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On Stokes' Problem for a Non-Newtonian Fluid

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1. Introduction

The fluids of the differential type (cf. Truesdell and Noll [1]) are amongst the many models which have been employed to describe the non-Newtonian behavior exhibited by certain fluids. In this note we extend the work of Stokes [2] on the flow due to an oscillating plate for a special subclass of the fluids of the differential type, namely the incompressible fluids of grade three. The stress $\boldsymbol{T}$ in such fluids is related to the fluid motion in the following manner (cf. Truesdell and Noll [1]):

$$\begin{aligned}\boldsymbol{T} = -p\boldsymbol{1} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 + \beta_1\boldsymbol{A}_3 \\ + \beta_2[\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1] + \beta_3(\operatorname{tr}\boldsymbol{A}_1^2)\,\boldsymbol{A}_1,\end{aligned} \tag{1}$$ [1]

where μ is the viscosity, α_1 and α_2 are material moduli which are usually referred to as the normal stress moduli, p is the pressure and the kinematical tensors $\boldsymbol{A}_1$, and $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ are defined recursively through

$$\boldsymbol{A}_1 = \operatorname{grad}\boldsymbol{v} + (\operatorname{grad}\boldsymbol{v})^T, \tag{2.1}$$

and

$$\boldsymbol{A}_n = \frac{d}{dt}(\boldsymbol{A}_{n-1}) + \boldsymbol{A}_{n-1}(\operatorname{grad}\boldsymbol{v}) + (\operatorname{grad}\boldsymbol{v})^T\boldsymbol{A}_1. \tag{2.2}$$

In the above equations, $\boldsymbol{v}$ denotes the velocity and $\frac{d}{dt}$ the material time derivative.

2. Equation of Motion

We are interested in the flow of a fluid modeled by (1) over an infinite flat plate which is either accelerating or oscillating. Thus, we seek a solution for the

[1] Fosdick and Rajagopal [3] have studied the thermodynamics of fluids modeled exactly by (1) in detail. They show that the assumption that all motions of the fluid obey the Clausius-Duhem inequality and the requirement that the specific Helmholtz free energy be a minimum when the fluid is locally at rest imply that

$$\mu \geqq 0, \quad \alpha_1 \geqq 0, \quad |\alpha_1 + \alpha_2| \leqq \sqrt{24\mu\beta_3}, \ \beta_1 = \beta_2 = 0 \quad \text{and} \quad \beta_3 \geqq 0.$$

In this note, in addition to studying the problem for the above range of material parameters, we also study the problem when $\alpha_1 < 0$, and $\beta_2 \neq 0$.

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velocity field of the form:

$$\boldsymbol{v} = u(y, t)\, \boldsymbol{i}, \tag{3}$$

where u is the velocity in the x-coordinate direction and $\boldsymbol{i}$ is the unit vector in the x-direction.

Substituting (1) into the balance of linear momentum

$$\operatorname{div} \boldsymbol{T} + \varrho \boldsymbol{b} = \varrho \frac{d\boldsymbol{v}}{dt},$$

and using (3) and the fact that the fluid is incompressible, we obtain (when $\beta_1 = 0$)

$$\mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2\, \partial t} + 6(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \varrho \frac{\partial u}{\partial t} = \frac{\partial p}{\partial x}. \tag{4.1}$$

$$\frac{\partial}{\partial y} \left[(2\alpha_1 + \alpha_2) \left(\frac{\partial u}{\partial y}\right)^2 + 2\beta_2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y\, \partial t}\right] = \frac{\partial p}{\partial y}, \tag{4.2}$$

$$0 = \frac{\partial p}{\partial z}. \tag{4.3}$$

For the problem in question the boundary conditions are

$$u(0, t) = U(t),$$

$$u \to 0 \quad \text{as} \quad y \to \infty.$$

Defining a modified pressure $\overset{*}{p}$ through

$$\overset{*}{p} = p - \left[(2\alpha_1 + \alpha_2) \left(\frac{\partial u}{\partial y}\right)^2 + 2\beta_2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y\, \partial t}\right],$$

we can rewrite Eqs. (4) as

$$\mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2\, \partial t} + 6(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \varrho \frac{\partial u}{\partial t} = \frac{\partial \overset{*}{p}}{\partial x}, \tag{5.1}$$

$$0 = \frac{\partial \overset{*}{p}}{\partial y} = \frac{\partial \overset{*}{p}}{\partial z}. \tag{5.2,3}$$

Equations (5.1, 2, 3) imply that $\frac{\partial \overset{*}{p}}{\partial x}$ is at most a function of time. The conditions at $y \to \infty$ imply $\frac{\partial \overset{*}{p}}{\partial x} = 0$ and, hence,

$$\mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2\, \partial t} + 6(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \varrho \frac{\partial u}{\partial t} = 0. \tag{6}$$

The above equation can be re-written in the following dimensionless form

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \bar{\alpha}_1 \frac{\partial^3 \bar{u}}{\partial \bar{y}^2\, \partial \bar{t}} + \varepsilon \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right), \tag{7}$$

where U_0 is a reference velocity

$$\bar{\alpha}_1 = \frac{\alpha_1 U_0^2}{\varrho \nu^2}, \quad \varepsilon = \frac{6(\beta_2 + \beta_3)}{\varrho \nu^2}$$

and

$$\bar{u} = \frac{u}{U_0}, \quad \bar{t} = \frac{U_0^2 t}{\nu} \quad \text{and} \quad \bar{y} = \frac{U_0 y}{\nu}.$$

The appropriate boundary conditions are

$$\bar{u} = \bar{U}(t) \quad \text{at} \quad t = 0,$$

$$\bar{u} \to 0 \quad \text{as} \quad \bar{y} \to \infty.$$

Let us suppose the non-dimensional velocity $\bar{u}$ can be expanded in power series in ε:

$$\bar{u}(t, y; \varepsilon) = \bar{u}_0(t, y) + \varepsilon \bar{u}_1(t, y) + \varepsilon^2 \bar{u}_2(t, y) + \cdots. \tag{8}$$

On substituting the expansion (8) for $\bar{u}$ and equating like powers of ε, we obtain the following equations at zeroth and first powers, respectively.

$$\frac{\partial \bar{u}_0}{\partial \bar{t}} = \frac{\partial^2 \bar{u}_0}{\partial \bar{y}^2} + \alpha_1 \frac{\partial^3 \bar{u}_0}{\partial \bar{y}^2\, \partial t} \tag{9}$$

$$\bar{u}_0 = \bar{U}(t) \quad \text{at} \quad \bar{y} = 0, \tag{10.1}$$

$$\bar{u}_0 \to 0 \quad \text{as} \quad \bar{y} \to \infty, \tag{10.2}$$

and

$$\frac{\partial \bar{u}_1}{\partial \bar{t}} = \frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2} + \alpha_1 \frac{\partial^3 \bar{u}_1}{\partial \bar{y}^2\, \partial t} + \left(\frac{\partial \bar{u}_0}{\partial \bar{y}}\right)^2 \left(\frac{\partial^2 \bar{u}_0}{\partial \bar{y}^2}\right), \tag{11}$$

$$\bar{u}_1 = 0 \quad \text{at} \quad \bar{y} = 0, \tag{12.1}$$

$$\bar{u}_1 \to 0 \quad \text{as} \quad \bar{y} \to \infty. \tag{12.2}$$

Introducing the similarity transformation

$$\eta = \bar{y}, \qquad f_0(\eta) = \frac{\bar{u}_0}{e^{\gamma t}}, \qquad f_1(\eta) = \frac{\bar{u}_1}{e^{3\gamma t}}, \tag{13}$$

we can re-write Eqs. (9) and (10) in the following manner:

$$(1 + \gamma \bar{\alpha}_1) f_0'' - \gamma f_0 = 0, \tag{14}$$

$$f_0(0) = 1, \tag{15.1}$$

$$f_0(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \tag{15.2}$$

where

$$\bar{U}(t) = e^{\gamma \bar{t}}.$$

Also, Eqs. (11) and (12) can be written as

$$(1 + 3\gamma\bar{\alpha}_1) f_1'' - 3\gamma f_1 = -(f_0')^2 f_0'', \tag{16}$$

$$f_1(0) = 0, \tag{17.1}$$

$$f_1(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{17.2}$$

It is straightforward to verify that the solutions to (14)—(15) and (16)—(17) are, respectively

$$f_0(\eta) = \exp\left\{-\sqrt{\frac{\gamma}{1+\gamma\bar{\alpha}_1}}\,\eta\right\}, \tag{18}$$

and

$$f_1(\eta) = \frac{\gamma}{6(1+\gamma\bar{\alpha}_1)(1+4\gamma\bar{\alpha}_1)}\left\{\exp\left(-\sqrt{\frac{3\gamma}{1+3\gamma\bar{\alpha}_1}}\,\eta\right) - \exp\left(-3\sqrt{\frac{\gamma}{1+\gamma\bar{\alpha}_1}}\,\eta\right)\right\}. \tag{19}$$

Since γ may be both real or imaginary, both cases are considered.

Case 1: γ is real

In this case Eq. (8) yields

$$\bar{u}\,(y, t; \varepsilon) = e^{\gamma\bar{t}} f_0(\eta) + \varepsilon e^{3\gamma\bar{t}} f_1(\eta) + \cdots. \tag{20}$$

The numerical values of $f_0(\eta)$ and $f_1(\eta)$, for various values of γ and $\bar{\alpha}_1$ are provided in Table 1. Thus, one can obtain the numerical value of $\bar{u}$ for various values of ε. Of course, when $\varepsilon = 0$, (18) implies that there is an exact solution for the problem.

The skin friction on the plate τ_ω for the problem is given by

$$\tau_\omega = \varrho U_0^2\{e^{\gamma\bar{t}} f_0'(0) + \varepsilon e^{3\gamma\bar{t}} f_1'(0) + \cdots\}. \tag{21}$$

It follows from (18) and (19) that

$$f_0'(\eta) = -\sqrt{\frac{\gamma}{1+\gamma\bar{\alpha}_1}}\, f_0(\eta), \tag{22}$$

and

$$f_1'(\eta) = \frac{\gamma}{6(1+\gamma\bar{\alpha}_1)(1+4\gamma\bar{\alpha}_1)}\left\{3\sqrt{\frac{\gamma}{1+\gamma\bar{\alpha}_1}}\exp\left(-3\sqrt{\frac{\gamma}{1+\gamma\bar{\alpha}_1}}\,\eta\right) - \sqrt{\frac{3\gamma}{1+3\gamma\bar{\alpha}_1}}\exp\left(-\sqrt{\frac{3\gamma}{1+3\gamma\bar{\alpha}_1}}\,\eta\right)\right\}. \tag{23}$$

The numerical values of $f_0'(\eta)$ and $f_1'(\eta)$ for values of γ and $\bar{\alpha}_1$, are provided in Table 1.

Case 2: γ is imaginary

In the case of an oscillating plate, i.e., when γ is imaginary, say $\gamma = i\omega$, a straightforward computation verifies

$$\bar{u}(\bar{t}, \bar{y}; \varepsilon) = \{f_{0R}\cos\omega\bar{t} - f_{0I}\sin\omega\bar{t}\} + \varepsilon\{f_{1R}\cos 3\omega\bar{t} - f_{1I}\sin 3\omega\bar{t}\} + \cdots, \tag{24}$$

where

$$f_{0R} = e^{-\delta_R\eta}\cos\delta_I\eta \tag{25}$$

Table 1

	η	$f_0(\eta)$	$f_0'(\eta)$	$f_1(\eta)$	$f_1'(\eta)$
$\bar{\alpha}_1 = -4$	0.0	1.0	−0.79056942	0.0	−0.22667550
	0.5	0.67348836	−0.53243930	0.0386788	−0.00565536
$\gamma = .5$	1.0	0.45358657	−0.35859167	0.02650408	−0.03017132
	2.0	0.20574078	−0.16265237	0.01367445	−0.01021715
	4.0	0.04232927	−0.03346422	0.00018575	−0.00034251
	6.0	0.00870086	−0.00688496	0.00000434	−0.00000826
	10.0	0.00036864	−0.00029144	0.00000000	−0.00000000
	15.0	0.00000708	−0.00000560	0.00000000	−0.00000000
	20.0	0.00000014	−0.00000011	0.00000000	−0.00000000
$\bar{\alpha}_1 = -2$	0.0	1.0	−0.74535599	0.0	−0.11916942
	0.5	0.68888711	−0.51346614	0.02377473	+0.00415646
$\gamma = .5$	1.0	0.47456545	−0.35372021	0.01920771	−0.01538062
	2.0	0.22521237	−0.24367329	0.01177957	−0.01307965
	4.0	0.05072061	−0.03780491	0.00042190	−0.00060205
	6.0	0.01142291	−0.00851413	0.00002343	−0.00003412
	10.0	0.00057938	−0.00043184	0.00000007	−0.00000010
	15.0	0.00001395	−0.00001039	0.00000000	−0.00000000
	20.0	0.00000034	−0.00000025	0.00000000	−0.00000000
$\bar{\alpha}_1 = 2$	0.0	1.0	−0.67419986	0.0	−0.05132182
	0.5	0.71383759	−0.48126920	0.01194280	−0.00583949
$\gamma = .5$	1.0	0.50956410	−0.34354805	0.01132410	−0.00537358
	2.0	0.25965557	−0.17505975	0.00536640	−0.00486598
	4.0	0.06742102	−0.04545524	0.00072008	−0.00045028
	6.0	0.01750624	−0.01180271	0.00008566	−0.00009174
	10.0	0.00118029	−0.00079575	0.00000117	−0.00000126
	15.0	0.00004055	−0.00002734	0.00000001	−0.00000001
	20.0	0.00000139	−0.00000094	0.00000000	−0.00000000

and

$$f_{0I} = -e^{-\delta_R\eta} \sin \delta_I\eta \tag{26}$$

are the real and imaginary parts of f_0. Also,

$$\begin{aligned} f_{1R} = &-\{A_R \cos m_I\eta + A_I \sin m_I\eta\}\, e^{-m_R\eta} \\ &+ \{A_R \cos 3\delta_I\eta + A_I \sin 3\delta_I\eta\}\, e^{-3\gamma_R\eta}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} f_{1I} = &-\{A_I \cos m_I\eta - A_R \sin m_I\eta\}\, e^{-m_R\eta} \\ &+ \{A_I \cos 3\delta_I\eta - A_R \sin 3\delta_I\eta\}\, e^{-3\delta_R\eta}, \end{aligned} \tag{28}$$

where f_{1R} and f_{1I} are the real and imaginary parts of f_1, respectively. In the above equations

$$\delta_R = \left\{\frac{a_1 + (a_1^2 + a_2^2)^{1/2}}{2}\right\}^{1/2}, \quad \delta_I = \frac{a_2}{\{2[a_1 + (a_1^2 + a_2^2)^{1/2}]\}^{1/2}}, \tag{29.1, 2}$$

$$m_R = \left\{\frac{b_1 + (b_1^2 + b_2^2)^{1/2}}{2}\right\}^{1/2}, \quad m_I = \frac{b_2}{\{2[b_1 + (b_1^2 + b_2^2)^{1/2}]\}^{1/2}}, \tag{30.1, 2}$$

Table 2

	η	$\Omega t = 1.5708$		$\Omega t = 3.1416$		$\Omega t = 6.2832$	
		$u_0(\eta)$	$u_1(\eta)$	$u_0(\eta)$	$u_1(\eta)$	$u_0(\eta)$	$u_1(\eta)$
$\bar{\alpha}_1 = -2$	0.0	1.0	0.0	1.0	0.0	1.0	0.0
	1.0	0.31116009	0.04315088	−0.53961863	−0.00164945	0.53962081	0.0164855
	2.0	0.33581557	0.01102422	−0.19436649	−0.02819471	0.19436885	0.02819448
	4.0	0.13054154	−0.00788366	0.07499422	0.00150078	−0.07499330	−0.00150061
	6.0	0.00018816	0.00102598	0.05841424	0.00154834	−0.05841424	−0.00154836
	10.0	−0.00763958	−0.00010041	−0.00435619	−0.00002961	0.00435614	0.00002961
	15.0	0.00082469	0.00000055	−0.00000664	−0.00000287	0.00000665	0.00000287
	20.0	−0.00006656	0.00000008	0.00003939	0.0000001	−0.00003939	−0.0000001
$\bar{\alpha}_1 = -4$	0.0	1.0	0.0	1.0	0.0	1.0	0.0
	1.0	0.33067138	−0.13878299	−0.54967557	−0.04245936	0.54967789	0.04246227
	2.0	0.36352387	−0.07321561	−0.19279840	−0.09002350	0.19280094	−0.09002197
	4.0	0.14017238	0.04087078	0.09497887	0.00368240	−0.09497789	−0.00368326
	6.0	−0.00750268	0.00439193	0.06926775	−0.01226276	−0.06926781	0.01226285
	10.0	−0.00899680	0.00125829	−0.00763067	0.00057150	0.00763061	−0.00057152
	15.0	0.00123499	−0.00001708	0.00034144	0.00008281	−0.00034143	−0.00008281
	20.0	−0.00013730	−0.00000517	0.00002271	0.00000010	−0.00002272	−0.00000010
$\bar{\alpha}_1 = 2$	0.0	1.0	0.0	1.0	0.0	1.0	0.0
	1.0	0.27020173	0.02785002	−0.52753049	−0.02130391	0.52753238	1.02130333
	2.0	0.28507951	−0.00082394	−0.20527844	−0.01952763	0.20528044	0.01952765
	4.0	0.11705065	−0.00305878	0.03913149	0.00054134	−0.03913068	−0.00054128
	6.0	0.01286996	0.00003569	0.04139879	0.00042237	−0.04139870	−0.00042237
	10.0	0.00534897	0.00000194	−0.00011371	−0.00000871	0.00011367	0.00000871
	15.0	0.00028540	0.00000002	−0.00026776	0.00000007	0.00026776	−0.00000007
	20.0	−0.00000122	−0.00000000	0.00002860	−0.00000000	−0.00002860	0.00000000

with

$$a_1 = \frac{\bar{\alpha}_1 \omega^2}{1 + \bar{\alpha}_1{}^2 \omega^2}, \qquad a_2 = \frac{\omega}{1 + \bar{\alpha}_1{}^2 \omega^2}, \tag{31.1, 2}$$

$$b_1 = \frac{9\bar{\alpha}_1 \omega^2}{1 + 9\bar{\alpha}_1{}^2 \omega^2}, \quad b_2 = \frac{3\omega}{1 + 9\bar{\alpha}_1{}^2 \omega^2}, \tag{32.1, 2}$$

$$A_R = \frac{-5\omega^2 \bar{\alpha}_1}{6[(1 - 4\omega^2 \bar{\alpha}_1{}^2)^2 - 25\bar{\alpha}_1{}^2 \omega^2]}, \tag{33}$$

and

$$A_I = \frac{-(1 - 4\omega^2 \alpha_1{}^2)\,\omega}{6[(1 - 4\omega^2 \bar{\alpha}_1{}^2)^2 - 25\bar{\alpha}_1{}^2 \omega^2]}. \tag{34}$$

Again, when $\varepsilon = 0$ one can establish an exact solution (cf. Rajagopal [4]).

The skin friction on the plate is given by the real part of

$$\tau_\omega = \varrho U_0{}^2 \{e^{i\omega t}[f'_{0R}(0) + i f'_{0I}(0)] + \varepsilon e^{i3\omega t}[f_1'(0) + i f'_{1I}(0)] + \cdots\}. \tag{35}$$

It follows from Eqs. (25)—(34) that

$$f'_{0R}(0) = -\delta_R, \tag{36}$$

$$f'_{0I}(0) = -\delta_I, \tag{37}$$

$$f'_{1I}(0) = -A_I m_I + A_R m_R + 3A_I \delta_I - 3A_R \delta_R, \tag{38}$$

and

$$f'_{1I}(0) = A_R m_I + A_I m_R - 3A_R \delta_I - 3A_I \delta_R. \tag{39}$$

In Table 2 the numerical values of

$$\bar{u}_0'(\eta) = f'_{0R}(\eta) \cos \omega \bar{t} - f'_{0I}(\eta) \sin \omega \bar{t} \tag{40}$$

and

$$\bar{u}_1'(\eta) = f'_{1R}(\eta) \cos 3\omega \bar{t} - f'_{1I}(\eta) \sin 3\omega \bar{t} \tag{41}$$

are provided for various values of $\omega \bar{t}$ and $\bar{\alpha}_1$.

References

[1] Truesdell, C., Noll, W.: The non-linear field theories of mechanics. Handbuch der Physik, III/3. Berlin—Heidelberg—New York: Springer 1965.

[2] Stokes, G. G.: On the effect of the internal friction of fluids on the notion of pendulums. Cambr. Phil. Trans. **9**, 8 (1851).

[3] Fosdick, R. L., Rajagopal, K. R.: Thermodynamics and stability of fluids of third grade. Proc. R. Soc. Lond. **A 339**, 351—377 (1980).

[4] Rajagopal, K. R.: Unsteady unidirectional flows of a non-Newtonian fluid. International Journal of Non-Linear Mechanics (in press).

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