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# Stress Distribution in a Rotating Elastic-Plastic Tube

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With 2 Figures

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**Summary**

Subject of the investigation is the distribution of displacement and stresses in a tube with fixed ends of elastic-plastic material under the assumption of Tresca's yield condition, its associated flow rule and linear strain-hardening.

## 1. Introduction

The interest in the stress distribution in a rotating cylinder goes back to J. C. Maxwell [1]. For elastic behavior, stresses and displacement can be found in A. E. H. Love's "Treatise on the Mathematical Theory of Elasticity" [1]. Rotating perfectly plastic cylinders were treated by A. Nádai [2] and by O. Hoffman and G. Sachs [3] under the simplifying assumption of incompressibility. Thereafter, E. A. Davis and F. M Conelly studied rotating cylinders and tubes of a strain-hardening material [4]. Interested primarily in the deformation of a rotating cylinder are P. G. Hodge, Jr., and M. Balaban [5] but their solution, which includes finite strain, lacks continuity of displacement at the elastic-plastic interface [6]. The latter two studies are based on a finite stress strain relation and not on an incremental constitutive relation, which is preferred nowadays.

In the following, the elastic-plastic stress distribution in a rotating tube with fixed ends is calculated under the assumption of Tresca's yield condition, its associated flow rule and linear strain-hardening. The more complicate plane strain problem of the rotating tube with free ends will be treated in a forthcoming paper.

## 2. The Elastic Tube with Fixed Ends

The equation of motion and the geometric relations are valid for any material. In cylindrical coordinates they read

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = -\varrho\omega^2 r, \tag{2.1}$$

$$\varepsilon_r = \frac{du}{dr}, \qquad \varepsilon_\theta = \frac{u}{r}, \tag{2.2}$$

where $\sigma_r$, $\sigma_\theta$, $\varepsilon_r$, $\varepsilon_\theta$ mean radial and circumferential stress and strain, respectively, $u$ indicates radial displacement, $\varrho$ density and $\omega$ angular velocity.

In the elastic case stresses and strains are related by Hooke's law, and the problem can be reduced to a differential equation in the displacement with the solution

$$u = \frac{A}{r} + Br - \frac{1 - 2\nu}{16G(1 - \nu)} \varrho\omega^2 r^3. \tag{2.3}$$

Therefrom the stresses

$$\sigma_r = -2G \frac{A}{r^2} + \frac{2G}{1 - 2\nu} B - \frac{3 - 2\nu}{8(1 - \nu)} \varrho\omega^2 r^2, \tag{2.4}$$

$$\sigma_\theta = 2G \frac{A}{r^2} + \frac{2G}{1 - 2\nu} B - \frac{1 + 2\nu}{8(1 - \nu)} \varrho\omega^2 r^2 \tag{2.5}$$

with the shear modulus $G$ and Poisson's ratio $\nu$ are arrived at. In the derivation, use has been made of the condition of fixed ends, $\varepsilon_z = 0$, which leads to

$$\sigma_z = \nu(\sigma_r + \sigma_\theta). \tag{2.6}$$

The preceding results can be adapted from the solution of the corresponding plane stress problem of the rotating disk by modification of the material constants [7].

The unknown quantities $A$ and $B$ are determined with the help of the boundary conditions of vanishing stress

$$\sigma_r(a) = 0, \tag{2.7}$$

$$\sigma_r(b) = 0 \tag{2.8}$$

at the inner and outer surface of radii $a$ and $b$, respectively.

The elastic solution shows that the stresses satisfy the inequality

$$\sigma_\theta > \sigma_z > \sigma_r \geqq 0. \tag{2.9}$$

The maximum circumferential stress occurs at the inner boundary $r = a$ (Fig. 2). There, plastic deformation appears for a certain value of the angular velocity.

## 3. Displacement and Stresses in the Plastic Region

Since $\sigma_\theta$ is the largest and $\sigma_r$ the smallest stress, Tresca's yield condition adopts the form [8]

$$\sigma_\theta - \sigma_r = \sigma_y. \tag{3.1}$$

The yield stress, $\sigma_y$, of an elastic-plastic material with linear strain-hardening grows with the equivalent plastic strain, $\varepsilon_{EQ}^p$, according to

$$\sigma_y = \sigma_0(1 + \eta\varepsilon_{EQ}^p), \tag{3.2}$$

where $\sigma_0$ is the initial yield limit and $\eta$ the hardening parameter [9]. The plastic strain depends on $\sigma_y$ as shown in the following.

Insertion of the yield condition, Eq. (3.1), into the equation of motion, Eq. (2.1), and integration gives

$$\sigma_r = \int \frac{\sigma_y}{r}\,dr - \frac{1}{2}\,\varrho\omega^2 r^2 + C, \tag{3.3}$$

$$\sigma_\theta = \sigma_y + \int \frac{\sigma_y}{r}\,dr - \frac{1}{2}\,\varrho\omega^2 r^2 + C. \tag{3.4}$$

According to the flow rule associated to Tresca's yield condition $d\varepsilon_r{}^p = -d\varepsilon_\theta{}^p$ and $d\varepsilon_z{}^p = 0$. Hence, $\varepsilon_z{}^e = 0$, and Eq. (2.6) holds in the entire tube. The sum of circumferential and radial strain is purely elastic and can be expressed by the above stresses via Hooke's law

$$\frac{du}{dr} + \frac{u}{r} = \frac{1-2\nu}{2G}\left[\sigma_y + 2\int \frac{\sigma_y}{r}\,dr - \varrho\omega^2 r^2 + 2C\right],$$

and integration yields

$$u = \frac{1-2\nu}{2G}\left[r\int \frac{\sigma_y}{r}\,dr - \frac{1}{4}\,\varrho\omega^2 r^3 + Cr + \frac{D}{r}\right]. \tag{3.5}$$

The equivalent plastic strain follows from the consideration of plastic work [8],

$$\sigma_{ij}\,d\varepsilon_{ji}^p = \sigma_y\,d\varepsilon_{EQ}^p.$$

In the case of monotonously increasing angular velocity

$$\varepsilon_{EQ}^p = \varepsilon_\theta{}^p \tag{3.6}$$

holds. $\varepsilon_\theta{}^p$ is the difference of the total strain, $\varepsilon_\theta$, derived from Eq. (3.5) and the elastic strain, $\varepsilon_\theta{}^e$, which is calculated with the help of Hooke's law and the stresses $\sigma_r$ and $\sigma_\theta$, Eqs. (3.3) and (3.4). These operations lead to

$$\varepsilon_\theta{}^p = -\frac{1-\nu}{2G}\,\sigma_y + \frac{1-2\nu}{2G}\left(\frac{1}{4}\,\varrho\omega^2 r^2 + \frac{D}{r^2}\right), \tag{3.7}$$

the equivalent plastic strain. Elimination of $\varepsilon_{EQ}^p$ from Eq.(3.2) gives the dependence of the yield stress on the radius,

$$\sigma_y = \frac{1}{2 + (1-\nu)\,H}\left[2\sigma_0 + (1-2\nu)\,H\left(\frac{1}{4}\,\varrho\omega^2 r^2 + \frac{D}{r^2}\right)\right] \tag{3.8}$$

with the hardening parameter $H = \eta\sigma_0/G$.

Now the integration in the stresses and the displacement can be performed, and Eqs. (3.5), (3.3) and (3.4) yield

$$\frac{2G}{1-2\nu}\,u = \frac{1}{8[2+(1-\nu)\,H]}\left[16\sigma_0 r\log r - (4+H)\,\varrho\omega^2 r^3 + 4(4+H)\,\frac{D}{r}\right] + Cr, \tag{3.9}$$

$$\sigma_r = \frac{1}{8[2+(1-\nu)\,H]}\left[16\sigma_0\log r - \{8+(3-2\nu)\,H\}\,\varrho\omega^2 r^2 - 4(1-2\nu)\,H\,\frac{D}{r^2}\right] + C, \tag{3.10}$$

$$\sigma_\theta = \frac{4}{8[2+(1-\nu)\,H]}\left[16\sigma_0(1+\log r) - \{8+(1+2\nu)H\}\,\varrho\omega^2 r^2 + 4(1-2\nu)\,H\frac{D}{r^2}\right] + C. \tag{3.11}$$

These general expressions for the displacement and the stresses in the plastic region of a material with linear strain hardening reduce to the corresponding functions in a perfectly plastic material for $H = 0$ and, on the other hand, for $H \to \infty$ they adopt the forms of Eqs. (2.3), (2.4) and (2.5) describing elastic behavior.

Eqs. (3.9), (3.10) and (3.11) do not apply for the plastic zone of a rotating solid cylinder, since the inequality (2.9) is not satisfied there (cf. also [6]).

For $\omega = 0$, the above equations represent the displacement and the stresses in the plastic zone of a tube with fixed ends under external pressure if not too thick-walled or under internal pressure for any diameter ratio.

## 4. The Elastic-Plastic Tube

After first occurence of yield a plastic zone spreads out from the inner surface of the tube. General expressions for displacement and stresses in the elastic region (superscript $e$) and the plastic region (superscript $p$) are given by Eqs. (2.3), (2.4), (2.5), (3.9), (3.10), and (3.11), respectively. They contain the unknowns $A$, $B$, $C$ and $D$. An additional unknown is the radius $z$ of the elastic-plastic interface. There, the elastic stresses reach the yield limit

$$\sigma_\theta^{(e)}(z) - \sigma_r^{(e)}(z) = \sigma_0, \tag{4.1}$$

and displacement and radial stress have to be continuous,

$$u^{(p)}(z) = u^{(e)}(z), \tag{4.2}$$

$$\sigma_r^{(p)}(z) = \sigma_r^{(e)}(z). \tag{4.3}$$

Further conditions are the ones of vanishing stress at the curved surfaces of the tube, Eqs. (2.7) and (2.8).

In the order (2.7), (4.2), (4.3), (4.1) and (2.8) the system of equations reads

$$\frac{1}{8[2 + (1 - \nu) H]} \left[16\sigma_0 \log a - \{8 + (3 - 2\nu) H\} \varrho\omega^2 a^2 - 4(1 - 2\nu) H \frac{D}{a^2}\right] + C = 0, \tag{4.4}$$

$$\frac{1}{8[2 + (1 - \nu) H]} \left[16\sigma_0 \log z - (4 + H) \varrho\omega^2 z^2 + 4(4 + H) \frac{D}{z^2}\right] + C = \frac{2G}{1 - 2\nu} \frac{A}{z^2} + \frac{2G}{1 - 2\nu} B - \frac{1}{8(1 - \nu)} \varrho\omega^2 z^2, \tag{4.5}$$

$$\frac{1}{8[2 + (1 - \nu) H]} \left[16\sigma_0 \log z - \{8 + (3 - 2\nu) H\} \varrho\omega^2 z^2 - 4(1 - 2\nu) H \frac{D}{z^2}\right] + C = -2G \frac{A}{z^2} + \frac{2G}{1 - 2\nu} B - \frac{3 - 2\nu}{8(1 - \nu)} \varrho\omega^2 z^2, \tag{4.6}$$

$$4G \frac{A}{z^2} + \frac{1 - 2\nu}{4(1 - \nu)} \varrho\omega^2 z^2 = \sigma_0, \tag{4.7}$$

$$-2G \frac{A}{b^2} + \frac{2G}{1 - 2\nu} B - \frac{3 - 2\nu}{8(1 - \nu)} \varrho\omega^2 b^2 = 0. \tag{4.8}$$

Redundant but not in contradiction to the other equations is

$$\frac{1}{2+(1-\nu)\,H}\left[2\sigma_0+(1-2\nu)\,H\left(\frac{1}{4}\varrho\omega^2z^2+\frac{D}{z^2}\right)\right]=\sigma_0. \tag{4.9}$$

It expresses the fact that, at the elastic-plastic interface, where no plastic strain has developed as yet, the yield stress is at the initial yield limit, $\sigma_y = \sigma_0$.

As functions of the interface radius one receives

$$2GA=\frac{1}{2}\sigma_0z^2-\frac{1-2\nu}{8(1-\nu)}\varrho\omega^2z^4, \tag{4.10}$$

$$\frac{2G}{1-2\nu}B=\frac{1}{2}\sigma_0\frac{z^2}{b^2}+\frac{1}{8(1-\nu)}\varrho\omega^2b^2\left[3-2\nu-(1-2\nu)\frac{z^4}{b^4}\right], \tag{4.11}$$

$$C=\frac{1}{8[2+(1-\nu)\,H]}\left[-16\sigma_0\log a+4(1-\nu)\,H\sigma_0\frac{z^2}{a^2}\right.$$
$$\left.+\{8+(3-2\nu)\,H\}\,\varrho\omega^2a^2-(1-2\nu)\,H\varrho\omega^2\frac{z^4}{a^2}\right], \tag{4.12}$$

$$D=\frac{1-\nu}{1-2\nu}\sigma_0z^2-\frac{1}{4}\varrho\omega^2z^4. \tag{4.13}$$

In the calculation of the above quantities no use has been made of Eqs. (4.5) and (4.6). There difference gives with $A$ and $D$ according to Eqs. (4.10) and (4.13) an identity. Insertion of $A, B, C$, and $D$ in either of Eqs. (4.5) or (4.6) yields the dependence of the nondimensional elastic-plastic interface radius, $\zeta = z/b$, on the nondimensional angular velocity, $\Omega^2 = \varrho\omega^2b^2/\sigma_0$,

$$\Big[\Big[(3-2\nu)\,[2+(1-\nu)\,H]-(1-\nu)\,[8+(3-2\nu)\,H]\,Q^2+4(1-2\nu)\,\zeta^2$$
$$-(1-2\nu)\,[2+(1-\nu)\,H]\,\zeta^4+(1-\nu)\,(1-2\nu)\,H\,\frac{\zeta^4}{Q^2}\Big]\Big]\,\Omega^2$$
$$=4(1-\nu)\,[2+(1-\nu)\,H]\,(1-\zeta^2)+4(1-\nu)^2\,H\left(\frac{\zeta^2}{Q^2}-1\right)$$
$$+16(1-\nu)\log\frac{\zeta}{Q} \tag{4.14}$$

with the diameter ratio $Q = a/b$ as a parameter.

Eq. (4.14) contains as special cases the angular velocities for which plastic flow begins ($\zeta = Q$),

$$[(1-2\nu)\,Q^2+3-2\nu]\,\Omega_{EP}^2=4(1-\nu), \tag{4.15}$$

and, on the other hand, the value for which the whole tube becomes plastic ($\zeta = 1$),

$$[(1 - 2\nu) H + 2(4 + H) Q^2 - [8 + (3 - 2\nu) H] Q^4] \Omega_{FP}^2 = 4(1 - \nu) H(1 - Q^2) - 16Q^2 \log Q. \tag{4.16}$$

$\Omega_{EP}$ can be calculated also from the elastic solution and does not depend on $H$.

## 5. Fully Plastic Tube

The boundary conditions of vanishing radial stress at the free surfaces of the tube, Eqs. (2.7) and (2.8), together with Eq. (3.10) yield

$$C = \frac{1}{8[2 + (1 - \nu) H]} \left[ -\frac{16\sigma_0}{b^2 - a^2} (b^2 \log b - a^2 \log a) + \{8 + (3 - 2\nu) H\} \varrho\omega^2(a^2 + b^2) \right], \tag{5.1}$$

$$D = \frac{a^2b^2}{4(1 - 2\nu) H} \left[ -\frac{16\sigma_0}{b^2 - a^2} \log \frac{b}{a} + \{8 + (3 - 2\nu) H\} \varrho\omega^2 \right], \tag{5.2}$$

and now, the displacement and the stresses can be calculated with the help of Eqs. (3.9), (3.10) and (3.11).

## 6. Numerical Results

A few numerical results are presented in the following. No complete study of the influence of the parameters $Q$ and $H$ on the distribution of displacement and stresses was made but only the behavior of a hardening tube with $H \equiv \eta\sigma_0/G = 1$ is compared with the perfectly plastic tube, $H = 0$, and the elastic tube, $H \to \infty$, for the diameter ratio $Q \equiv a/b = 0.5$. The Poisson number, $\nu$, equals 0.3.

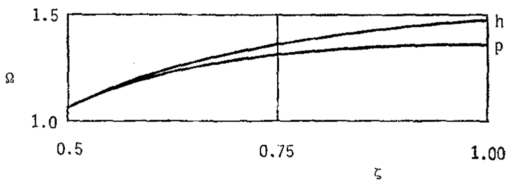


Fig. 1. Nondimensional angular velocity vs. nondimensional elastic-plastic interface radius

Fig. 1 shows the relation between the nondimensional angular velocity $\Omega = \omega b \sqrt{\varrho/\sigma_0}$ and the nondimensional elastic-plastic interface radius $\zeta = z/b$ for perfectly plastic material and hardening material. The former curve shows the characteristic extremum of the load, in this case the angular velocity, in the fully plastic state.

Fig. 2 exhibits the stress distribution in elastic, perfectly plastic and hardening material for $\Omega = 1.359555987$. This angular velocity corresponds to the fully

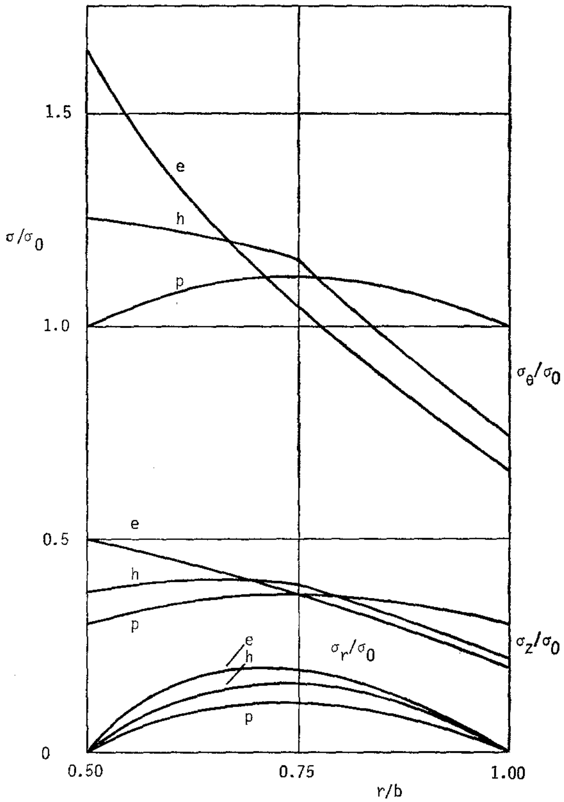


Fig. 2. Stress distribution for elastic behavior ($e$), perfectly plastic behavior ($p$) and material with hardening ($h$)

plastic state, $\zeta = 1$, of the perfectly plastic tube, i.e., it is the bursting speed. For hardening material the elastic-plastic boundary is at $\zeta = 0{,}750161877$.

The difference between the three materials is largest for the circumferential stress, $\sigma_\theta/\sigma_0$, and smallest for the radial stress, $\sigma_r/\sigma_0$. In the elastic region as well as in the plastic region, $\sigma_z = \nu(\sigma_r + \sigma_\theta)$. As expected the stresses in the hardening material are, besides a small region near the elastic plastic interface, smaller than the elastic stresses but larger than the perfectly plastic stresses.

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