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A REMARK ON H^1 MAPPINGS

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With $\mathbb{B} = \{x \in \mathbb{R}^3 : |x| < 1\}$, we here construct, for each positive integer N, a smooth function $g : \partial\mathbb{B} \to \mathbf{S}^2$ of degree zero so that there must be at least N singular points for any map that minimizes the energy $\mathcal{E}(u) = \int_{\mathbb{B}} |\nabla u|^2 dx$ in the family

$\mathfrak{U}(g) = \{u \in H^1(\mathbb{B}, \mathbf{S}^2) : u \,|\, \partial\mathbb{B} = g\}$.

The infimum of $\mathcal{E}$ over $\mathfrak{U}(g)$ is strictly smaller than the infimum of $\mathcal{E}$ over the continuous functions in $\mathfrak{U}(g)$. There are some generalizations to higher dimensions.

<u>INTRODUCTION.</u> Any smooth map $g : \partial\mathbb{B} \to \mathbf{S}^2$ admits a finite energy extension $u : \overline{\mathbb{B}} \to \mathbf{S}^2$; for example, $u(x) = g(x/|x|)$. Thus the existence of an energy minimizing function v in $\mathfrak{U}(g)$ follows from elementary properties of the space $H^1(\mathbb{B})$. By [6, Th.2] and [7, Th. 2.7], such a v defines a real analytic function on $\overline{\mathbb{B}} \sim Z$ for some finite subset Z of $\mathbb{B}$. If g has nonzero degree (for example, g = identity) , then v may not, by elementary topology, be continuous everywhere on $\mathbb{B}$, and the singular set Z must be nonempty. If g has degree zero, then g does admit some smooth extension to $\overline{\mathbb{B}}$. Nevertheless,

<u>THEOREM A.</u> <u>For any positive integer</u> N , <u>there exists a smooth function</u> $g : \partial\mathbb{B} \to \mathbf{S}^2$ <u>that has degree zero so that any map</u> $v \in \mathfrak{U}(g)$ <u>that minimizes energy in</u> $\mathfrak{U}(g)$ <u>must have at least</u> N <u>singular points</u>.

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THEOREM B. For any positive integer N, there exists a smooth function $g : \mathbf{S}^2 \to \mathbf{S}^2$ that has degree zero for which there is the gap

$$\inf_{u \in \mathcal{U}(g)} \mathcal{E}(u) \leq 1/N < 1 \leq \inf_{u \in \mathcal{U}(g) \cap C^0(\overline{\mathbf{B}})} \mathcal{E}(u) .$$

Our work on these problems followed interesting discussions with J. Ericksen and H. Brezis concerning some experimentally observed liquid crystals. By using the regularity theorems [3, 2.6, 5.6] and changing constants in the proofs, suitable analogues of Theorems A and B may be obtained for the general liquid crystal functional $\tilde{\mathcal{W}}$ considered in [3, 1.2].

PROOF OF THEOREM A. We will work with certain "lens-shaped" domains obtained by intersecting the unit ball with larger balls. For any nonnegative t, let

$$U_t = \mathbf{B} \cap V_t \quad \text{where} \quad V_t = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 + (z+t)^2 < 1 + t^2\} .$$

Since $\partial\mathbf{B} \cap \partial V_t = \mathbf{S}^1 \times \{0\}$, ∂U_t is the union of the lower half-sphere

$$\mathbf{S}^2_- = \{(x,y,z) \in \mathbf{S}^2 : z \leq 0\}$$

and the spherical cap

$$\partial V_t^+ = \{(x,y,z) \in \partial V_t : z \geq 0\} .$$

Note that $U_0 = \mathbf{B}$ and that $\lim_{t \to \infty} U_t$ is the lower unit half-ball. One may easily obtain a constant L along with bilipschitz maps

$$f_t : \overline{U}_t \to \overline{\mathbf{B}}$$

so that $\mathrm{Lip}(f_t) \leq L$, $\mathrm{Lip}(f_t^{-1}) \leq L$, and

$$f_t(a) = a \quad \text{for} \quad a \in \mathbf{S}^2 ,$$

for all $t \geq 0$. We will also use the Lipschitz map $h : \mathbf{R}^3 \to \mathbf{R}^3$ which is described in spherical coordinates by

$$h(\rho,\varphi,\theta) = (\rho,2\varphi,\theta) \quad \text{for } 0 \le \varphi \le \pi/2 ,$$
$$h(\rho,\varphi,\theta) = (\rho,\pi,\theta) \quad \text{for } \pi/2 \le \varphi \le \pi .$$

The expression

$$u_t = (h \circ f_t)/|(h \circ f_t)|$$

defines a function in $H^1(\overline{U}_t, \mathbf{S}^2)$. In fact, noting that

$$\int_{\mathbb{B}} |\nabla\omega|^2 dx = 8\pi \quad \text{where } \omega(x) = x/|x| ,$$

and that

$$|\xi| \le \langle Dh(x,y,z),\xi\rangle \le 2|\xi| \quad \text{for } \xi \in \mathbb{R}^3 \text{ and } z > 0 ,$$

we readily obtain the absolute bound

$$\int_{U_t} |\nabla u_t|^2 dx \le 32\pi L^5 .$$

The function u_t is continuous away from the point $f^{-1}\{0\}$. Concerning its behavior on ∂U_t, note that, on the bottom,

$$u_t | \mathbf{S}^2 \equiv (0,0,-1) ,$$

while, on the top, $u_t | \partial V_t^+$ is an orientation-preserving, degree one map onto $\mathbf{S}^2$.

We now construct a small energy function $\tilde{u} : \mathbb{B} \to \mathbf{S}^2$ by placing along $\partial\mathbb{B}$ copies of u_t (for suitable t) that are transformed by translation, rotation, and homothety. The trace of u on $\partial\mathbb{B}$ will then provide the function g that satisfies the theorem.

Fix a positive number $r < 1/(64\pi N^2 L^5)$, and, for $i = 1, \cdots, N$, let

$$\xi_i = (0, (1-\lambda_i^2)^{1/2}, \lambda_i) \quad \text{where } \lambda_i = i/2N .$$

There exist (easily computed) $s > 0$, $t > 0$, $a_i \in \mathbb{B}$, and $\Phi_i \in \mathbf{SO}(3)$ so that

$$\mathbb{S}^2 \cap \overline{\mathbb{B}}_r(\xi_i) = \{s\Phi_i(x) + a_i : x \in \partial V_t^+\}.$$

Note that the scale factor s satisfies $r < s < 2^{1/2} r$. Let

$$U_i = \{s\Phi_i(x) + a_i : x \in U_t\} \text{ and}$$

$$u_i(y) = u_t \circ \Phi^{-1}((y - a_i)/s) \quad \text{for } y \in U_i .$$

Then

$$\int_{U_i} |\nabla u_i|^2 dx \le 16\pi L^5 s < 32\pi L^5 r \le 1/2N^2 .$$

Let

$$\begin{aligned} \tilde{u}(x) &= u_i(x) && \text{for } x \in U_i , \\ \tilde{u}(x) &= u_i(-x) && \text{for } x \in -U_i \text{ (i.e. } -x \in U_i), \\ \tilde{u}(x) &= (0,0,-1) && \text{for } x \in \mathbb{B} \sim \cup_{i=1}^N [U_i \cup (-U_i)] . \end{aligned}$$

Then $\tilde{u} \in H^1(\mathbb{B}, \mathbb{S}^2)$ with

$$\mathcal{E}(\tilde{u}) = \int_{\mathbb{B}} |\nabla \tilde{u}|^2 dx < 2N \cdot (1/2N^2) = 1/N .$$

Also, $\tilde{g} = \tilde{u} \mid \partial\mathbb{B}$ is a Lipschitz map that has degree zero because

$$\tilde{g} \mid \partial\mathbb{B} \sim \cup_{i=1}^N [\mathbb{B}_r(\xi_i) \cup \mathbb{B}_r(-\xi_i)] = (0,0,-1)$$

and, for each i, $\tilde{g} \mid \partial\mathbb{B} \cap \overline{\mathbb{B}}_r(\xi_i)$ is an orientation-preserving (degree 1) map onto $\mathbb{S}^2$ while $\tilde{g} \mid \partial\mathbb{B} \cap \overline{\mathbb{B}}_r(-\xi_i)$ is an orientation-reversing (degree -1) map onto $\mathbb{S}^2$.

Next we can choose an approximation g of $\tilde{g}$ so that $g \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, g has degree zero, and

$$\inf_{u \in \mathcal{U}(g)} \mathcal{E}(u) < \mathcal{E}(\tilde{u}) + [1/N - \mathcal{E}(\tilde{u})] = 1/N .$$

From the inequalities

$$(\lambda_i - r) - (\lambda_j + r) = (1/2N) - 2r \ge 1/4N ,$$

$$\mathcal{E}(v) = \inf_{u \in \mathcal{U}(g)} \mathcal{E}(u) < 1/N ,$$

[6, Th.2], [7, Th.2.7], and Fubini's theorem, we may choose numbers

$$0 < \mu_0 < \lambda_1 - r < \lambda_1 + r < \mu_1 < \lambda_2 - r < \cdots < \lambda_N + r < \mu_N < 1$$

so that, on each slice,

$$S_i = \{(x,y,z) \in \mathbb{B} : z = \mu_i\} ,$$

v is smooth with

$$\int_{S_i} |\nabla v|^2 \, d\mathcal{H}^2 < (1/N)/(1/4N) = 4 .$$

From the area estimate

$$\mathcal{H}^2(v(S_i)) \le \int_{S_i} \|\Lambda_2 Dv\| \, d\mathcal{H}^2 \le \tfrac{1}{2} \int_{S_i} |\nabla v|^2 \, d\mathcal{H}^2 < 2 < 4\pi ,$$

we infer that each image $v(S_i)$ is a proper subset of $\mathbb{S}^2$. Inasmuch as $v|\partial S_i \equiv (0,0,-1)$, $v|S_i$ is thus homotopic (relative to ∂S_i) to the constant vector $(0,0,-1)$. Letting Ω_i denote the slab

$$\Omega_i = \{(x,y,z) \in \mathbb{B} : \mu_{i-1} < z < \mu_i\} ,$$

we deduce that $v|\partial\Omega_i$ has degree one because

$$\partial\Omega_i = S_{i-1} \cup S_i \cup T_i \quad \text{with } T_i = \{(x,y,z) \in \partial\mathbb{B} : \mu_{i-1} \le z \le \mu_i\} ,$$

$$T_i \supset \partial\mathbb{B} \cap \overline{\mathbb{B}}_r(\xi_i) , \quad \text{and} \quad v|T_i \sim (\partial\mathbb{B} \cap \overline{\mathbb{B}}_r(\xi_i)) \equiv (0,0,-1) .$$

Since $\overline{\Omega}_i$ is topologically a closed ball, v must have at least one discontinuity in Ω_i. □

<u>PROOF OF THEOREM B.</u> Suppose $u \in \mathcal{U}(g) \cap C^0(\overline{\mathbb{B}})$ with g and r as above. Assuming for contradiction that

$$\mathcal{E}(u) = \int_{\mathbb{B}} |\nabla u|^2 dx < 1 ,$$

we would find that

$$\int_{\{(x,y,z)\in\mathbb{B} \,:\, \lambda_{i-1} < z < \lambda_i\}} |\nabla u|^2 dx < 1/N$$

for some i. Then we could choose a number $\mu \in [\lambda_{i-1} + r, \lambda_i - r]$ so that

$$\mathcal{H}^2(u(S)) \le ½ \int_S |\nabla u|^2 d\mathcal{H}^2 < 2 , \text{ hence } \mathbf{S}^2 \sim u(S) \ne \emptyset ,$$

where

$$S = \{(x,y,z)\in\mathbb{B} : z = \mu\} .$$

As before, we would then infer that, for

$$\Omega = \{(x,y,z)\in\mathbb{B} : z > \mu\} ,$$

$u|\partial\Omega$ has positive degree and that u would have to have a discontinuity in Ω . This contradiction establishes the desired inequalities

$$\mathcal{E}(u) \ge 1 > 1/N \ge \inf_{\mathcal{U}(g)} \mathcal{E} \qquad \square$$

Next we will sketch the proofs of two results that indicate how Theorems A or B generalize to some higher dimensional problems.

THEOREM C. For $n \in \{3, 4, \cdots\}$ and for any positive integer N, there exists a smooth function $g : \partial\mathbb{B}^n \to \mathbf{S}^{n-1}$ that has degree zero so that any map v in

$$\mathcal{U}(g) = \{u \in H^1(\mathbb{B}^n, \mathbf{S}^{n-1}) : u|\partial\mathbb{B}^n = g\}$$

that minimizes energy in $\mathcal{U}(g)$ must have at least N singular points.

THEOREM D. For $n \in \{3,4,\cdots\}$ and for any positive integer N, there exists a smooth convex domain Ω in $\mathbb{R}^n$ and a smooth map $w : \overline{\Omega} \to \mathbf{S}^2$ so that, for $g = w \mid \partial\Omega$, the singular set of any energy minimizing map v in

$$\mathfrak{U}(g) = \{u \in H^1(\Omega, \mathbf{S}^2) : u \mid \partial\Omega = g\}$$

must have positive $n-3$ dimensional Hausdorff measure. Moreover,

$$\inf_{u \in \mathfrak{U}(g)} \mathcal{E}(u) \leq 1/N < 1 \leq \inf_{u \in \mathfrak{U}(g) \cap \tilde{\mathcal{C}}^0(\overline{\Omega})} \mathcal{E}(u)$$

where

$$\tilde{\mathcal{C}}^0(\overline{\Omega}) = \{u : u \in \mathcal{C}^0(\overline{\Omega} \sim X) \text{ for some } X \subset \overline{\Omega} \text{ with } \mathcal{H}^{n-3}(X) = 0\}.$$

PROOF OF THEOREM C. Here we can essentially repeat the argument from the proof of Theorem A with one modification. For $n \geq 4$, we can no longer estimate, for any smooth function $u : \mathbb{B}^n \to \mathbf{S}^{n-1}$, the slice volume $\mathcal{H}^{n-1}(u(S_i))$ by the slice energy $\int_{S_i} |\nabla u|^2 \, d\mathcal{H}^{n-1}$. However, for the energy minimizer v, we can assume, for contradiction, that v has no singularity in

$$\{(x,y,z) \in \mathbb{B}^n : \lambda_{i-1} < z < \lambda_{i+1}\},$$

and then use Schoen's interior estimate [5, Th. 2.2] (for regular harmonic maps) along with a boundary regularity estimate (that follows from [5, Th.2.6] or [7, Th.2.7]) to obtain a small volume estimate for $\mathcal{H}^{n-1}(v(S_i))$. With r chosen sufficiently small (depending on the constants of these estimates), the proof then proceeds as before. □

PROOF OF THEOREM D. We use induction on n. The case $n = 3$ follows from Theorems A and B.

We now assume that $n \geq 4$ and, by induction, that Ω', w', and g' satisfy Theorem D with n replaced by $n-1$ and N replaced by $N' = 2N$. Then we may find a function $u' \in \mathfrak{U}(g')$ with

$$\mathcal{E}(u') = \int_{\Omega'} |\nabla u'|^2 \, dx = 1/N' = 1/2N .$$

Choose a positive concave function $\omega \in C^0(\overline{\Omega}') \cap C^\infty(\Omega')$ so that $\omega | \partial\Omega' \equiv 0$ and so that

$$\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \approx \mathbb{R}^n : x' \in \Omega' \text{ and } |x_n| < \omega(x')\},$$

is a smooth domain in $\mathbb{R}^n$. Then

$$U = \{(x', x_n) : x' \in \Omega' \text{ and } 0 < x_n < \omega(x')\}$$

is a Lipschitz domain, and we may fix a function $\zeta \in H^1(U, \mathbf{S}^2)$ such that (in the sense of traces)

$$\zeta(x', 0) = u'(x') \text{ and } \zeta(x', \omega(x')) = w'(x') \quad \text{for } x' \in \Omega' .$$

With $\mathcal{E} = \mathcal{E}(\zeta)$, we now let

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x' \in \Omega' \text{ and } |x_n| < \omega(x') + 2NE\} ,$$

$$w(x', x_n) = w'(x') \text{ for } (x', x_n) \in \overline{\Omega} .$$

Clearly Ω is a smooth domain and w is a smooth function on $\overline{\Omega}$. Moreover, the function $\tilde{u}$, defined by

$$\begin{aligned}
\tilde{u}(x', x_n) &= \zeta(x', x_n - 2NE) && \text{for } (x', x_n) \in \Omega \text{ and } x_n > 2NE , \\
\tilde{u}(x', x_n) &= u'(x') && \text{for } (x', x_n) \in \Omega \text{ and } |x_n| \le 2NE , \\
\tilde{u}(x', x_n) &= \zeta(x', -x_n - 2NE) && \text{for } (x', x_n) \in \Omega \text{ and } x_n < -2NE ,
\end{aligned}$$

belongs to $\mathcal{U}(g)$ where $g = w | \partial\Omega$. Thus we have the energy upper bound

$$\begin{aligned}
\inf_{\mathcal{U}(g)} \mathcal{E} \le \mathcal{E}(\tilde{u}) &\le 2E + 4NE \cdot \mathcal{E}(u') \\
&= 2E + 4NE(1/N') = 4E .
\end{aligned}$$

Suppose, on the other hand, that $u \in \mathcal{U}(g) \cap C^0(\overline{\Omega} \sim X)$ for some $X \subset \overline{\Omega}$ with $\mathcal{H}^{n-3}(X) = 0$. For almost all $t \in [-2NE, 2NE]$,

$u(x',t) = u'(x') = g'(x')$ for $x' \in \partial\Omega'$ and

$\int_{S_t} |\nabla u|^2 \, d\mathcal{H}^{n-1} < \infty$ where $S_t = \{(x',x_n) \in \overline{\Omega} : x_n = t\}$.

Since $u \,|\, (S_t \sim X) \in \mathcal{C}^0(S_t \sim X)$ and since, by [2, 2.10.25],

$\mathcal{H}^{n-4}(X \cap S_t) = 0$ for almost all t,

we deduce by induction that

$\int_{S_t} |\nabla u|^2 \, d\mathcal{H}^{n-1} \geq 1$ for almost all $t \in [-2NE, 2NE]$,

and obtain from Fubini's theorem the energy lower bound

$\mathcal{E}(u) \geq 4NE \cdot 1$.

Finally replacing Ω , w , and u by

$\{rx : x \in \Omega\}$, $w((\cdot)/r)$, and $u((\cdot)/r)$ with $r = (1/4NE)^{1/(n-2)}$,

and taking the infimum over all such u , we see that the energy upper and lower bounds scale to give the desired inequalities

$\inf_{\mathcal{U}(g)} \mathcal{E} \leq 1/N < 1 \leq \inf_{u \in \mathcal{U}(g) \cap \tilde{\mathcal{C}}^0(\overline{\Omega})} \mathcal{E}(u)$.

In particular, $v \notin \tilde{\mathcal{C}}^0(\overline{\Omega})$ because $\mathcal{E}(v) = \inf_{\mathcal{U}(g)} \mathcal{E}$. □

QUESTIONS

(1) Are there a smooth domain Ω in $\mathbb{R}^n$ and a smooth map $g : \partial\Omega \to \mathbf{S}^m$ with $n \geq m+1 \geq 4$ for which there is the gap

$\inf_{\mathcal{U}(g)} \mathcal{E} < \inf_{u \in \mathcal{U}(g) \cap \mathcal{C}^0(\overline{\Omega})} \mathcal{E}$?

(2) If there is a gap (as in Theorem B or Theorem D), then what can be said about a weak limit w of an energy minimizing sequence in $\mathcal{U}(g) \cap \mathcal{C}^0(\overline{\Omega})$? Which of the inequalities

$$\inf_{\mathcal{U}(g)}\mathcal{E} \leq \mathcal{E}(w) \leq \inf_{u\in\mathcal{U}(g)\cap C^0(\bar{\Omega})}\mathcal{E}$$

is strict? Can they both be strict? The search for non-minimizing critical points (as in [1]) seems quite challenging. See also [4] and [5] for other interesting related questions on harmonic maps.

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