinations of (17); hence we can write

$$(18)\qquad G^{(0)}(x, t, \varrho) = \frac{1}{\varrho - \varrho_0} \sum_{i=1}^{\mu} \theta_i(x)\psi_i(t),$$

where

$$(19)\qquad \theta_i(x) = \sum_{k=1}^{\mu} c_{ik}\varphi_k(x).$$

It remains only to compute the matrix of the constants c_{ik}. Using the properties of the Green's function and the equations:

$$L'(\psi_j, \varrho_0) = 0; \quad L_i'(\psi_j, \varrho_0) = 0 \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, \mu)$$

and integrating by parts, we easily obtain:

$$\psi_j(t) = \int_a^b \{G(x, t, \varrho) L'(\psi_j, \varrho) - \psi_j(x) L(G, \varrho)_x\}\, dx + Q_L(G, \psi_j)$$

$$= \int_a^b G(x, t, \varrho)\{L'(\psi_j, \varrho) - L'(\psi_j, \varrho_0)\}\, dx + Q_L(G, \psi_j)_x.$$

The expression (36) 9 of the bilinear form $Q_L(u, v)$ gives us now:

$$\psi_j(t) = \int_a^b G(x, t, \varrho)\{L'(\psi_j, \varrho) - L'(\psi_j, \varrho_0)\}\, dx$$

$$+ \sum_{s=1} L_{n+s}(G, \varrho)\{L_s'(\psi_j, \varrho) - L_s'(\psi_j, \varrho_0)\}.$$

If we substitute here

$$G(x, t, \varrho) = G^{(0)}(x, t, \varrho) + \Gamma(x, t, \varrho),$$

where $G^{(0)}(x, t, \varrho)$ is determined by (13) and $\Gamma(x, t, \varrho)$ is analytic in ϱ in the neighbourhood of $\varrho = \varrho_0$, we obtain, as $\varrho \to \varrho_0$:

$$\psi_j(t) = \sum_{i=1}^{\mu} \psi_i(t) \int_a^b \theta_i(x) \frac{\partial}{\partial \varrho} L'(\psi_j, \varrho)/_{\varrho=\varrho_0} dx$$
$$+ \sum_{i=1}^{\mu} \psi_i(t) \sum_{s=1}^{n} L_{n+s}(\theta_i, \varrho_0) \frac{\partial}{\partial \varrho} L_s'(\psi_j, \varrho)/_{\varrho=\varrho_0} \qquad (j = 1, 2, \ldots, \mu).$$

Now, $\theta_i(x)$ are given by (19) and $\psi_j(t)$ are linearly independent, so that finally we obtain a system of μ^2 equations for the determination of the matrix (c_{ik}):

$$(20) \qquad \sum_{k=1}^{n} c_{ik} \left\{ \int_a^b \varphi_k(x) \frac{\partial}{\partial \varrho} L'(\psi_j, \varrho)/_{\varrho=\varrho_0} dx \right\}$$
$$+ \sum_{s=1}^{n} L_{n+s}(\varphi_k, \varrho_0) \frac{\partial}{\partial \varrho} L_s'(\psi_j, \varrho)/_{\varrho=\varrho_0} = \delta_{ij}\,[13] \qquad (i, j = 1, 2, \ldots, \mu).$$

This system, which admits of a unique solution for c_{ik}, is simplified if the operators $L_i(y)$ and the bilinear form $Q_L(u, v)$ are independent of ϱ. In this case the operators $L_i'(y)$ do not depend on ϱ either, and we get instead of (20):

$$(21) \qquad \sum_{k=1}^{n} c_{ik} \int_a^b \varphi_k(x) \frac{\partial}{\partial \varrho} L'(\psi_j, \varrho)/_{\varrho=\varrho_0} dx = \delta_{ij}.$$

Suppose finally that

$$L(y) \equiv y^{(n)} + p_1(x) y^{(n-1)} + \ldots + p_n(x) y + \varrho^n y.$$

The system (21) reduces then to

$$\sum_{k=1}^{\mu} c_{ik} \int_a^b n \varrho_0^{n-1} \varphi_k(x) \psi_j(x) dx = \delta_{ij}.$$

Without loss of generality we may suppose that the sets (16) and (17) are biorthogonal and normal, which yields:

$$(22) \qquad G^{(0)}(x, t, \varrho) = \frac{1}{n \varrho_0^{n-1} (\varrho - \varrho_0)} \sum_{i=1}^{\mu} \varphi_i(x) \psi_i(t).$$

---

[13]) δ_{ij} denotes, as usually, Kronecker's symbol: $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

16. In some cases it is possible to prove that all the poles of the Green's function are simple and real. Take for instance the self-adjoint problem (L) of the form:

$$(23)\qquad \begin{aligned} L(y) &\equiv y^{(n)} + p_1(x)\,y^{(n-1)} + \dots + p_n(x)\,y + \lambda q(x)\,y \\ &\equiv L^{(0)}(y) + \lambda q(x)\,y, \end{aligned}$$

$$L_i(y) \equiv L_i^{(0)}(y) + \lambda L_i^{(1)}(y) \qquad (i = 1, 2, \dots, n),$$

where λ is written instead of ϱ^n, and the operators $L^{(0)}(y)$, $L_i^{(0)}(y)$, $L_i^{(1)}(y)$ do not contain λ.

In addition to the conditions 1° and 2° of **9** we suppose that

The matrix of the coefficients of the forms $L_i(y)$ contains at least one non-vanishing determinant of the order n, whose elements do not depend on λ.

In this case it is easy to show[1]) that

$$(24)\qquad Q_L(u, v) \equiv \sum_{i=1}^{n} \{L_i(u)\, M_i(v) - L_i(v)\, M_i(u)\},$$

where

$$(25)\qquad M_i(y) \qquad (i = 1, 2, \dots, n)$$

are linearly independent forms in $2n$ variables

$$(26)\qquad y(a), y'(a), \dots, y^{(n-1)}(a), \quad y(b), y'(b), \dots, y^{(n-1)}(b),$$

whose coefficients do not depend on λ, and which, taken together with the forms $L_i^{(0)}(y)$, constitute a complete system of $2n$ linearly independent forms. Since $Q_L(u, v)$ does not depend on λ, we have for $\lambda = 0$:

$$(27)\qquad Q_L(u, v) \equiv \sum_{i=1}^{n} \{L_i^{(0)}(u)\, M_i(v) - L_i^{(0)}(v)\, M_i(u)\}.$$

Hence

$$\sum_{i=1}^{n} \{L_i^{(1)}(u)\, M_i(v) - L_i^{(1)}(v)\, M_i(u)\} \equiv 0$$

and the bilinear form

$$(28)\qquad T(u, v) \equiv \sum_{i=1}^{n} L_i^{(1)}(u)\, M_i(v)$$

is symmetric.

Using all these facts the following theorem can be proved:

Theorem 7. *Suppose that the problem (L), whose operators are given by (1) and (2), is self-adjoint and satisfies the conditions of* **16.** *Suppose also that the coefficients of the operators $L(y)$, $L_i(y)$ are real. The Green's function $G(x, t, \lambda)$ has only simple and real poles (in λ), if*

[1]) pp. 133–134.

1°. *The function $q(x)$ is of constant sign on (a, b) and is different from zero almost everywhere on (a, b).*

2°. *The quadratic form*

$$(29) \qquad T(y) \equiv T(y, y) \equiv \sum_{i=1}^{n} L_i^{(1)}(y)\, M_i(y)$$

is semidefinite and has the sign opposite to that of $q(x)$.

Introducing some supplementary conditions it is even possible to get rid of any restriction concerning the sign of $q(x)$.

The method of proof used[1]) is in essential a generalization of Stekloff's methods[14]).

The formulas of 15 can be considerably simplified in the case of Theorem 7. Suppose

$$\varphi_1(x), \varphi_2(x), \ldots, \varphi_\mu(x)$$

be the given set of fundamental functions of the problem (L) corresponding to $\varrho = \varrho_0$. We always can "orthogonalize" them according to the conditions:

$$(30) \qquad \int_a^b q(x)\, \varphi_k(x)\, \varphi_j(x)\, dx - T(\varphi_k, \varphi_j) = \delta_{kj} \quad (k, j = 1, 2, \ldots, \mu),$$

and then we obtain easily:

$$(31) \qquad G^{(0)}(x, t, \varrho) = \frac{1}{n \varrho_0^{n-1}(\varrho - \varrho_0)} \sum_{i=1}^{\mu} \varphi_i(x)\, \varphi_i(t).$$

§ 4.

Existence and asymptotic expression of the characteristic values.

17. This section is devoted to the discussion of the existence and approximate representation of the poles of the Green's function. We assume now that the operator $L(y)$ of the problem (L) is of the form

$$(1) \qquad L(y) \equiv L(y, \varrho) \equiv y^{(n)} + P_1(x, \varrho)\, y^{(n-1)} + \ldots + P_n(x, \varrho)\, y,$$

$$(2) \qquad P_i(x, \varrho) \equiv \varrho^i p_i(x, \varrho) \equiv \varrho^i \sum_{j=0}^{i} \varrho^{-j} p_{ij}(x) \quad (i = 1, 2, \ldots, n).$$

We shall first discuss the case in which the operators $L_i(y)$ contain neither singular values of t, nor integrals, and are polynomials in ϱ:

$$(3) \qquad L_i(y) \equiv L_i(y, \varrho) \equiv \sum_{s=0}^{n} \varrho^s L_i^{(s)}(y),$$

where $L_i^{(s)}(y)$ do not depend on ϱ:

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[1]) pp. 131—141.

[14]) Sur l'existence des fonctions fondamentales, Atti d. R. Acc. dei Lincei (5^a), 8 (1910), pp. 166—167.

$$L_i^{(s)}(y) \equiv A_i^{(s)}(y) + B_i^{(s)}(y);$$

(4) $$A_i^{(s)}(y) \equiv \sum_{k=1}^{n} a_{ik}^{(s)} y^{(k-1)}(a); \qquad B_i^{(s)}(y) \equiv \sum_{k=1}^{n} b_{ik}^{(s)} y^{(k-1)}(b)$$

and $a_{ik}^{(s)}$, $b_{ik}^{(s)}$ are given constants. We shall write also

$$L_i(y) \equiv A_i(y, \varrho) + B_i(y, \varrho);$$

(5) $$A_i(y, \varrho) \equiv \sum_{s=0}^{n} \varrho^s A_i^{(s)}(y); \qquad B_i(y, \varrho) \equiv \sum_{s=0}^{n} \varrho^s B_i^{(s)}(y).$$

As to the functions $p_{ij}(x)$, we suppose:

1°. *The functions*

(6) $$\frac{d^2 p_{10}(x)}{dx^2}, \quad \frac{dp_{11}(x)}{dx}, \quad p_{ij}(x) \qquad (i = 1, 2, \ldots, n;\ j = 2, \ldots, n)$$

are continuous on (a, b).

2°. *The characteristic equation*

$$\Phi(\theta) \equiv \theta^n + p_{10}(x)\theta^{n-1} + \ldots + p_{n-10}(x)\theta + p_{n0}(x) = 0$$

has only simple roots

(7) $$\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x).$$

3°. *The roots* (7) *of the characteristic equation are different from zero and their arguments as well as those of their differences are constant.*

It is easy to prove that the suppositions 1°—3° imply that either:

4_1°. *The functions* (7) *are of the form*

$$\varphi_i(x) = \pi_i q(x) \qquad (i = 1, 2, \ldots, n),$$

where π_i *are constants which are distinct and different from zero, and* $q(x)$ *is a positive function which possesses a second derivative continuous on* (a, b) *and which has a lower bound* q_0 *different from zero*:

$$q(x) \geqq q_0 > 0.$$

Or: 4_2°. *The functions* (7) *are of the form:*

$$\varphi_i(x) = \pm \pi_0 q_i(x)$$

where π_0 *is a constant different from zero and* $q_i(x)$ *are positive and distinct for all values of* x *in* (a, b) *and have the same properties of continuity as* $q(x)$.

Both cases may be unified in one notation:

(8) $$\varphi_i(x) = \pi_i q_i(x).$$

The suppositions $1^\circ - 3^\circ$ restrict considerably the nature of the functions $p_{i0}(x)$.[15]) So, for instance, in the case 4_1° we must have

$$p_{i0}(x) = C_i \{q(x)\}^i \qquad (i = 1, 2, \ldots, n),$$

where C_i are constants such that π_i are the roots of the equation

$$\Phi_0(\xi) \equiv \xi^n - C_1 \xi^{n-1} + \ldots + (-1)^n C_n = 0.$$

The conditions of Theorem 2 are satisfied under the conditions $1^\circ - 3^\circ$, and the region $(\mathfrak{D})$ of Theorem 2 coincides with one of the sectors

(9) $$(\mathfrak{S}_1), (\mathfrak{S}_2), \ldots, (\mathfrak{S}_N),$$

which are made by the straight lines

(10) $$\Re e(\varrho \varphi_i) = \Re e(\varrho \varphi_k) \qquad (i, k = 1, 2, \ldots, n;\ i \neq k).$$

18. Theorem 2 ensures the existence of a fundamental system of solutions

(11) $$y_{ij}(x, \varrho) = e^{\varrho \int_a^x \varphi_i(x)\,dx} \left\{ \eta_i(x) + \frac{E}{\varrho} \right\} \qquad (i = 1, 2, \ldots, n)$$

on every sector $(\mathfrak{S}_j)$[16]) (cfr. 2).

At the same time we can use the fundamental system

(12) $$u_i(x, \varrho) \qquad (i = 1, 2, \ldots, n)$$

of §§ 2, 3, which, in our case, is determined on the whole ϱ-plane. This system can be chosen in various ways, for instance we can impose the conditions:

$$\left. \frac{d^{s-1} u_i}{dx^{s-1}} \right|_{x=a} = \delta_{is} \qquad (i, s = 1, 2, \ldots, n).$$

The Green's function $G(x, t, \varrho)$ may be expressed in terms of either of the systems (11) and (12). We denote by $\Delta_j(\varrho)$ the determinant which we obtain from $\Delta(\varrho)$, if the functions (12) are replaced by (11) in the formula (12) 6 for $\Delta(\varrho)$.

We find:

(13) $$\Delta(\varrho) = [C] \varrho^{-\frac{n(n-1)}{2}} \Delta_j(\varrho),$$ [17])

where $C \neq 0$ is a numerical constant which does not depend on ϱ.

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[15]) The restriction concerning the arguments of the functions (7) is essential for the expansion problem only. Most of the results of this section can be proved supposing only that the arguments of all the differences of (7) are constant. Cfr. [1]) Ch. IV.

[16]) In what follows the term "sector $(\mathfrak{S}_j)$" shall be understood to mean "the part of this sector which is outside the circle $|\varrho| \leqq R_0$".

[17]) We use here a notation due to G. D. Birkhoff; $[\alpha]$ denotes an expression of the form $\alpha + \frac{E}{\varrho}$, where E is bounded for large values of $|\varrho|$.

It is not difficult to obtain an expression for $\Delta_j(\varrho)$. We set

$$(14) \qquad w_k = \int_a^b \varphi_k(x)\,dx$$

and denote by $l_i^{(s)}$ the number which is determined as follows: suppose we have

$$a_{il}^{(s)} = b_{il}^{(s)} = 0 \quad \text{when} \quad l > l_i^{(s)},$$

and at least one of the numbers

$$a_{il}^{(s)}, \quad b_{il}^{(s)} \qquad (l = l_i^{(s)})$$

is different from zero. If all the cofficients $a_{il}^{(s)}$, $b_{il}^{(s)}$ are equal to zero $(l = 1, 2, \ldots, n)$ we set $l_i^{(s)} = 0$.

Now we define the number l_i as the maximum term of the sequence

$$(15) \qquad l_i^{(s)} + s - 1 \qquad (s = 0, 1, \ldots n)$$

in which only terms with $l_i^{(s)} > 0$ are taken into account.

It is easy to show that

$$(16) \qquad u'_{ik} = L_i(y_{kj}) = \varrho^{l_i}\{[A_{ik}] + e^{\varrho w_k}[B_{ik}]\},$$

where

$$(17) \qquad \begin{cases} A_{ik} = \eta_k(a) \sum_{(s)} a_{i\,l_i^{(s)}}^{(s)} \varphi_k(a)^{l_i^{(s)}-1}, \\ B_{ik} = \eta_k(b) \sum_{(s)} b_{i\,l_i^{(s)}}^{(s)} \varphi_k(b)^{l_i^{(s)}-1} \end{cases}$$

and the summation is made over all values of s for which

$$l_i^{(s)} + s - 1 = l_i.$$

Replacing u_{ik} by u'_{ik} in the expression (12) 6 for $\Delta(\varrho)$, we get:

$$(18) \qquad \Delta_j(\varrho) = \varrho^l \begin{vmatrix} [A_{11}] + e^{\varrho w_1}[B_{11}], & \ldots, & [A_{1n}] + e^{\varrho w_n}[B_{1n}] \\ \cdots & \cdots & \cdots \\ [A_{n1}] + e^{\varrho w_1}[B_{n1}], & \ldots, & [A_{nn}] + e^{\varrho w_n}[B_{nn}] \end{vmatrix},$$

where

$$(19) \qquad l = l_1 + l_2 + \ldots + l_n,$$

and (13) gives then:

$$(20) \qquad \Delta(\varrho) = C\varrho^{l - \frac{n(n-1)}{2}} \Delta^{(0)}(\varrho),$$

$$(21) \qquad \Delta^{(0)}(\varrho) = \begin{vmatrix} [A_{11}] + e^{\varrho w_1}[B_{11}], & \ldots, & [A_{1n}] + e^{\varrho w_n}[B_{1n}] \\ \cdots & \cdots & \cdots \\ [A_{n1}] + e^{\varrho w_1}[B_{n1}], & \ldots, & [A_{nn}] + e^{\varrho w_n}[B_{nn}] \end{vmatrix}.$$

It ought to be noted that the formula (18) is valid only on the sector $(\mathfrak{S}_j)$, but the formulas (20) and (21) are valid on the whole ϱ-plane.

19. It remains now to discuss the roots of the equation:

$$(22)\qquad \Delta^{(0)}(\varrho) = 0.$$

Each of the equations

$$(23)\qquad \Re e\, \varrho\, w_i = 0 \qquad (i = 1, 2, \ldots, n)$$

determines two rays $(0 \to \infty)$ on the ϱ-plane, so that the equations (23) together determine 2μ $(\leqq 2n)$ *distinct* rays. We shall denote these rays by

$$(24)\qquad d_1, d_2, \ldots, d_{2\mu},$$

and by $\left(\alpha_j + \frac{\pi}{2}\right)$ the argument of the ray d_j. We suppose then that the rays are ordered so that

$$0 \leqq \alpha_1 < \alpha_2 < \ldots < \alpha_{2\mu} < 2\pi.$$

Let

$$(25)\qquad d'_1, d'_2, \ldots, d'_{2\mu}$$

be a second set of rays which are different from (24) and arbitrary, subject only to the condition that the sequence

$$(26)\qquad d'_1\, d_1\, d'_2\, d_2 \ldots d'_{2\mu}\, d_{2\mu}\, d'_1$$

progresses in the counter-clockwise sense.

The rays (26) divide the whole ϱ-plane into 2μ sectors

$$(27)\qquad (\mathfrak{T}_1), (\mathfrak{T}_2), \ldots, (\mathfrak{T}_{2\mu}).$$

Consider one of these sectors, say $(\mathfrak{T}_j)$, and let

$$w_1^{(j)}, w_2^{(j)}, \ldots, w_{\nu_j}^{(j)}$$

be those numbers of the set

$$(28)\qquad w_1, w_2, \ldots, w_n$$

which lie on the rays perpendicular to d_j and $d_{j+\mu}$.[18] Now, if we set

$$(29)\qquad w_k^{(j)} = \lambda_k^{(j)} e^{\alpha_j \sqrt{-1}}$$

it is always possible to arrange the numbers (28) so that

$$(30)\qquad \lambda_1^{(j)} < \lambda_2^{(j)} < \ldots < \lambda_{\tau_j}^{(j)} < 0 < \lambda_{\tau_j+1}^{(j)} < \ldots < \lambda_{\nu_j}^{(j)}.$$

If all the numbers $\lambda_k^{(j)}$ are > 0, we set $\tau_j = 0$; if all the numbers $\lambda_k^{(j)}$ are < 0, we set $\tau_j = \nu_j$, and we must modify the inequalities (30)

[18]) If k is an integer different from $1, 2, \ldots, 2\mu$, the symbol d_k denotes the ray d_j where j is the positive residuum of k mod 2μ. The rays d_j, $d_{j+\mu}$ constitute the entire straight line through the origin.

accordingly. The remaining numbers of the set (28) can be divided into two groups (w') and (w''), which are characterized by the conditions:

(31) $$\Re e(\varrho w') \to -\infty; \quad \Re e(\varrho w'') \to +\infty,$$

as $|\varrho| \to \infty$ on $(\mathfrak{T}_j)$.

20. The substitution

$$\varrho e^{-\alpha_j \sqrt{-1}} = \zeta = \xi + \eta \sqrt{-1}$$

transforms the ray d_j to the positive part of the imaginary ζ-axis, and the sector $(\mathfrak{T}_j)$ to a sector $(\mathfrak{T})$ of the ζ-plane containing the positive part of the imaginary ζ-axis. We have on this sector $(\mathfrak{T})$:

(32) $$\Delta^{(0)}(\varrho) = e^{\Sigma \varrho w''} \mathsf{H}_j(\zeta),$$

where

(33) $$\mathsf{H}_j(\zeta) = [M_1^{(j)}]\, e^{m_1^{(j)} \zeta} + \ldots + [M_{\sigma_j}^{(j)}]\, e^{m_{\sigma_j}^{(j)} \zeta}.$$

We denote here by

$$m_1^{(j)} < m_2^{(j)} < \ldots < m_{\sigma_j}^{(j)}$$

the real exponents and by

$$M_1^{(j)}, \ldots, M_{\sigma_j}^{(j)}$$

certain constants which may be easily expressed in terms of the exponents $\lambda_k^{(j)}$ and of the constants A_{ik}, B_{ik} respectively. We note only:

(34) $$\begin{cases} m_1^{(j)} = \begin{cases} \lambda_1^{(j)} + \ldots + \lambda_{\tau_j}^{(j)}, & \text{if } \tau_j > 0, \\ 0, & \text{if } \tau_j = 0. \end{cases} \\[2ex] m_{\sigma_j}^{(j)} = \begin{cases} \lambda_{\tau_j+1}^{(j)} + \ldots + \lambda_{\nu_j}^{(j)}, & \text{if } \tau_j < \nu_j, \\ 0, & \text{if } \tau_j = \nu_j \end{cases} \end{cases}$$

$$M_1^{(j)} = \begin{vmatrix} A_{11} A_{12} \ldots A_{1\varkappa_j} B_{1\varkappa_j+1} \ldots B_{1n} \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ A_{n1} A_{n2} \ldots A_{n\varkappa_j} B_{n\varkappa_j+1} \ldots B_{nn} \end{vmatrix}, \qquad M_{\sigma_j}^{(j)} = \begin{vmatrix} A_{11} \ldots A_{1\tau_j} B_{1\tau_j+1} \ldots B_{1n} \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ A_{n1} \ldots A_{n\tau_j} B_{n\tau_j+1} \ldots B_{nn} \end{vmatrix}$$

where the index $\varkappa_j$ is determined as follows: if some of the quantities $\Re e\, \varrho w_i$ remain always < 0 on $(\mathfrak{T}_j)$, then $\varkappa_j$ is the greatest value of the index i for which $\Re e\, \varrho w_i < 0$ on $(\mathfrak{T}_j)$. Otherwise $\varkappa_j = 0$. It is obvious that $\varkappa_j \leqq \tau_j$.

21. Theorem 8. *Suppose the conditions* 1°—3° *of* 17 *to be satisfied and all the constants*

$$M_1^{(j)}, \quad M_{\sigma_j}^{(j)} \qquad (j = 1, 2, \ldots, 2\mu)$$

to be different from zero. In this case the problem (L) *has infinitely many characteristic values, which can be distributed into* 2μ *groups.*

The values of the j^{th} group lie in a strip (D_j) of finite width, parallel to the ray d_j and including this ray. Denoting the numbers of the j^{th} group by

$$\varrho_1^{(j)}, \varrho_2^{(j)}, \ldots, \varrho_k^{(j)}, \ldots \qquad (|\varrho_1^{(j)}| \leqq |\varrho_2^{(j)}| \leqq \cdots)$$

we have

$$(35) \qquad |\varrho_k^{(j)}| = \frac{2k\pi}{m_{\sigma_j}^{(j)} - m_1^{(j)}} \left\{1 + O\left(\frac{1}{k}\right)\right\}.$$ [19]

We omit the index j in (33) for the sake of brevity, so that

$$(36) \qquad \mathsf{H}_j(\zeta) \equiv \mathsf{H}(\zeta) \equiv [M_1]\, e^{m_1\zeta} + \ldots + [M_\sigma]\, e^{m_\sigma \zeta},$$

$$M_1 \neq 0; \quad M_\sigma \neq 0; \quad m_1 < m_2 < \ldots < m_\sigma.$$

In order to prove Theorem 8 it is enough to show that: 1°. All the roots of the equation

$$(37) \qquad \mathsf{H}(\zeta) = 0$$

which are in the sector $(\mathfrak{T})$ can be included in a strip (D) of finite width, parallel to the imaginary ζ-axis and including the positive part of it.

2°. If these roots are denoted by

$$\zeta_1, \zeta_2, \ldots, \zeta_k, \ldots \quad (|\zeta_1| \leqq |\zeta_2| \leqq \cdots)$$

we have

$$(38) \qquad |\zeta_k| = \frac{2k\pi}{m_\sigma - m_1} \left\{1 + O\left(\frac{1}{k}\right)\right\}.$$

22. The proof is based upon the following important

Lemma. *Given a function*

$$(39) \qquad F(z, x_1, \ldots, x_n)$$

continuous in $x_1, \ldots, x_n$ on a finite closed region $(\mathfrak{D}_x)$ of n-dimensional space and analytic in z on a finite closed region $(\mathfrak{D}_z)$ of the complex z-plane, and such that for every fixed point $(x_1, \ldots, x_n)$ in $(\mathfrak{D}_x)$ the equation

$$F(z, x_1, \ldots, x_n) = 0$$

has no more than N distinct roots, N being independent of the position

[19]) This fact was proved independently and by different methods by Ch. E. Wilder and by the Author [[6]) pp. 419–433; [1]) pp. 160–176]. Our proof is reproduced in a note which is going to appear in the Journal of the London Mathematical Society. We use here Wilder's proof because of its greater simplicity and because of its applicability to more general cases. In 1920 analogous results were obtained subsequently by G. Pólya, Geometrisches über die Verteilung der Nullstellen etc., Münch. Ber. 1920. Cfr. also the Thesis of E. Schwengler, Geometrisches über die Verteilung der Nullstellen etc., Zürich 1925.

of this point in $(\mathfrak{D}_x)$. *If for any point in* $(\mathfrak{D}_x)$ z *is at a distance greater than* δ *from zeros of* (39) *and from the boundary of* $(\mathfrak{D}_z)$, *then*

$$|F(z, x_1, \ldots, x_n)| \geqq F_0 > 0,$$

where F_0 *is a positive constant which depends only on* δ, *but does not depend on the position of the point* $(x_1, \ldots, x_n)$ *in* $(\mathfrak{D}_x)$. [5])

23. Theorem 8 can be proved immediately for the simplified equation

$$(40) \qquad Z(\zeta) \equiv M_1 e^{m_1 \zeta} + \ldots + M_\sigma e^{m_\sigma \zeta} = 0.$$

Let $\zeta = \xi + \eta\sqrt{-1}$. Since

$$(41) \qquad Z(\zeta) = \begin{cases} M_\sigma e^{m_\sigma \zeta}\left(1 + \sum_{i=1}^{\sigma-1} e^{(m_i - m_\sigma)\zeta}\right), & \text{if } \xi \geqq 0, \\ M_1 e^{m_1 \zeta}\left(1 + \sum_{i=2}^{\sigma} e^{(m_i - m_1)\zeta}\right), & \text{if } \xi \leqq 0, \end{cases}$$

a positive constant h can be determined such that

$$(42) \qquad Z(\zeta) \neq 0 \quad \text{for} \quad |\xi| \geqq \frac{h}{2},$$

which proves that all the roots of the equation (40) are within the strip (D) of the width h between the two straight lines

$$\Re e\, \zeta = \xi = \pm \frac{h}{2}.$$

Hence we can confine ourselves to the discussion of the values of ζ inside the strip (D). Now it is easy to prove that in every rectangle of (D) of the form

$$(\Pi_h) \qquad |\xi| \leqq \frac{h}{2}, \quad \eta_1 \leqq \eta \leqq \eta_2$$

the number N of the roots of (40) is contained between the limits

$$(43) \qquad \frac{1}{2\pi}(m_\sigma - m_1)(\eta_2 - \eta_1) \pm \sigma. \text{[20])}$$

The number in question is expressed by the ratio of the increase of the argument of $Z(\zeta)$ to 2π, when ζ describes the contour of (Π_h) counter-clockwise. In other words, if we set

$$Z(\zeta) = R e^{i\Theta} = X + iY,$$

we have

$$(44) \qquad N = \frac{1}{2\pi} \int_{(\Pi_h)} d\Theta = \frac{1}{2\pi} \int_{(\Pi_g)} d\Theta,$$

where g is any constant greater than h.

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[5]) p. 422.

[20]) We suppose that no roots are on the boundary of (Π_h). Cfr. [1]) 168–170; 297–298; [5]) pp. 420–422.

The expressions (41) show that the parts of the integral $\int\limits_{(\Pi_g)}$ taken over the vertical sides of (Π_g) contribute

$$\frac{1}{2\pi}(m_\sigma - m_1)(\eta_2 - \eta_1) + \varepsilon_g; \quad \varepsilon_g \to 0 \text{ when } g \to \infty.$$

On the other hand, if we write the integral (44) in the form:

$$\frac{1}{2\pi}\int d \arctan\frac{Y}{X},$$

we see at once that each of the horizontal sides contributes in absolute value no more than $\frac{\tilde{\omega}+1}{2}$, where $\tilde{\omega}$ denotes the number of real roots of the equation

$$X(\xi, \eta) = 0$$

the left hand member being considered as a function of ξ alone, for fixed η. This equation is of the type:

$$\sum_{i=1}^{\sigma} A_i e^{\lambda_i \xi} = 0, \tag{45}$$

where A_i and λ_i are real constants. Or, using the complete induction, it is easy to show that the equation (45) has at most $\sigma - 1$ real roots, so that $\tilde{\omega} + 1 \leqq \sigma$. The number N being independent of g, we have

$$\frac{1}{2\pi}(m_\sigma - m_1)(\eta_2 - \eta_1) - \sigma \leqq N \leqq \frac{1}{2\pi}(m_\sigma - m_1) + \sigma, \quad \text{Q. E. D.} \tag{46}$$

24. Now we can prove that, if the interiors of small circles of the radius δ centered at the zeros of $\mathsf{Z}(\zeta)$ are excluded from the strip (D), then in the remaining part (D_δ) of the strip:

$$|\mathsf{Z}(\zeta)| \geqq \mathsf{Z}_\delta > 0, \tag{47}$$

where Z_δ is a positive constant depending only on δ. This fact can be proved by a simple application of the lemma of 22, if we observe that, ζ being in any of rectangles

$$|\xi| \leqq \frac{h}{2}, \quad 2l\pi \leqq \eta < 2(l+1)\pi,$$

the function $\mathsf{Z}(\zeta)$ can be brought to the form:

$$\mathsf{Z}(\zeta) = \sum_{i=1}^{\sigma} M_i e^{m_i z + x_i\sqrt{-1}} = F(z, x_1, x_2, \ldots, x_\sigma),$$

where z denotes the corresponding point of the rectangle

$$|\xi| \leqq \frac{h}{2}, \quad 0 \leqq \eta < 2\pi \qquad (z = \zeta - 2l\pi)$$

and $x_1, x_2, \ldots, x_\sigma$ are real parameters, whose values are on the interval $(0, 2\pi)$ and which depend only on l.

Returning to our function $\mathsf{H}(\zeta)=[\mathsf{Z}(\zeta)]$, the same reasoning as before shows that all the roots of the equation

$$(48)\qquad \mathsf{H}(\zeta)=0,$$

which are in the sector $(\mathfrak{T})$, lie in the interior of the strip (D) for h sufficiently large. Moreover, if ζ is in (D_δ), we have

$$(49)\qquad \mathsf{H}(\zeta)=\mathsf{Z}(\zeta)\left\{1+\frac{\psi(\zeta)}{\zeta}\right\},$$

where $\psi(\zeta)$ is bounded on (D_δ). It we take a rectangle (Π_h) whose boundary has no points in common with excluded small circles, and which is so far from the origin, that on its boundary

$$\left|\frac{\psi(\zeta)}{\zeta}\right|<1,$$

a known theorem asserts that equations (48) and (40) have the same number of roots in this rectangle. The formula (46), thus being proved for the number of roots of (48), a simple geometric consideration proves the evaluation (38) of $|\zeta_k|$. [1])

25. Denote by $(\mathfrak{T}_j^{(\delta)})$ the part of the sector $(\mathfrak{T}_j)$, which remains after the interiors of small circles of radius δ centered at the roots of (48) are excluded from $(\mathfrak{T}_j)$. The ray d_j divides $(\mathfrak{T}_j^{(\delta)})$ into two parts; denote by w the sum of those of the numbers

$$w_1, w_2, \ldots, w_n,$$

which satisfy the condition

$$\Re e\, \varrho w_i \geqq 0,$$

when ϱ remains in one of those parts. Using the formulas (32), (33), (36), (41), (49) and the property (47) of the function $\mathsf{Z}(\zeta)$, we have

$$(50)\qquad |\Delta^{(0)}(\varrho)\, e^{-\varrho w}| \geqq N_\delta > 0,$$

where N_δ is a positive constant which depends only on δ.

26. We need to make but slight modifications in the preceding arguments, in order to discuss the more general case of the operators $L_i(y)$: [21])

$$(51)\quad L_i(y)\equiv \sum_{s=0}^{n}\varrho^s L_i^{(s)}(y)\equiv A_i(y,\varrho)+B_i(y,\varrho)+\int_a^b \alpha_i(x,\varrho)\, y^{(n-1)}(x)\,dx$$

where now

$$L_i^{(s)}(y)\equiv A_i^{(s)}(y)+B_i^{(s)}(y)+\int_a^b \alpha_{i_s}(x)\, y^{(n-1)}(x)\,dx.$$

In this case we replace the condition 1° of 17 by a more restrictive one:

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[1]) pp. 170—171.

[21]) Cfr. (5) of 5.

The functions (6) *are continuous and of bounded variation on* (a, b), *and the functions* $\alpha_{is}(x)$ *possess first derivatives continuous and of bounded variation on* (a, b).

In order to evaluate

$$u'_{ik} = L_i(y_{kj})$$

we can use now (14) 3 and apply the following

Lemma. *If the function* $\psi(z)$ *is of bounded variation on* $(0, Z)$, *and the constant* $c \neq 0$, *then*

$$(52) \qquad \int_0^Z \psi(z) e^{c\varrho z} dz = \frac{E}{\varrho} + e^{\varrho c Z} \frac{E}{\varrho};$$

if $\psi(z)$ *is merely integrable, then*

$$(53) \qquad \int_0^Z \psi(z) e^{c\varrho z} dz = E + E e^{\varrho c Z}.$$

Using the notations of 18 and integrating by parts, we obtain:

$$(54) \qquad \int_a^b \alpha_{is}(x) y_{kj}^{(n-1)}(x) dx$$
$$= \varrho^{n-2} \{[-\alpha_{is}(a) \eta_k(a) \varphi_k(a)^{n-2}] + e^{\varrho w_k} [\alpha_{is}(b) \eta_k(b) \varphi_k(b)^{n-2}]\}.$$

The number l_i is defined now as the maximum term of the sequence

$$(55) \qquad s_i + n - 2, \quad l_i^{(s)} + s - 1 \qquad (s = 0, 1, \ldots, n)$$

where s_i denotes the maximum value of the index s, for which

$$\alpha_{is}(x) \not\equiv 0 \quad \text{on} \quad (a, b)$$

and only terms with $l_i^{(s)} \neq 0$ are taken into account. The constants A_{ik}, B_{ik} must be changed accordingly.

After these modifications Theorem 8 remains true in the more general case (51) of the operators $L_i(y)$. [22])

Theorem 8 remains true for all the problems $(\bar{L})$ with the same functions $p_{i0}(x)$, $p_{i1}(x)$ $(i = 1, 2, \ldots, n)$ and the same numbers $l_i^{(s)}$, s_i, A_{ik}, B_{ik}. We can even say that, if δ is any given positive number, arbitrarily small, and circles of radius δ are described around all the characteristic values of the problem (L) then all the characteristic values of the problem $(\bar{L})$ sufficiently large, lie inside these circles, the

[22]) The case where the operators $L_i(y)$ contain singular values of t, instead of integral terms, has been discussed by Wilder [5]). In the same paper a theorem analogous to the theorem 8 has been proved under shlightly more general suppositions concerning the coefficients M. For a detailed discussion of some of the most important special cases see [1]) Ch. IV.

number of the characteristic values of either of the problems (L), $(\bar{L})$ being the same.

§ 5.

Expansion of the Green's function in partial fractions.

27. The results of § 4 enable us to deduce a formula for the expansion of the Green's function in partial fractions.

Theorem 9. *Suppose the conditions of Theorem* 8 (**21, 26**) *to be satisfied and denote by*

$$(1) \qquad \varrho_1, \varrho_2, \ldots, \varrho_m, \ldots$$

the characteristic values of the problem (L) *ordered so that*

$$|\varrho_1| \leqq |\varrho_2| \leqq \cdots \leqq |\varrho_m| \leqq \cdots.$$

If the interiors of small circles of radius δ *around each of the points* (1) *are excluded from the* ϱ*-plane, then on the remaining part* $(\mathfrak{D}_\delta)$ *of the plane we have*

$$(2) \qquad G(x, t, \varrho) \leqq \frac{G_\delta}{|\varrho|^{n-1}},$$

where G_δ *is a positive constant depending only on* δ.

For all values of ϱ *different from* (1):

$$(3) \qquad G(x, t, \varrho) = \sum_{\nu=1}^{\infty} G^{(\nu)}(x, t, \varrho),$$

that is the Green's function is equal to the sum of all its principal parts. If $n \geqq 2$, *the series* (3) *converges uniformly in* x, t *on* (a, b) *and in* ϱ *on* $(\mathfrak{D}_\delta)$. *If* $n = 1$, *the series* (3) *converges uniformly in* ϱ *on* $(\mathfrak{D}_\delta)$ *and in* x, t *on every portion of the region*

$$a \leqq x \leqq b, \quad a \leqq t \leqq b$$

which has no points in common with the lines $x = t, x = a, x = b$.

28. The boundaries of the sectors $(\mathfrak{S})$ and $(\mathfrak{T})$ divide the whole ϱ-plane into sectors $(\mathfrak{R})$, each of them being simultaneously and entirely in one of the sectors $(\mathfrak{S})$ and in one of the sectors $(\mathfrak{T})$. Consider one of these sectors $(\mathfrak{R})$ and denote by $(\mathfrak{R}_\delta)$ the part of $(\mathfrak{R})$ remaining after the small circles of Theorem 9 are excluded. The numbers

$$(4) \qquad w_1, w_2, \ldots, w_n$$

can be ordered so that on $(\mathfrak{R})$

$$(5) \qquad \Re\mathfrak{e}\, \varrho w_1 \leqq \Re\mathfrak{e}\, \varrho w_2 \leqq \ldots \leqq \Re\mathfrak{e}\, \varrho w_\tau \leqq 0 \leqq \Re\mathfrak{e}\, \varrho w_{\tau+1} \leqq \ldots \leqq \Re\mathfrak{e}\, \varrho w_n.$$

Then the number w of **25** is equal to

$$(6) \qquad w = w_{\tau+1} + \ldots + w_n.$$

There exists also a fundamental system of solutions of $L(y)=0$, which is of the form [on $(\mathfrak{R})$]:

$$(7)\qquad \begin{cases} y_i(x,\varrho)=e^{\varrho\int_a^x \varphi_i(x)\,dx}[\eta_i(x)], \\ \dfrac{d^s y_i(x,\varrho)}{dx^s}=e^{\varrho\int_a^x \varphi_i(x)\,dx}\varrho^s\varphi_i(x)^s[\eta_i(x)] \end{cases}$$

$$(i=1,2,\ldots,n;\ s=1,2,\ldots,n-1).$$

We shall use this system instead of the system $u_i(x,\varrho)$ of § 3, in order to express the Green's function, and we shall conserve other notations of § 3. We have:

$$(8)\qquad G(x,t,\varrho)=(-1)^n\frac{\Delta(x,t,\varrho)}{\Delta(\varrho)},$$

where

$$(9)\qquad \Delta(x,t,\varrho)=\begin{vmatrix} y_1(x,\varrho),\ldots, & y_n(x,\varrho), & g(x,t,\varrho) \\ u_{11}(\varrho), \quad\ldots, & u_{1n}(\varrho), & L_1(g)_x \\ \cdots & \cdots & \cdots \\ u_{n1}(\varrho), \quad\ldots, & u_{nn}(\varrho), & L_n(g)_x \end{vmatrix}.$$

The function $g(x,t,\varrho)$, by virtue of (10) 6, can be written in the form:

$$(10)\qquad g(x,t,\varrho)=\pm\frac{1}{2}\sum_{k=1}^{n} y_k(x,\varrho)\,z_k(t,\varrho)\qquad \begin{pmatrix} +\text{ if } x>t \\ -\text{ if } x<t\end{pmatrix},$$

where

$$(11)\qquad z_k(t,\varrho)=\frac{Y_k(t,\varrho)}{\delta(t,\varrho)}$$

and $Y_k(t,\varrho)$ denotes the cofactor of the element $y_k^{(n-1)}(t,\varrho)$ in the determinant $\delta(t,\varrho)$. It is easy to show that

$$(12)\qquad z_k(t,\varrho)=e^{-\varrho\int_a^t \varphi_k(x)\,dx}\frac{[\psi_k(t)]}{\varrho^{n-1}}\qquad (k=1,2,\ldots,n),$$

where

$$(13)\qquad \psi_k(t)=\frac{1}{\eta_k(t)\,\Phi'(\varphi_k(t))}.$$

Adding to the last column of $\Delta(x,t,\varrho)$ the

$$1^{\text{st}},\ 2^{\text{nd}},\ldots,\tau^{\text{th}},\ (\tau+1)^{\text{th}},\ldots,n^{\text{th}}$$

columns multiplied respectively by

$$\frac{1}{2}z_1(t,\varrho),\ \frac{1}{2}z_2(t,\varrho),\ldots,\frac{1}{2}z_\tau(t,\varrho),\ -\frac{1}{2}z_{\tau+1}(t,\varrho),\ \ldots,\ -\frac{1}{2}z_n(t,\varrho),$$ [8])

[8]) p. 392.

we obtain

$$(14)\qquad \Delta(x,t,\varrho)=\begin{vmatrix} y_1(x,\varrho),\ \dots,\ y_n(x,\varrho), & g_0(x,t,\varrho)\\ u_{11}(\varrho),\ \dots,\ u_{1n}(\varrho), & g_1(t,\varrho)\\ \dots & \dots\\ u_{n1}(\varrho),\ \dots,\ u_{nn}(\varrho); & g_n(t,\varrho)\end{vmatrix},$$

where

$$(15)\qquad g_0(x,t,\varrho)=\begin{cases}\sum_{k=1}^{\tau} y_k(x,\varrho)\,z_k(t,\varrho) & \text{if}\quad x>t,\\ -\sum_{k=\tau+1}^{n} y_k(x,\varrho)\,z_k(t,\varrho) & \text{if}\quad x<t.\end{cases}$$

$$(16)\qquad \left\{\begin{aligned} g_i(t,\varrho) &= L_i(g)_x+\frac{1}{2}\sum_{k=1}^{\tau}u_{ik}(\varrho)z_k(t,\varrho)-\frac{1}{2}\sum_{k=\tau+1}^{n}u_{ik}(\varrho)z_k(t,\varrho)\\ &= -\sum_{k=\tau+1}^{n}A_i(y_k,\varrho)\,z_k(t,\varrho)+\sum_{k=1}^{\tau}B_i(y_k,\varrho)\,z_k(t,\varrho)\\ &\quad -\sum_{k=\tau+1}^{n}z_k(t,\varrho)\int_a^t \alpha_i(x,\varrho)\,y_k^{(n-1)}(x,\varrho)\,d\varrho\\ &\quad +\sum_{k=1}^{\tau}z_k(t,\varrho)\int_t^b \alpha_i(x,\varrho)\,y_k^{(n-1)}(x,\varrho)\,dx.\end{aligned}\right.$$

The functions

$$e^{\varrho\int_t^x \varphi_k(x)\,dx}\quad (k=1,2,\dots,\tau;\ x\geqq t),\qquad e^{\varrho\int_t^x \varphi_k(x)\,dx}\quad (k=\tau+1,\dots,n;\ x\geqq t)$$

being bounded on $(\mathfrak{R})$, the preceding formulas show that:

$$(17)\qquad g_0(x,t,\varrho)=\frac{E}{\varrho^{n-1}},$$

$$(18)\qquad g_i(t,\varrho)=\varrho^{l_i}\frac{E}{\varrho^{n-1}}\qquad (i=1,2,\dots,n),$$

$$(19)\qquad \Delta(x,t,\varrho)=\varrho^{l-n+1}e^{\varrho w}E.$$

Since the determinant $\Delta(\varrho)$ in this case coincides with $\Delta_j(\varrho)$ of 18, we have

$$(20)\qquad \Delta(\varrho)=\varrho^{l}\Delta^{(0)}(\varrho),$$

and finally, using (50) 25:

$$|G(x,t,\varrho)|=\frac{|\Delta(x,t,\varrho)|}{\Delta(\varrho)}\leqq\frac{|E|}{N_\delta|\varrho|^{n-1}},$$

which proves the inequality (2).

Now we can construct a system of closed contours

$$(C_1),\ (C_2),\ \dots,\ (C_N),\ \dots$$

such that the length C_N of (C_N) is of the order $O(R_N)$, R_N denoting the shortest distance of (C_N) from the origin, and such that one and only one characteristic value of the problem (L) lies between two consecutive contours (C_N), (C_{N+1}). Such a choice of (C_N) is always possible because of the theorem 8, 21.

Let

$$\varrho_1, \varrho_2, \ldots, \varrho_N$$

be all the distinct poles of the Green's function, within (C_N). On the one hand we have

$$J(x, t, \varrho) = \frac{1}{2\pi\sqrt{-1}} \int_{(C_N)} \frac{G(x, t, \varrho')}{\varrho' - \varrho} d\varrho' = G(x, t, \varrho) - \sum_{\nu=1}^{N} G^{(\nu)}(x, t, \varrho).$$

On the other hand, if $n \geqq 2$, by virtue of (2) we have

$$|J(x, t, \varrho)| \leqq \frac{G_\delta}{2\pi} \int_{(C_N)} \frac{|d\varrho'|}{|\varrho' - \varrho| \cdot |\varrho'|^{n-1}} \leqq \frac{G_\delta}{2\pi} \frac{C_N}{R_N^n \left(1 - \frac{|\varrho|}{R_N}\right)} \to 0,$$

when $N \to \infty$, uniformly for x, t in (a, b) and ϱ in $(\mathfrak{D}_\delta)$, which proves Theorem 9 in the case $n \geqq 2$.

If $n = 1$, the Green's function $G(x, t, \varrho)$ has a very simple expression and the corresponding statement of Theorem 9 may be easily proved in this case by applying Lemma 1, 38.

29. Theorem 10. *If $F(x)$ is an arbitrary function which has an absolutely continuous derivative of the order $(n-1)$ on (a, b) and satisfies the conditions:*

$$L_i(F) = 0 \qquad (i = 1, 2, \ldots, n) \tag{21}$$

for a certain value of ϱ, different from the characteristic values of the problem (L), then

$$F(x) = \sum_{\nu=1}^{\infty} \int_a^b L(F, \varrho)_t G^{(\nu)}(x, t, \varrho) dt, \tag{22}$$

this expansion being uniformly convergent on (a, b). The conditions of Theorem 9 are supposed to be satisfied.

The formula (22) follows immediately from the expansion (3) of the Green's function and from the formula [(22), 8]:

$$F(x) = \int_a^b L(F, \varrho)_t G(x, t, \varrho) dt.$$

We shall call (22) the "preliminary form of the expansion of an arbitrary function".

§ 6.

Equiconvergence Theorem.

30. In this section we shall transform the preliminary form of the expansion of an arbitrary function $F(x)$ so as to get rid of most of the restrictions imposed on $F(x)$, and even as to be able to draw some general conclusions concerning the general case of an integrable $F(x)$.

We shall suppose here that *all the conditions of Theorem* 8, **21** *are satisfied, that the operators $L_i(y)$ of the problem (L) do not contain integrals, and that the functions*

$$p_{ij}(x) \qquad (j = 0, 1, \ldots, i;\ i = 1, 2, \ldots, n)$$

possess continuous derivatives of the order $(n-i)$, so that the adjoint problem exists (**9**).

Let $\varrho = 0$ be not a characteristic value, and $F(x)$ be an arbitrary function which satisfies the conditions of Theorem 10 (**29**) for $\varrho = 0$. The function $F(x)$ can be expanded according to Theorem 10 as follows

$$(1) \qquad F(x) = \sum_{\nu=1}^{\infty} \int_a^b L_0(F)_t\, G^{(\nu)}(x, t, 0)\, dt; \qquad L_0(y) \equiv L(y)\big|_{\varrho=0},$$

the series being uniformly convergent on (a, b). Denote by

$$\Sigma_N(F)$$

the sum of the N first terms of the expansion (1). Applying Green's identity (**9**) we have:

$$\int_a^b G^{(\nu)}(x, t, 0)\, L_0(F)_t\, dt = \int_a^b F(t)\, L_0'(G^{(\nu)})_t\, dt + Q_{L_0}(F, G^{(\nu)})_t,$$

whence

$$(2) \qquad \Sigma_N(F) = \Sigma_N^{(1)}(F) + \Sigma_N^{(2)}(F),$$

where

$$(3) \qquad \Sigma_N^{(1)}(F) = \sum_{\nu=1}^{N} \int_a^b F(t)\, L_0'(G^{(\nu)})_t\, dt,$$

$$(4) \qquad \Sigma_N^{(2)}(F) = \sum_{\nu=0}^{N} Q_{L_0}(F, G^{(\nu)})_t.$$

31. Since

$$(5) \qquad G^{(\nu)}(x, t, \varrho) = \operatorname*{Res}_{\varrho'=\varrho_\nu} \frac{G(x, t, \varrho')}{\varrho - \varrho'},$$

we have:

$$(6)\quad \begin{cases} \Sigma_N^{(1)}(F) = -\sum_{\nu=1}^{N} \int_a^b F(t)\, L_0'\left\{ \operatorname*{Res}_{\varrho'=\varrho_\nu} \frac{G(x,t,\varrho')}{\varrho'} \right\}_t dt \\ \qquad = -\sum_{\nu=1}^{N} \int_a^b F(t) \operatorname*{Res}_{\varrho'=\varrho_\nu} \frac{1}{\varrho'} L_0'\{G(x,t,\varrho')\}_t\, dt \\ \qquad = \frac{-1}{2\pi\sqrt{-1}} \int_{(C_N)} \frac{d\varrho}{\varrho} \int_a^b F(t)\, L_0'\{G(x,t,\varrho)\}_t\, dt \end{cases}$$

where (C_N) denotes the contour of 28 [23]).

The function $G(x, t, \varrho)$, as function of t, satisfies the adjoint problem (L'), and therefore

$$L'\{G(x,t,\varrho), \varrho\}_t = 0,$$

so that in (6) we can replace

$$-L_0'\{G(x,t,\varrho)\}_t \quad \text{by} \quad L'\{G(x,t,\varrho),\varrho\}_t - L_0'\{G(x,t,\varrho)\}_t.$$

Taking into account the definition of the operator $L'(y)$ we obtain easily:

$$(7)\quad \Sigma_N^{(1)}(F)$$
$$= \frac{1}{2\pi\sqrt{-1}} \int_{(C_N)} d\varrho \int_a^b F(t) \sum_{m=1}^{n} (-1)^{n-m} \varrho^{m-1} p_{m0}(t) \frac{\partial^{n-m} G(x,t,\varrho)}{\partial t^{n-m}}\, dt + \sigma_N^{(1)}(F),$$

where $\sigma_N^{(1)}(F)$ denotes the sum of terms of the form:

$$(8)\quad \frac{1}{2\pi\sqrt{-1}} \int_{(C_N)} d\varrho \int_a^b \chi(t) F(t) \varrho^{\varkappa} \frac{\partial^{\mu} G(x,t,\varrho)}{\partial t^{\mu}}\, dt,$$

$\chi(t)$ being a continuous function and integers $\varkappa$ and μ satisfying the condition:

$$(9)\quad \mu + \varkappa \leqq n - 2.$$

32. In an analogous way we shall transform the sum $\Sigma_N^{(2)}(F)$. Here we have to consider three different cases:

1°. The matrix of the coefficients of the operators $L_i(y)$ contains at least one determinant of the n^{th} order, which is free from the elements of

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[23]) The integration with respect to ϱ must be taken also over a small circle around the origin, because of the singularity of the integrand at the point $\varrho = 0$. The corresponding term vanishes, however, because of

$$L_0'\{G(x,t,0)\}_t = 0.$$

either of the columns corresponding to $y(a)$, $y(b)$ and which is not identically zero.

2°. All the determinants of 1° are identically zero, but there exists at least one determinant of the n^{th} order which is free from the elements of one of the two columns corresponding to $y(a)$ or to $y(b)$, and which is not identically zero.

3°. Among all the determinants of the n^{th} order of the matrix in question only those are not identically zero which contain both columns corresponding to $y(a)$ and $y(b)$.

Without loss of generality we may suppose that the determinant in question is different from zero for $\varrho = 0$.

33. Suppose we have the case 1°. The formula (36), 9 shows:

$$(10) \qquad Q_{L_0}(u, v) = \sum_{i=1}^{n} \{L_i^{(0)}(u) L_{n+i}^{(0)'}(v) + L_{n+i}^{(0)}(u) L_i^{(0)'}(v)\},$$

where

$$L_i^{(0)'}(y) \qquad (i = 1, 2, \ldots, n)$$

are the operators of the adjoint problem (L') for $\varrho = 0$, and where each of two sets of linear forms

$$L_i^{(0)}(y), \quad L_i^{(0)'}(y) \qquad (i = 1, 2, \ldots, 2n)$$

represent a complete set of $2n$ linearly independent forms in $2n$ variables:

$$(11) \qquad y(a), \ldots, y^{(n-1)}(a), \quad y(b), \ldots, y^{(n-1)}(b).$$

Suppose that the function $u(x)$ satisfies the conditions:

$$(12) \qquad L_i^{(0)}(u) = 0 \qquad (i = 1, 2, \ldots, n).$$

We can express n of the $2n$ quantities

$$(13) \qquad u(a), \ldots, u^{(n-1)}(a), \quad u(b), \ldots, u^{(n-1)}(b)$$

in terms of the n remaining ones, which may be taken as

$$L_{n+i}^{(0)}(u) \qquad (i = 1, 2, \ldots, n)$$

and we can set:

$$L_{n+1}^{(0)}(u) \equiv u(a), \quad L_{n+2}^{(0)}(u) \equiv u(b).$$

If we suppose that the function $u(x)$ satisfies the conditions

$$(14) \qquad L_{n+i}^{(0)}(u) = 0 \qquad (i = 3, 4, \ldots, n)$$

in addition to the conditions (12), the values $u(a), u(b)$ remain arbitrary and we obtain:

$$Q_{L_0}(u, v) = u(a) L_1^{(0)'}(v) + u(b) L_2^{(0)'}(v),$$

which entirely determines the operators

$$L_1^{(0)'}(v), \quad L_2^{(0)'}(v).$$

The same calculations may be made for any value of ϱ, and so we can define the operators

$$L_1'(v, \varrho), \quad L_2'(v, \varrho)$$

as the coefficients of $u(a), u(b)$ in the expressions of $Q_L(u, v)$, if all the quantities

$$u'(a), \ldots, u^{(n-1)}(a), \quad u'(b), \ldots, u^{(n-1)}(b)$$

are expressed in terms of $u(a)$ and $u(b)$, according to the conditions

$$L_i(u) = 0 \ \ (i = 1, 2, \ldots, n); \quad L_{n+i}(u) = 0 \ \ (i = 3, 4, \ldots, n).$$ [24]

Suppose now that $F(x)$ satisfies the conditions

$$(15) \quad L_i^{(0)}(F) = 0 \ \ (i = 1, 2, \ldots, n); \quad L_{n+i}^{(0)}(F) = 0 \ \ (i = 3, 4, \ldots, n).$$

On substituting

$$u(t) = F(t), \quad v(t) = G^{(\nu)}(x, t, 0)$$

in (10) we have:

$$(16) \qquad Q_{L_0}(F, G^{(\nu)})_t = F(a)\, L_1^{(0)'}(G^{(\nu)})_t + F(b)\, L_2^{(0)'}(G^{(\nu)})_t .$$

The Green's function $G(x, t, \varrho)$, as function of t, is a solution of the adjoint problem (L'). Hence

$$L_i'(G, \varrho)_t = 0 \qquad (i = 1, 2)$$

and

$$(17) \qquad L_i^{(0)'}(G^{(\nu)})_t = L_i^{(0)'}\left\{\operatorname*{Res}_{\varrho'=\varrho_\nu} \frac{G(x, t, \varrho')}{-\varrho'}\right\}_t = \operatorname*{Res}_{\varrho'=\varrho_\nu}\left\{\frac{L_i'(G, \varrho')_t - L_i^{(0)'}(G)_t}{\varrho'}\right\}.$$

The function under the sign Res is a linear combination of the function $G(x, t, \varrho')$ and of its first $(n-1)$ derivatives with respect to t, taken at $t = a$, $t = b$, the coefficients being rational functions in ϱ'. Expanding every coefficient in descending powers of ϱ', we shall retain only those terms for which the exponent of ϱ' plus the order of the derivative is at least $(n-2)$. Computing the Res of the sum of these terms we obtain a perfectly defined function of x, which we denote by

$$\Xi_\nu^{(i)}(x) \qquad (i = 1, 2;\ \nu = 1, 2, \ldots).$$

From this definition of $\Xi_\nu^{(i)}(x)$ it is obvious that

$$(18) \qquad \Sigma_N^{(2)}(F) = \sum_{\nu=1}^{N} \{F(a)\, \Xi_\nu^{(1)}(x) + F(b)\, \Xi_\nu^{(2)}(x)\} + \sigma_N^{(2)}(F),$$

[24]) The coefficients of the operators are in general rational functions in ϱ, so that it is necessary to suppose that ϱ is different from any root of their denominators.

where $\sigma_N^{(2)}(F)$ is a sum of products of $F(a)$ or $F(b)$ by the terms of the form:

$$(19) \qquad \frac{1}{2\pi\sqrt{-1}} \int_{(C_N)} d\varrho\, \theta_\varkappa(\varrho) \frac{\partial^\mu G(x,t,\varrho)}{\partial t^\mu}\Big|_{t=a \text{ or } t=b},$$

$\theta_\varkappa(\varrho)$ denoting a function of the order $O(\varrho^\varkappa)$ for large values of $|\varrho|$, and

$$(20) \qquad \varkappa + \mu < n - 2.$$

34. In the case 2° of 32, if the function $u(x)$ satisfies the conditions (12), only one of the quantities $u(a)$, $u(b)$ can be considered as arbitrary, and this quantity may be taken as equal to $L_{n+1}^{(0)}(u)$.

It is obvious also that, if $F(x)$ satisfies the conditions

$$(21) \qquad L_i^{(0)}(F) = 0 \quad (i = 1, 2, \ldots, n); \qquad L_{n+i}^{(0)}(F) = 0 \quad (i = 2, 3, \ldots, n),$$

there must exist a relation of the form:

$$(22) \qquad F(b) = a_0 F(a) \quad \text{or} \quad F(a) = b_0 F(b),$$

a_0, b_0 beeing constant factors, which in special cases may reduce to zero. Instead of (16) now we have respectively

$$(23) \qquad Q_{L_0}(F, G^{(\nu)})_t = F(a) L_1^{(0)\prime}(G^{(\nu)})_t$$

or

$$(24) \qquad Q_{L_0}(F, G^{(\nu)})_t = F(b) L_1^{(0)\prime}(G^{(\nu)})_t$$

Using the operator $L_1^{(0)\prime}$ we can construct the function $\Xi_\nu(x)$ in the same way as the functions $\Xi_\nu^{(i)}(x)$ have been constructed above using the operators $L_i^{(0)\prime}$. According as we have the first or the second case of (22) we set:

$$(25) \qquad \Xi_\nu^{(1)}(x) = \Xi_\nu(x), \qquad \Xi_\nu^{(2)}(x) = 0,$$

or

$$(26) \qquad \Xi_\nu^{(1)}(x) = 0, \qquad \Xi_\nu^{(2)}(x) = \Xi_\nu(x),$$

and we obtain for $\Sigma_N^{(2)}(F)$ the same expression (18) as in the case 1°.

In the case 3° we set

$$\Xi_\nu^{(1)}(x) \equiv 0, \qquad \Xi_\nu^{(2)}(x) \equiv 0,$$

and the formula (18) remains true, if $F(x)$ satisfies the conditions

$$(27) \qquad L_i^{(0)}(F) = 0, \qquad L_{n+i}^{(0)}(F) = 0 \qquad (i = 1, 2, \ldots, n).$$

35. We see finally that for every function $F(x)$ which possesses an absolutely continuous derivative of the $(n-1)^{\text{th}}$ order, and which satisfies the conditions (15) in the case 1°, (21) in the case 2° and (27) in the case 3°, the sum $\Sigma_N(F)$ can be transformed as follows:

$$(28)\quad \begin{cases} \Sigma_N(F) = \sum_{\nu=1}^{N} \int_a^b dt\, F(t) \sum_{m=1}^{n} (-1)^{n-m} p_{m0}(t) \operatorname*{Res}_{\varrho=\varrho_\nu} \varrho^{m-1} \dfrac{\partial^{n-m} G(x,t,\varrho)}{\partial t^{n-m}} \\ \qquad + \sum_{\nu=1}^{N} \{F(a)\, \Xi_\nu^{(1)}(x) + F(b)\, \Xi_\nu^{(2)}(x)\} + \sigma_N^{(1)}(F) + \sigma_N^{(2)}(F). \end{cases}$$

The right hand member of this equality has a definite sense for every integrable function $F(x)$, which takes on definite values at the points a, b. We shall prove moreover that $\sigma_N^{(1)}(F)$ and $\sigma_N^{(2)}(F)$ tend uniformly to zero for every integrable function $F(x)$, so that the whole discussion is reduced to that of the two first terms.

It is important to note that the assumption that $\varrho = 0$ is not a characteristic value of the problem (L), is not essential. If $\varrho = 0$ is a characteristic value, we can start from a certain initial value $\varrho = \varrho_0$ different from any characteristic value, and it is obvious that all the terms of the first sum of the right-hand member of (28) are independent of the particular choice of this initial value of ϱ_0 and the same can be proved concerning the limit of the second sum, as $N \to \infty$.[25])

The following discussion is based upon a detailed study of the asymptotic character of the Green's function $G(x, t, \varrho)$ and of its derivatives for large values of $|\varrho|$.

36. We shall use the notation of **17**, **18**, **28** taking into account that the operators $L_i(y)$ do not contain integrals and that, under the conditions of **30**, the expressions (12) **28** for the functions $z_k(t, \varrho)$ can be differentiated $(n-1)$ times with respect to t.

We have

$$(29)\qquad \frac{d^m z_k(t,\varrho)}{dt^m} = \varrho^{m-n+1} e^{-\varrho \int_a^t \varphi_k(x)\,dx} [\psi_{k,m}(t)],$$

where

$$(30)\qquad \psi_{k,m}(t) = \frac{(-1)^m \{\varphi_k(t)\}^m}{\eta_k(t)\, \Phi'\{\varphi_k(t)\}}; \quad \psi_{k,0}(t) = \psi_k(t).$$

We saw in **17** that the conditions $1^\circ - 3^\circ$ involve either 4_1° or 4_2°, and in both cases we can write:

$$(31)\qquad \varphi_i(x) = \pi_i q_i(x); \quad q_i(x) \geqq q_0 > 0.$$

For the sake of brevity we set:

$$(32)\qquad X_i = \int_a^x q_i(x)\,dx; \quad \xi_i = \int_a^t q_i(x)\,dx; \quad X_{0i} = \int_a^b q_i(x)\,dx;$$

[25]) Cfr. **45** below.

$$(33)\qquad g_{0,m}(x,t,\varrho)\equiv g_{0,m}\equiv\frac{\partial^m g_0(x,t,\varrho)}{\partial t^m};\quad g_{0,0}\equiv g_0(x,t,\varrho);$$

$$(34)\qquad g_{i,m}(t,\varrho)\equiv g_{i,m}\equiv\frac{\partial^m g_i(t,\varrho)}{\partial t^m};\quad g_{i,0}\equiv g_i(t,\varrho).$$

Then we have

$$(35)\qquad w_i=\int_b^a\varphi_i(x)\,dx=\pi_i X_{0i},$$

$$(36)\quad \Re(\varrho\pi_1)\leqq\cdots\leqq\Re(\varrho\pi_\tau)\leqq 0\leqq\Re(\varrho\pi_{\tau+1})\leqq\cdots\leqq\Re(\varrho\pi_n)\text{ on }(\Re),$$

$$(37)\quad g_{0,m}=\begin{cases}\varrho^{m-n+1}\sum\limits_{k=1}^{\tau}e^{\varrho\pi_k(X_k-\xi_k)}\,\eta_k(x)\,[\psi_{k,m}(t)], & \text{if } x>t,\\ -\varrho^{m-n+1}\sum\limits_{k=\tau+1}^{n}e^{\varrho\pi_k(X_k-\xi_k)}\,\eta_k(x)\,[\psi_{k,m}(t)], & \text{if } x<t.\end{cases}$$

$$(38)\quad g_{i,m}=\varrho^{l_i+m-n+1}\Big\{-\sum_{k=\tau+1}^{n}e^{-\varrho\pi_k\xi_k}[A_{ik}\psi_{k,m}(t)] + \sum_{k=1}^{\tau}e^{\varrho\pi_k(X_{0k}-\xi_k)}[B_{ik}\,\psi_{k,m}(t)]\Big\}.$$

37. In what follows we shall use the notation:

$$(39)\quad (-1)^n\begin{vmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n & \chi\\ u_{11} & u_{12} & \dots & u_{1n} & \beta_1\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ u_{n1} & u_{n2} & \dots & u_{nn} & \beta_n\end{vmatrix}\equiv\varDelta(\alpha_1,\alpha_2,\dots,\alpha_n;\beta_1,\beta_2,\dots,\beta_n;\chi)\equiv\varDelta(\alpha_i;\beta_j;\chi).$$

In particular the numerator of the **Green's** function is:

$$(40)\qquad (-1)^n\varDelta(x,t,\varrho)=\varDelta\{y_i(x,\varrho);\ g_j(t,\varrho);\ g_0(x,t,\varrho)\},$$

and, more generally,

$$(41)\qquad \frac{\partial^m G(x,t,\varrho)}{\partial t^m}=\frac{\varDelta\{y_i;g_{j,m};g_{0,m}\}}{\varDelta(\varrho)}\qquad(m=0,1,\dots,n-1).$$

We have obviously:

$$(42)\qquad \varDelta(\alpha_i;\beta_j;\chi)=\chi\varDelta(\varrho)+\sum_{i,j=1}^{m}\alpha_i\beta_j\varDelta_{ji}(\varrho)$$

where $\varDelta_{ji}(\varrho)$ denotes the cofactor of the element $u_{ji}(\varrho)$ [26] in the determinant $\varDelta(\varrho)$. Now (16) 18 shows:

$$(43)\qquad \varDelta_{ji}(\varrho)=\begin{cases}\varrho^{l-l_j}e^{\varrho w}E_{ji}(\varrho) & \text{if } i=1,2,\dots,\tau,\\ \varrho^{l-l_j}e^{\varrho(w-w_i)}E_{ji}(\varrho) & \text{if } i=\tau+1,\dots,n,\end{cases}$$

[26]) Here we write $u_{ji}(\varrho)$ instead of $u'_{ji}(\varrho)$ of 18.

$E_{ji}(\varrho)$ depending only on ϱ and being bounded on $(\mathfrak{R})$. Substituting in (41) we have.

$$(44)\qquad \frac{\partial^m G(x,t,\varrho)}{\partial t^m} = g_{0,m} + \varrho^{m-n+1} K_m(x,t,\varrho),$$

where $K_m(x,t,\varrho)$ is a bilinear form in the two sets of quantities:

$$(45)\qquad e^{\varrho \pi_i X_i}\ (i=1,2,\ldots,\tau), \qquad e^{-\varrho \pi_i (X_{0i}-X_i)}\ (i=\tau+1,\ldots,n);$$

$$(46)\qquad e^{\varrho \pi_i (X_{0i}-\xi_i)}\ (i=1,2,\ldots,\tau), \qquad e^{-\varrho \pi_i \xi_i}\ (i=\tau+1,\ldots,n).$$

Denoting these quantities respectively by

$$(47)\qquad \tilde{\omega}_i', \quad \tilde{\omega}_i'' \qquad (i=1,2,\ldots,n)$$

we have:

$$(48)\qquad K_m(x,t,\varrho) = \sum_{i,k=1}^{n} \Omega_{i,k}^{(m)}(x,t,\varrho)\, \tilde{\omega}_i' \tilde{\omega}_k'',$$

where

$$(49)\qquad \Omega_{i,k}^{(m)}(x,t,\varrho) = [\omega_{i,k}^{(m)}(x,t)]\, E_{i,k}^{(m)}(\varrho).$$

Here the functions $E_{i,k}^{(m)}(\varrho)$ depend only on ϱ and remain bounded on $(\mathfrak{R}_\delta)$. The functions $\omega_{i,k}^{(m)}(x,t)$ depend only on x, t and each of them is equal to a sum of products of a function of x by a function of t, both factors having a second derivative continuous on (a, b).

38. Theorem 11. *Under the conditions of 30 the integrals*

$$(50)\qquad I_N^{(\varkappa,\mu)} = \frac{1}{2\pi\sqrt{-1}} \int_{(C_N)} d\varrho \int_a^b f(t)\, \varrho^{\varkappa} \frac{\partial^\mu G(x,t,\varrho)}{\partial t^\mu}\, dt \qquad (\varkappa+\mu \leqq n-2)$$

$f(x)$ being an arbitrary integrable function, and the integrals

$$(51)\qquad j_N^{(\varkappa,\mu)} = \frac{1}{2\pi\sqrt{-1}} \int_{(C_N)} \theta_\varkappa(\varrho) \frac{\partial^\mu G(x,t,\varrho)}{\partial t^\mu}\Big/_{t=a \text{ or } t=b}\, dt \qquad (\varkappa+\mu < n-2)$$

where

$$\theta_\varkappa(\varrho) = O(\varrho^\varkappa) \text{ for large } |\varrho|,$$

tend to zero uniformly on (a, b) as $N \to \infty$.

First let us state certain lemmas, which are necessary for further considerations[1]).

Lemma 1. *Let $\varepsilon(\varrho, z, x_1, \ldots, x_m)$ be a function of a complex variable ϱ, of a real variable z, and of a certain number of parameters $x_1, \ldots, x_m$, which is determined on the half-plane*

$$(52)\qquad \Re e(c\varrho) \leqq 0 \qquad (c \text{ constant} \neq 0),$$

for all values of z in the interval $(0, Z)$ and for all values of $x_1, \ldots, x_m$

[1]) pp. 216–217.

in a closed region (D). *Let* (Γ_ν) $(\nu = 1, 2, \ldots)$ *be a sequence of circular arcs in* (52) *with centers at the origin and respective radii* R_ν, *where* $R_\nu \to \infty$ *as* $\nu \to \infty$. *Then, if on the arc* (Γ_ν), $\varepsilon(\varrho, z, x_1, \ldots, x_m)$ *is analytic in* ϱ *and tends to zero uniformly in* $\varrho, z, x_1, \ldots, x_m$, *as* $\nu \to \infty$, *the integral*

$$\int_{(\Gamma_\nu)} \varepsilon(\varrho, z, x_1, \ldots, x_m) e^{c\varrho z} d\varrho \to 0 \quad as \quad \nu \to \infty,$$

uniformly with respect to $x_1, \ldots, x_m$ *in* (D) *and with respect to* z *in the interior of* $(0, Z)$.[27]

We have on (Γ_ν):

$$|\varepsilon(\varrho, z, x_1, \ldots, x_m)| \leqq \varepsilon_\nu,$$

where ε_ν does not depend on $(\varrho, z, x_1, \ldots, x_m)$ and $\lim \varepsilon_\nu = 0$.

Hence, setting

$$c\varrho = -R_\nu(\cos\varphi + \sqrt{-1}\sin\varphi)$$

we have

$$\left|\int_{(\Gamma_\nu)} \varepsilon(\varrho, z, x_1, \ldots, x_m) e^{c\varrho z} d\varrho\right| \leqq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varepsilon_\nu e^{-R_\nu z \cos\varphi} R_\nu d\varphi \leqq \pi \varepsilon_\nu \frac{1 - e^{-R_\nu z}}{z}$$

which proves Lemma 1.

Lemma 2. *Under the conditions of Lemma* 1, *if* $E(\varrho, z, x_1 \ldots x_m)$ *is uniformly bounded and* $\psi(z)$ *is an arbitrary integrable function, the integral*

$$\int_\alpha^\beta \psi(z) dz \int_{(\Gamma_\nu)} E(\varrho, z, x_1 \ldots x_m) e^{c\varrho z} \frac{d\varrho}{\varrho} \to 0 \qquad as \quad \nu \to \infty,$$

uniformly with respect to $x_1, \ldots, x_m$ *and to* α, β *in* $(0, Z)$.

If E_0 denotes the upper bound of $|E(\varrho, z, x_1 \ldots x_m)|$, we have:

$$\left|\int_\alpha^\beta \psi(z) dz \int_{(\Gamma_\nu)} E e^{c\varrho z} \frac{d\varrho}{\varrho}\right| \leqq E_0 \pi \int_0^Z |\psi(z)| \frac{1 - e^{-R_\nu z}}{R_\nu z} dz$$

$$\leqq E_0 \pi \left\{\frac{1}{R_\nu \delta_\nu} \int_0^Z |\psi(z)| dz + \int_0^{\delta_\nu} |\psi(z)| dz\right\} \to 0,$$

if δ_ν tends to zero so that

$$R_\nu \delta_\nu \to \infty \quad as \quad \nu \to \infty.$$

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[27]) That is uniformly with respect to z on any interval which is in the interior of $(0, Z)$.

Lemma 3. *If $\psi(z)$ is an arbitrary integrable function, the integral*

$$\int_\alpha^\beta \psi(z)\, e^{c\varrho z}\, dz \to 0$$

as $|\varrho| \to \infty$ *and* $\Re(c\varrho) \leqq 0$, *the convergence being uniform for all* α, β *in* $(0, Z)$.

This lemma can be easily proved if we approximate the function $\psi(z)$ by a suitable step-function, for which the proof is immediate.

It is important to observe that Lemmas 1 and 2 hold true when arcs of bounded length of the contours (Γ_ν) are slightly deformed, the variation of length being also bounded. It is obvious that the contours (C_ν) of 28 satisfy this condition and they can replace (Γ_ν) in the statements of Lemmas 1 and 2.

Now the first part of Theorem 11 concerning the integral (50) follows from the formulas (44), (37), (48), (49), (50), if we apply Lemma 2 in a suitable way. The statement of Theorem 11 concerning the integral (51) is obvious, since the integrand is of the order $O\left(\frac{1}{\varrho^2}\right)$.

39. The preceding discussion shows that if $F(x)$ is an arbitrary function which possesses an absolutely continuous $(n-1)^{\text{th}}$ derivative and which satisfies suitable boundary conditions, we always have the expansion

$$(53)\quad \left\{ \begin{aligned} F(x) &= \lim_{N\to\infty} \Sigma_N(F) \\ &= \sum_{\nu=1}^{\infty} \int_a^b F(t) \sum_{m=1}^{n} (-1)^{n-m} p_{m0}(t) \operatorname*{Res}_{\varrho=\varrho_\nu} \varrho^{m-1} \frac{\partial^{n-m} G(x,t,\varrho)}{\partial t^{n-m}}\, dt \\ &\quad + F(a) \sum_{\nu=1}^{\infty} \Xi_\nu^{(1)}(x) + F(b) \sum_{\nu=1}^{\infty} \Xi_\nu^{(2)}(x). \end{aligned} \right.$$

We now shall consider the first term of this expansion separately, and we shall suppose only that $F(x)$ is integrable. Denote

$$(54)\quad \left\{ \begin{aligned} J_N(F) &= \frac{1}{2\pi\sqrt{-1}} \int_{(C_N)} d\varrho \int_a^b F(t) \sum_{m=1}^{n} (-1)^{n-m} p_{m0}(t)\, \varrho^{m-1} \frac{\partial^{n-m} G(x,t,\varrho)}{\partial t^{n-m}}\, dt \\ &= \sum_{\nu=1}^{N} \int_a^b F(t) \sum_{m=1}^{n} (-1)^{m-n} p_{m0}(t) \operatorname*{Res}_{\varrho=\varrho_\nu} \varrho^{m-1} \frac{\partial^{n-m} G(x,t,\varrho)}{\partial t^{n-m}}\, dt. \end{aligned} \right.$$

Theorem 12. *Let $(\bar{L})$ be a problem analogous to the problem (L) with the same functions*

$$p_{i0}(x), \quad p_{i1}(x) \qquad (i = 1, 2, \ldots, n)$$

and with the same numbers $l_i^{(s)}$ and the same coefficients $a^{(s)}_{i l_i^{(s)}}$, $b^{(s)}_{i l_i^{(s)}}$.

Denoting by $\bar{J}_N(F)$ *the integral* $J_N(F)$ *for the problem* $(\bar{L})$, *the difference*

$$(55) \qquad J_N(F) - \bar{J}_N(F)$$

tends to zero uniformly on (a, b) *as* $N \to \infty$, *for every integrable function* $F(x)$. *Denoting by* $\bar{\Xi}_\nu^{(i)}(x)$ *the functions corresponding to* $\Xi_\nu^{(i)}(x)$ *for the problem* $(\bar{L})$, *the series*

$$\sum_{\nu=1}^{\infty} \{ \Xi_\nu^{(i)}(x) - \bar{\Xi}_\nu^{(i)}(x) \} \qquad (i = 1, 2)$$

converge to zero uniformly on (a, b), *if the operators* $L_i^{(0)'}$ *of* **33, 34** *coincide for both problems* (L) *and* $(\bar{L})$[1].

Under the conditions of Theorem 12, the principal terms in the expressions (44) of

$$\frac{\partial^m G(x, t, \varrho)}{\partial t^m}, \qquad \frac{\partial^m \bar{G}(x, t, \varrho)}{\partial t^m} \text{ [28]}$$

are the same, so that the difference (55) can be reduced to a sum of terms, to each of which Lemma 2 is applicable. This proves the first statement of Theorem 12.

The second statement of Theorem 12 will be proved, if we observe that, according to **33, 34** both series

$$\sum_{\nu=1}^{\infty} \{ L_i^{(0)'}(G^{(\nu)})_t - \Xi_\nu^{(i)}(x) \}, \qquad \sum_{\nu=1}^{\infty} \{ L_i^{(0)'}(\bar{G}^{(\nu)})_t - \bar{\Xi}_\nu^{(i)}(x) \}$$

converge uniformly to zero and that the functions

$$L_i^{(0)'} \left\{ \frac{G(x, t, \varrho) - \bar{G}(x, t, \varrho)}{\varrho} \right\}$$

are of the order $O\left(\frac{1}{\varrho^2}\right)$ on (C_N).

Theorem 12 enables us to compare the expansions of an arbitrary function $F(x)$ for various problems (L) and therefore this theorem may be called *equiconvergence theorem.*

For the special case of Sturm-Liouville's functions an analogous theorem was proved by W. Stekloff[29]) and by A. Haar[30]). For the equations of the n^{th} order, a theorem which essentially coincides with the equiconvergence theorem (for interior points of (a, b)) was proved first by the Author[31]).

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[1]) Ch. V. This result was not explicitly stated but might be derived immediately from the formulas therein given.

[28]) $\bar{G}(x, t, \varrho)$ denotes the Green's function of the problem $(\bar{L})$.

[29]) Sur les expressions asympt. etc., Proc. Charcoff Math. Soc. **10** (1907—1908), pp. 96—97.

[30]) Zur Theorie der orthog. Funktionen-Systeme, Math. Ann. **69** (1912), p. 355.

[31]) Sur quelques points de la theorie des equat. differ. etc., Rend. di Pal. **34** (1912), pp. 345—382.

§ 7.

Expansion Problem.

40. Now we can compare our expansion $J_N(F)$ with the classical Dirichlet's integral of the theory of Fourier's series, or with the analogous integrals. This comparison is furnished by the following

Theorem 13. *Under the conditions of* **30** *there exists a set of constants* $N_1, N_2, \ldots, N_n$ *depending on* N *such that if*

$$(1)\qquad \mathrm{T}_N(F) = \frac{1}{\pi}\int_a^b dt\, F(t) \sum_{k=1}^{n} \omega_k(t)\, q_k(t) \frac{\sin N_k (X_k - \xi_k)}{X_k - \xi_k},$$

$$(2)\qquad \omega_k(t) = \frac{\{\varphi_k(t)\}^{n-1}}{\Phi'\{\varphi_k(t)\}} \qquad (k = 1, 2, \ldots, n),$$

the difference

$$(3)\qquad J_N(F) - \mathrm{T}_N(F)$$

tends to zero as $N \to \infty$, *for every integrable function* $F(x)$, *and uniformly on the interior of* (a, b)[32].

We have

$$J_N(F) = J'_N(F) + J''_N(F),$$

where

$$(4)\quad J'_N(F) = \frac{1}{2\pi\sqrt{-1}} \sum_{(\mathfrak{R})} \int_{(c_N)} d\varrho \int_a^b F(t) \sum_{m=1}^{n} (-1)^{n-m} p_{m0}(t)\, \varrho^{m-1} g_{0,\,n-m}(x, t, \varrho)\, dt,$$

$$(5)\quad \begin{cases} J''_N(F) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\sum_{(\mathfrak{R})} \int_{(c_N)} \mathfrak{K}(F, x, \varrho)\, d\varrho; \\[2ex] \mathfrak{K}(F, x, \varrho) = \displaystyle\int_a^b F(t) \sum_{m=1}^{n} (-1)^{n-m} p_{m0}(t) K_{n-m}(x, t, \varrho)\, dt. \end{cases}$$

Here (c_N) denotes the part of (C_N) intercepted by $(\mathfrak{R})$, and the summation is extended over all the sectors $(\mathfrak{R})$. Using (48), (49) of **37** and Lemma 3, we see at once that $\mathfrak{K}(x, t, \varrho)$ reduces to a sum of terms of the form

$$\varepsilon(x, \varrho)\, e^{\varrho \pi_i X_i} \quad (i = 1, 2, \ldots, \tau); \qquad \varepsilon(x, \varrho)\, e^{-\varrho \pi_i (X_{0i} - X_i)} \quad (i = \tau + 1, \ldots, n),$$

where $\varepsilon(x, \varrho) \to 0$ uniformly on (c_N). A suitable application of Lemma 1 proves that the integral $J''_N(F)$ tends to zero uniformly on the interior of (a, b).

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[32]) It follows immediately from here that the difference (55) **39**, $\to 0$ as $N \to \infty$, uniformly on the interior of (a, b), for any integrable function $F(x)$ and for any pair of problems (L) satisfying the conditions of **30**, provided the functions $p_{i0}(x)$ are the same for both problems.

Now, using the formulas of **36** and remembering that $\varphi_k(x)$ satisfies the characteristic equation, we easily obtain

$$J'_N(F) = \frac{1}{2\pi\sqrt{-1}} \sum_{(\Re)} \Big\{ -\int_a^x dt \cdot F(t) \sum_{k=1}^{\tau} \omega_k(t) \frac{\eta_k(x)}{\eta_k(t)} \int_{(c_N)} e^{\varrho \pi_k (X_k - \xi_k)} [\varphi_k(t)]\, d\varrho$$
$$+ \int_x^b dt \cdot F(t) \sum_{k=\tau+1}^{n} \omega_k(t) \frac{\eta_k(x)}{\eta_k(t)} \int_{(c_N)} e^{\varrho \pi_k (X_k - \xi_k)} [\varphi_k(t)]\, d\varrho \Big\}.$$

A simple application of Lemma 2 shows that the omission of the brackets introduces an error which $\rightarrow 0$, uniformly on (a, b), so that setting

$$(6) \quad \begin{cases} J_N^{(0)}(F) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\sum_{(\Re)} \Big\{ -\int_a^x dt \cdot F(t) \sum_{k=1}^{\tau} \omega_k(t) \frac{\eta_k(x)}{\eta_k(t)} \int_{(c_N)} e^{\varrho \pi_k (X_k - \xi_k)} \varphi_k(t)\, d\varrho \\ \qquad + \displaystyle\int_x^b dt \cdot F(t) \sum_{k=\tau+1}^{n} \omega_k(t) \frac{\eta_k(x)}{\eta_k(t)} \int_{(c_N)} e^{\varrho \pi_k (X_k - \xi_k)} \varphi_k(t)\, d\varrho \Big\}, \end{cases}$$

the difference

$$J'_N(F) - J_N^{(0)}(F) \rightarrow 0, \quad \text{uniformly on } (a, b).$$

In (6) the integration with respect to ϱ can be performed immediately, and a simple consideration shows that

$$(7) \qquad J_N^{(0)}(F) = \frac{1}{\pi} \int_a^b F(t) \sum_{k=1}^{n} \omega_k(t) \frac{\eta_k(x)}{\eta_k(t)} \, \frac{\sin r_N |\pi_k| (X_k - \xi_k)}{X_k - \xi_k} \, \varphi_k(t)\, dt,$$

where r_N denotes the distance from the origin to either of two points of intersection of (C_N) with the imaginary ϱ-axis[33]). Now, using well known properties of Fourier's coefficients of an integrable function, we can replace $\frac{\eta_k(x)}{\eta_k(t)}$ by 1, the error thus introduced tending to zero uniformly on (a, b), so that finally

$$J_N(F) - \mathsf{T}_N(F) \rightarrow 0 \qquad (N_k = r_N |\pi_k|)$$

uniformly on the interior of (a, b).

41. The integral $\mathsf{T}_N(F)$ replaces the classical integral

$$(8) \qquad S_N(F) = \frac{1}{\pi} \int_a^b F(t) \frac{\sin N(x-t)}{x-t}\, dt$$

of the theory of Fourier's series. Taking into account the relation

$$\sum_{k=1}^{n} \omega_k(t) = 1,$$

[33]) It is tacitly assumed here that these points are equidistant from the origin. Obvious modifications in the arguments of the text suffice to meet the general case.

it is easy to show that if the integral $S_N(F)$ tends to a limit as $N \to \infty$, the integral $T_N(F)$ tends to the same limit[1]. Analogous conclusions can be drawn concerning the summability of the integral $T_N(F)$. On the other hand it is possible to show that, ir general, the difference

$$T_N(F) - S_N(F)$$

does not tend to zero for *every* integrable $F(x)$. This means, that the integral $T_N(F)$ has its own theory of convergence and of summability, which is not essentially different, however, from that of the ordinary Fourier's series. The various details of this theory will be ommitted here. They have been previously discussed by the Author[1].

42. With this we leave the case of the interior points of (a, b), and proceed to the discussion of the integral $J_N(F)$ at the end points $x = a$, $x = b$. We suppose that *the function $F(x)$ is of bounded variation in the neigbourhood of $x = a$, $x = b$*, so that there exists a positive number δ, arbitrarily small but fixed, such that $F(x)$ is of bounded variation on both intervals $(a, a+\delta)$, $(b-\delta, b)$.

The results of 40 concerning the integral $J'_N(F)$ hold true at the points $x = a$, $x = b$ also, and since

$$S_N(F)/_{x=a} \to \frac{1}{2} F(a+0); \quad S_N(F)/_{x=b} \to \frac{1}{2} F(b-0),$$

we have

$$(9) \qquad J'_N(F)/_{x=a} \to \frac{1}{2} F(a+0); \quad J'_N(F)/_{x=b} \to \frac{1}{2} F(b-0).$$

It remains only to discuss the integral $J''_N(F)/_{x=a,b}$. We denote by

$$J_N^{\alpha,\beta}(F, x)$$

the integral which we obtain from $J''_N(F)$, if we integrate between the limits α and β, instead of between a and b.

Using the previous notation we have obviously

$$(10) \qquad J_N^{\alpha,\beta}(F, x) = \frac{1}{2\pi\sqrt{-1}} \sum_{(\Re)} \int_{(c_N)} \Re_{\alpha,\beta}(F, x, \varrho)\, d\varrho,$$

where

$$(11) \quad \Re_{\alpha,\beta}(F, x, \varrho) = \int_\alpha^\beta F(t) \sum_{m=1}^{n} (-1)^{n-m} p_{m0}(t)\, K_{n-m}(x, t, \varrho)\, dt.$$

43. The integral (10) enjoys following properties:

1°. *For every integrable function $F(x)$ and for every pair of values of α, β, satisfying the condition*

$$a < \alpha \leqq \beta < b,$$

[1]) pp. 234—236.

[1]) Chapter V.

the integral $J_N^{\alpha,\beta}(F,x)$ *tends to zero as* $N\to\infty$, *uniformly with respect to* x *in* (a,b).

2°. *The integral* $J_N^{\alpha,\beta}(1,x)$ *is uniformly bounded for all values of* α,β,x *in* (a,b).

3°. *If* α,β *have any fixed values in the interior of* (a,b), *the expressions*

$$(12)\qquad J_N^{a,\alpha}(1,a);\quad J_N^{a,\alpha}(1,b);\quad J_N^{\beta,b}(1,a);\quad J_N^{\beta,b}(1,b)$$

tend to definite limits which are independent of α,β.

The formulas (48), (49) of **37** and Lemma 3 show that (10) reduces to a sum of terms of the form

$$\int\limits_{(c_N)}\varepsilon(x,\varrho)\,e^{c\varrho(b-\beta)}\,d\varrho;\qquad \int\limits_{(c_N)}\varepsilon(x,\varrho)\,e^{c\varrho(a-\alpha)}\,d\varrho,$$

$$\Re e(c\varrho)\leqq 0;\qquad \varepsilon(x,\varrho)\to 0 \text{ uniformly},$$

whereupon the proof of the statement 1° is given by Lemma 1.

The proof of the statement 2° follows if we observe that the integral

$$J_N^{\alpha,\beta}(1,x)=\int\limits_{(C_N)}E\frac{d\varrho}{\varrho}$$

is bounded.

In order to prove the statement 3°, let us take for instance

$$J_N^{a,\alpha}(1,a)=\sum_{(\Re)}\frac{1}{2\pi\sqrt{-1}}\int\limits_{(c_N)}d\varrho\int\limits_0^{\alpha}\sum_{m=1}^{n}(-1)^{n-m}\,p_{m0}(t)\,K_{n-m}(a,t,\varrho)\,dt.$$

Using the expression (48) **37** for $K_m(x,t,\varrho)$ and integrating by parts, we obtain under the sign $\int d\varrho$ an expression of the form

$$\frac{1}{\varrho}\{\mathsf{A}_{\Re}+\varepsilon(\varrho)\},$$

where $\mathsf{A}_{\Re}$ is a constant which depends on the sector $(\Re)$, and $\varepsilon(\varrho)$ tends to zero uniformly on (c_N).

Hence, denoting by $\theta_{\Re}$ the angle of the sector $(\Re)$, we have:

$$J_N^{a,\alpha}(1,a)=\sum_{(\Re)}\frac{1}{2\pi\sqrt{-1}}\int\limits_{(c_N)}\frac{d\varrho}{\varrho}\{\mathsf{A}_{\Re}+\varepsilon(\varrho)\}\to\sum_{(\Re)}\mathsf{A}_{\Re}\cdot\theta_{\Re}.$$

The constants $\mathsf{A}_{\Re},\theta_{\Re}$ depend on the sector $(\Re)$ and therefore they depend on the arbitrary choice of the rays d_i' (**19**). Nevertheless the final result $\sum_{(\Re)}\mathsf{A}_{\Re}\theta_{\Re}$ does not involve any arbitrariness, because the order of the numbers $w_1,w_2,\ldots,w_n$ can change only when ϱ crosses one of the rays

$$(13)\qquad \Re e(\varrho w_i)=0;\qquad \Re e(\varrho w_i)=\Re e(\varrho w_k),$$

so that the constants $A_{\mathfrak{R}}$ remain the same for all the sectors $(\mathfrak{R})$ which lie between any two consecutive rays of the set (13).

44. Using the results of **43** and a known theorem of Lebesgue[34]) we can be sure, that for every integrable function $F(x)$, which is of bounded variation in the neighbourhood of $x = a$, $x = b$,

$$(14) \qquad \begin{cases} J_N(F)/_{x=a} \to A_a F(a+0) + B_a F(b-0), \\ J_N(F)/_{x=b} \to A_b F(a+0) + B_b F(b-0) \end{cases}$$

where A_a, B_a, A_b, B_b are perfectly determined constants, which do not depend on the function $F(x)$. Moreover, it is easy to show that *if. $\Phi(x)$ is a continous function of bounded variazion on (a, b) and*

$$(15) \qquad \Phi(a) = 0, \quad \Phi(b) = 0,$$

the integral $J_N(\Phi)$ converges to $\Phi(x)$ uniformly on (a, b).

Taking into account that

$$J_N(\Phi) = J'_N(\Phi) + J''_N(\Phi)$$

and that the difference

$$J'_N(\Phi) - \mathrm{T}_N(\Phi)$$

tends uniformly to zero and the integral $\mathrm{T}_N(\Phi)$ tends uniformly to $\Phi(x)$, it remains only to prove, that for our function $\Phi(x)$, the integral

$$J''_N(\Phi) = J_N^{a,b}(\Phi; x)$$

tends uniformly to zero. This fact follows immediately from the formula

$$J_N^{a,b}(\Phi, x) = J_N^{a,a+\delta}(\Phi, x) + J_N^{a+\delta, b-\delta}(\Phi, x) + J_N^{b-\delta, b}(\Phi, x),$$

where the middle term tends uniformly to zero for fixed δ, and the extreme terms can be made as small as we please by choosing δ sufficiently small (applying the second law of the mean).

45. Now everything is prepared for the proof of the

Theorem 14. *Let $F(x)$ be an arbitrary function integrable on (a, b), and denote by*

$$(16) \qquad \begin{cases} \sigma_N(F, x) = \sum_{\nu=1}^{N} \int_a^b F(t) \sum_{m=1}^{n} (-1)^{n-m} p_{m0}(t) \operatorname*{Res}_{\varrho=\varrho_\nu} \varrho^{m-1} \dfrac{\partial^{n-m} G(x, t, \varrho)}{\partial t^{n-m}} \, dt \\ \qquad + \sum_{\nu=1}^{N} \{F(a+0)\, \Xi_\nu^{(1)}(x) + F(b-0)\, \Xi_\nu^{(2)}(x)\}, \end{cases}$$

where the factors $F(a+0)$, $F(b-0)$ may be replaced by zero in case these two limits do not exist. Then

[34]) Sur les intégrales singulières, Ann. Fac. Toulouse 1 (ser. 3), 1909, pp. 65, 69–70.

1°. *The difference*

$$\sigma_N(F, x) - \mathsf{T}_N(F) \to 0 \quad as \quad N \to \infty,$$

uniformly on the interior of (a, b).

2°. *If* $F(x)$ *is of bounded variation in the neighbourhood of* $x = a, b$, *we have*

$$\sigma_N(F, a) \to F(a+0), \qquad \sigma_N(F, b) \to F(b-0), \tag{17}$$

with no further restrictions for $F(x)$ *in the case* 1° *of* **32.**

In the case 2° *of* **32** *the equalities* (17) *hold true if* $F(x)$ *satisfies one of the corresponding boundary conditions*

$$\begin{aligned} F(b-0) &= a_0 F(a+0), \\ F(a-0) &= b_0 F(b-0). \end{aligned} \tag{18}$$

In the case 3° *of* **32**, (17) *is true if*

$$F(a+0) = 0, \qquad F(b-0) = 0.$$

3°. *The series*

$$\sigma(F, x) = \lim_{N \to \infty} \sigma_N(F, x)$$

converges to $F(x)$ *uniformly on* (a, b) *for every continuous function* $F(x)$ *of bounded variation, satisfying the conditions of the preceding statement* 2°.

The statement 1° is obvious. Denote now by $F_1(x)$ the auxiliary function, which was used in **33—35**. The results of **33—35, 38** show immediately that

$$\begin{aligned} F_1(x) &= \sigma(F_1, x) \\ &= \lim_{N \to \infty} J_N(F_1) + F_1(a) \sum_{\nu=1}^{\infty} \Xi_\nu^{(1)}(x) + F_1(b) \sum_{\nu=1}^{\infty} \Xi_\nu^{(2)}(x). \end{aligned} \tag{19}$$

Putting here $x = a$ or $x = b$, we have in virtue of (14):

$$\left\{ \begin{aligned} &\sum_{\nu=1}^{\infty} \Xi_\nu^{(1)}(a) + A_a = 1; \qquad \sum_{\nu=1}^{\infty} \Xi_\nu^{(2)}(a) + B_a = 0; \\ &\sum_{\nu=1}^{\infty} \Xi_\nu^{(1)}(b) + A_b = 0; \qquad \sum_{\nu=1}^{\infty} \Xi_\nu^{(2)}(b) + B_b = 1. \end{aligned} \right. \tag{20}$$

These relations prove the assertion 2° of our theorem. The proof of the assertion 3° follows if we set

$$\Phi(x) = F(x) - F_1(x)$$

and use the results 44 and the fact that the sum $\sigma_N(\Phi, x)$ reduces to $J_N(\Phi)$, because of $\Phi(a) = \Phi(b) = 0$, and that the series $\sigma(\Phi, x)$, $(\sigma F_1, x)$ are uniformly convergent.

46. It is interesting to observe that the statement 1° of Theorem 14 holds true if we replace $\sigma_N(F, x)$ by the simpler expression

$$(21)\quad \begin{cases} \sum_{\nu=1}^{N}\int_a^b F(t)\,p_{n0}(t)\operatorname*{Res}_{\varrho=\varrho_\nu}\varrho^{n-1}G(x,t,\varrho) \\ = \frac{1}{2\pi\sqrt{-1}}\int_{(C_N)}\varrho^{n-1}d\varrho\int_a^b F(t)\,p_{n0}(t)\,G(x,t,\varrho)\,dt = I_N(F,x) \end{cases}$$ [35]

and at the same time replace $\omega_\varkappa(x)$ in (1) by

$$\chi_k(x) = -\frac{p_{n0}(x)}{\varphi_k(x)\,\Phi'\{\varphi_k(x)\}} \qquad (k = 1, 2, \ldots, n).$$

The statements 2° and 3° of Theorem 14, however, do not hold true after this replacement in the general case, unless $F(a+0) = 0$, $F(b-0) = 0$. If these conditions are not satisfied, the integral $I_N(F, x)$ will not converge uniformly on (a, b), even for $F(x)$ continuous and of bounded variation.

47. Some special cases of the expansion of Theorem 14 deserve to be mentioned separately:

1°. The operators of the problem (L) are of the form:

$$L(y) \equiv L_0(y) + \varrho^n q(x)\,y; \quad q(x) \geqq q_0 > 0;$$
$$L_i(y) \equiv L_i^{(0)}(y) + \varrho^n L_i^{(1)}(y) \qquad (i = 1, 2, \ldots, n)$$

and all the conditions of **16** are satisfied. We suppose moreover that the matrix of the forms $L_i^{(0)}(y)$ has at least one determinant of the n^{th} order which is $\neq 0$, and which is free from the elements corresponding to $y(a)$, $y(b)$. For the sake of brevity we shall consider a special case only, namely that in which

$$L_1^{(1)}(y) = y(a), \quad L_2^{(1)}(y) \equiv y(b); \quad L_i^{(1)}(y) \equiv 0 \quad (i = 3, 4, \ldots, n).$$

The quadratic form (**16**)

$$T(y) \equiv T(y, y) = \sum_{i=1}^{n} L_i^{(1)}(y)\,M_i(y) \equiv y(a)\,M_1(y) + y(b)\,M_2(y)$$

being semidefinite, we must have:

$$M_1(y) \equiv k_{11}\,y(a) + k_{12}\,y(b); \quad M_2(y) = k_{21}\,y(a) + k_{22}\,y(b); \quad k_{12} = k_{21},$$

so that in this case

$$\Xi_\nu^{(1)}(x) = -\operatorname*{Res}_{\varrho=\varrho_\nu}\varrho^{n-1}\{k_{11}G(x,a,\varrho) + k_{21}G(x,b,\varrho)\},$$
$$\Xi_\nu^{(2)}(x) = -\operatorname*{Res}_{\varrho=\varrho_\nu}\varrho^{n-1}\{k_{12}G(x,a,\varrho) + k_{22}G(x,b,\varrho)\},$$

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[35]) This integral was used by G. D. Birkhoff.

and finally

$$(22)\quad \begin{cases} \sigma(F,x)=\sum_{\nu=1}^{\infty}\left\{\int_a^b q(t)\,F(t)\operatorname*{Res}_{\varrho=\varrho_\nu}\varrho^{n-1}G(x,t,\varrho)\,dt-\operatorname*{Res}_{\varrho=\varrho_\nu}\varrho^{n-1}T(F,G)_t\right\} \\ \qquad =\sum_{\nu=1}^{\infty}\varphi_\nu(x)\{\int_a^b q(t)\,F(t)\,\varphi_\nu(t)\,dt-T(F,\varphi_\nu)\}\,^{36)}, \end{cases}$$

where $\varphi_1(x), \ldots, \varphi_\nu(x), \ldots$ denote the set of fundamental functions of the problem (L), which are supposed to be orthogonalized according to (30), **16**.

The series

$$\sum_{\nu=1}^{\infty}\varphi_\nu(x)\,T(1,\varphi_\nu)$$

enjoys the remarkable property that its sum is zero at every interior point of (a, b), and is not zero at the end-points a, b.

This fact shows that the theorems of Cantor and Du Bois Reymond are not true for our general expansions in series of the fundamental functions.

2°. When all the operators $L_i^{(1)}(y)$ reduce to zero, the form $T(u, v)$ vanishes and we get the case, which was treated by Birkhoff[8]) and by the Author[31]):

$$\sigma(F,x)=\sum_{\nu=1}^{\infty}\varphi_\nu(x)\frac{\int_a^b q(t)\,F(t)\,\varphi_\nu(t)\,dt}{\int_a^b q(t)\,\varphi_\nu(t)^2\,dt}$$

3°. A curious example of an expansion containing fundamental and principal functions simultaneously is furnished by the problem:

$L(y)\equiv y''+\varrho^2 y;\quad L_1(y)\equiv y'(1)-y'(0);\quad L_2(y)\equiv y(1)+2y(0).$

Here we obtain:

$$(23)\quad \begin{cases} \sigma(F,x)=2(3x-1)\left\{\int_0^1 F(t)\,dt+2\sum_{\nu=1}^{\infty}\cos 2\nu\pi x\int_0^1 F(t)\cos 2\nu\pi t\,dt\right\} \\ \qquad +4\sum_{\nu=1}^{\infty}\sin 2\nu\pi x\int_0^1(2-3t)\,F(t)\sin 2\nu\pi t\,dt, \end{cases}$$

the fundamental functions being

$$3x-1,\ \sin 2\nu\pi x,$$

and the principal functions

$$(3x-1)\cos 2\nu\pi x.$$

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36) In the expression of the bilinear form $T(F, v)$ we must replace $F(a)$, $F(b)$ by $F(a+0)$, $F(b-0)$ respectively.

The series (23) represents the arbitrary function $F(x)$ on the interior of (0, 1) under the same conditions as the classical Fourier's series, the difference between the sums of the N first terms of both expansions tending to zero for any integrable function, uniformly on the interior of (0, 1). At the end-points the sum of (23) is equal to $F(+0)$, $F(1-0)$, if

$$2F(+0)+F(1-0)=0.$$

We have not touched here the expansion problem in the case where the operators $L_i(y)$ contain integrals. This problem, however, has been discussed by the Author[1]).

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[1]) pp. 231–235.

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