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An Application of the Extended Kantorovich Method to the Stress Analysis of a Clamped Rectangular Plate[1]

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With 12 Figures

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Summary. The extended KANTOROVICH method discussed recently by A. D. KERR, is used to analyze a clamped rectangular plate subjected to a uniform lateral load. It was found that the generated one term solution approximates very closely, throughout the plate region, not only the deflections but also the bending moments and shearing forces. It is shown that the final form of the solution is independent of the initial choice, and that the convergence of the iterative procedure is very rapid. Because of the lack of a closed form exact solution in the technical literature, the coefficients occurring in the obtained solution were evaluated for various plate side ratios and are presented in graphs in order to simplify the utilization of the obtained results in engineering practice.

Zusammenfassung. Die erweiterte Methode von KANTOROWITSCH, die kürzlich von A. D. KERR diskutiert wurde, wird benützt, um die eingespannte Rechteckplatte unter Gleichlast zu untersuchen. Es ergab sich, daß die erzielte eingliedrige Lösung nicht nur die Durchbiegung, sondern auch die Biegemomente und Querkräfte im Bereich der Platte sehr gut approximiert. Es wird gezeigt, daß die Endgestalt der Lösung unabhängig ist von der anfangs getroffenen Wahl und daß die Konvergenz des Iterationsverfahrens sehr rasch fortschreitet. Da in der technischen Literatur eine exakte Lösung in geschlossener Form fehlt, wurden die in der erhaltenen Lösung auftretenden Koeffizienten für verschiedene Seitenverhältnisse der Platte ausgerechnet und in Schaubildern dargestellt, um die Benützung der erhaltenen Resultate in der Ingenieurpraxis zu erleichtern.

Introduction and Statement of Problem

As a step towards eliminating the arbitrariness in the choice of coordinate functions, a shortcoming inherent in the methods of RITZ and GALERKIN, L. V. KANTOROVICH assumed as approximate solution

$$w_m = \sum_{n=1}^{m} a_n(x_1)\, \psi_n(x_1, x_2, \ldots, x_r) \tag{1}$$

where ψ_n are, also here, a priori chosen functions but a_n are no longer constants but unknown functions of one of the independent variables, x_1.

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The condition that the a_n's have to make the functional under consideration stationary leads to m ordinary differential equations for the determination of the m functions $a_n(x_1)$.

Recently, A. D. KERR [1], [2] extended the KANTOROVICH method with the aim in view of completely eliminating the arbitrariness in the choice of the coordinate functions. It was suggested to assume

$$w_m = \sum_{n=1}^{m} \underbrace{a_{n1}(x_1)}_{\substack{\text{determined}\\ \text{previously}\\ \text{as } a_n(x_1)}} \; a_{n2}^*(x_2) \underbrace{\psi'_n(x_3, x_4, \ldots, x_r)}_{\substack{\text{a priori chosen}\\ \text{functions}}} \tag{2}$$

to determine a_{n2}^* from a set of m ordinary differential equations, then to set

$$w_m = \sum_{n=1}^{m} \underbrace{a_{n1}(x_1)\, a_{n2}(x_2)}_{\substack{\text{determined}\\ \text{previously as}\\ a_n(x_1) \text{ and } a_{n2}^*(x_2)\\ \text{respectively}}} \; a_{n3}^*(x_3) \underbrace{\psi''_n(x_4, x_5, \ldots, x_r)}_{\substack{\text{a priori chosen}\\ \text{functions}}} \tag{3}$$

to determine a_{n3}^* from a set of m ordinary differential equations, and to continue this process until for each x_k a set of $a_{nk}(x_k)$ functions is determined and w_m becomes

$$w_m^{(I)} = \sum_{n=1}^{m} a_{n1}(x_1)\, a_{n2}(x_2)\, a_{n3}(x_3) \ldots a_{nr}(x_r). \tag{4}$$

After completing the first cycle, which yields $w_m^{(I)}$, this procedure is continued assuming that in (4) the $a_{n1}(x_1)$ are unknown functions, determining them as described previously, substituting the determined function into (4), then assuming that $a_{n2}(x_2)$ are unknown functions, etc. It was conjectured that if this procedure is continued indefinitely it should yield a function w_m which will very closely approximate the exact solution w.

In Ref. [2], [3] the suggested method was demonstrated on a torsion problem of a beam of rectangular cross section. It was found that the iterative process converges very rapidly to a final form irrespective of the initial assumption. The numerical results showed that even a one term approximation of PRANDTL's stress function generated by the procedure described above, yields stresses which agree very closely with the corresponding values obtained from the exact solution.

In the present paper the above method is used to solve the problem of a clamped rectangular thin plate subjected to a uniform lateral load shown in Fig. 1, for which a closed form solution is not available in the literature. The solution will be restricted to a one term expression, that is, $m = 1$. Special attention is focused on how close the generated solution does approximate not only the deflection surface but also the bending moments and shearing forces which are obtained as higher order derivatives of the deflection surface.

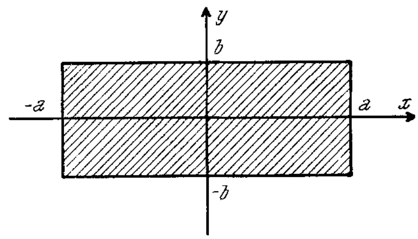

Fig. 1. Clamped rectangular plate subjected to a uniform lateral load

Derivation of the One Term Approximation

For a clamped rectangular plate subjected to a lateral distributed load, $q(x, y)$, the principle of virtual displacements yields

$$\int_{-a}^{+a} \int_{-b}^{+b} (D \nabla^4 w - q)\, \delta w \, dx \, dy = 0 \tag{5}$$

where w is the lateral deflection which satisfies the boundary conditions, D is the flexural rigidity of the plate, and

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \tag{6}$$

Assuming the deflection in the form

$$w(x, y) = w_{ij}(x, y) = f_i(x)\, g_j(y) \tag{7}$$

it follows that when g_j is prescribed a priori, equ. (5) may be written as

$$\int_{-a}^{+a} \left[\int_{-b}^{+b} (D \nabla^4 w_{ij} - q)\, g_j \, dy \right] \delta f_i \, dx = 0. \tag{8}$$

Equ. (8) is satisfied when

$$\int_{-b}^{+b} (D \nabla^4 w_{ij} - q)\, g_j \, dy = 0. \tag{9}$$

When f_i is prescribed a priori, equ. (5) may be written as

$$\int_{-b}^{+b} \left[\int_{-a}^{+a} (D \nabla^4 w_{ij} - q)\, f_i \, dx \right] \delta g_i \, dy = 0. \tag{10}$$

It is satisfied when

$$\int_{-a}^{+a} (D\nabla^4 w_{ij} - q) f_i \, dx = 0. \tag{11}$$

Equations (9) and (11) are the GALERKIN equations of the iterative procedure.

We start the present investigation by extending, in the manner discussed above, the problem presented by L. V. KANTOROVICH and V. I. KRYLOV [4] (see also [5]) by choosing the first approximation as

$$w_{10} = f_1(x)\, g_0(y), \tag{12}$$

where

$$g_0 = \left(\frac{y^2}{b^2} - 1\right)^2. \tag{13}$$

The assumed w_{10} satisfies the boundary conditions

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial y} &= 0 \end{aligned}\right\} \begin{aligned} &\text{at } y = \pm b \\ &-a \leqq x \leqq +a. \end{aligned} \tag{14}$$

Equation (9) becomes

$$\int_{-b}^{+b} \left(\nabla^4 w_{10} - \frac{q}{D}\right)\left(\frac{y^2}{b^2} - 1\right)^2 dy = 0 \tag{15}$$

which, after performing the indicated integrations, yields

$$b^4 \frac{d^4 f_1}{dx^4} - 6\, b^2 \frac{d^2 f_1}{dx^2} + \frac{63}{2} f_1 = \frac{21}{16} \frac{q\, b^4}{D}. \tag{16}$$

The general solution of (16) is

$$\begin{aligned} f_1(x) = {} & C_1 \sinh\left(\alpha_1 \frac{x}{a}\right) \sin\left(\beta_1 \frac{x}{a}\right) + C_2 \cosh\left(\alpha_1 \frac{x}{a}\right) \cos\left(\beta_1 \frac{x}{a}\right) + \\ & + C_3 \sinh\left(\alpha_1 \frac{x}{a}\right) \cos\left(\beta_1 \frac{x}{a}\right) + C_4 \cosh\left(\alpha_1 \frac{x}{a}\right) \sin\left(\beta_1 \frac{x}{a}\right) + f_p \end{aligned} \tag{17}$$

where

$$\left.\begin{aligned} \alpha_1 &= \frac{a}{b}\left(\sqrt{\frac{63}{8}} + \frac{3}{2}\right)^{\frac{1}{2}} \\ \beta_1 &= \frac{a}{b}\left(\sqrt{\frac{63}{8}} - \frac{3}{2}\right)^{\frac{1}{2}} \end{aligned}\right\} \tag{18}$$

and

$$f_p = \frac{1}{24} \frac{q\, b^4}{D}. \tag{19}$$

From the B.C.'s

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial x} &= 0 \end{aligned}\right\} \begin{aligned} & x = \pm a \\ & -b \leqq y \leqq +b \end{aligned} \tag{20}$$

it follows that

$$C_1 = \frac{q\, b^4}{24\, D} \frac{K_1}{K_0}; \quad C_2 = \frac{q\, b^4}{24\, D} \frac{K_2}{K_0}; \quad C_3 = C_4 = 0 \tag{21}$$

where

$$\left.\begin{aligned} K_0 &= +\,(\alpha_1 \sin \beta_1 \cos \beta_1 + \beta_1 \sinh \alpha_1 \cosh \alpha_1) \\ K_1 &= +\,(\alpha_1 \sinh \alpha_1 \cos \beta_1 - \beta_1 \cosh \alpha_1 \sin \beta_1) \\ K_2 &= -\,(\alpha_1 \cosh \alpha_1 \sin \beta_1 + \beta_1 \sinh \alpha_1 \cos \beta_1) \end{aligned}\right\}. \tag{22}$$

Thus

$$\begin{aligned} w_{10}(x, y) = \frac{q\, b^4}{24\, D\, K_0} \Big[& K_2 \cosh\left(\alpha_1 \frac{x}{a}\right) \cos\left(\beta_1 \frac{x}{a}\right) + \\ & + K_1 \sinh\left(\alpha_1 \frac{x}{a}\right) \sin\left(\beta_1 \frac{x}{a}\right) + K_0 \Big] \left(\frac{y^2}{b^2} - 1\right)^2. \end{aligned} \tag{23}$$

The next step is to assume

$$w_{11} = \left[K_2 \cosh\left(\alpha_1 \frac{x}{a}\right) \cos\left(\beta_1 \frac{x}{a}\right) + K_1 \sinh\left(\alpha_1 \frac{x}{a}\right) \sin\left(\beta_1 \frac{x}{a}\right) + K_0\right] g_1(y). \tag{24}$$

Substituting (24) into equ. (11) we obtain

$$\begin{aligned} \int_{-a}^{+a} \left(\nabla^4 w_{11} - \frac{q}{D}\right) \Big[& K_1 \sinh\left(\alpha_1 \frac{x}{a}\right) \sin\left(\beta_1 \frac{x}{a}\right) + \\ & + K_2 \cosh\left(\alpha_1 \frac{x}{a}\right) \cos\left(\beta_1 \frac{x}{a}\right) + K_0 \Big] dx = 0 \end{aligned} \tag{25}$$

which after performing the integrations yields the differential equation for $g_1(y)$.

Before proceeding with the details of the iterations it is useful to note that equation (9) may be rewritten as

$$\begin{aligned} \left[\int_{-b}^{+b} g_j^2\, dy\right] \frac{d^4 f_i}{dx^4} + \left[2 \int_{-b}^{+b} \frac{d^2 g_j}{dy^2} g_j\, dy\right] \frac{d^2 f_i}{dx^2} + & \\ + \left[\int_{-b}^{+b} \frac{d^4 g_j}{dy^4} g_j\, dy\right] f_i = \frac{1}{D} \int_{-b}^{+b} q\, g_j\, dy. & \end{aligned} \tag{26}$$

Integration by parts of two of the coefficients in (26), yields

$$\left.\begin{aligned} \int_{-b}^{+b} \frac{d^2 g_j}{dy^2} g_j\, dy &= \left[\frac{d g_j}{dy} g_j\right]_{-b}^{+b} - \int_{-b}^{+b} \left(\frac{d g_j}{dy}\right)^2 dy \\ \int_{-b}^{+b} \frac{d^4 g_j}{dy^4} g_j\, dy &= \left[\frac{d^3 g_j}{dy^3} g_j\right]_{-b}^{+b} - \left[\frac{d^2 g_j}{dy^2} \frac{d g_j}{dy}\right]_{-b}^{+b} + \int_{-b}^{+b} \left(\frac{d^2 g_j}{dy^2}\right)^2 dy \end{aligned}\right\}. \tag{27}$$

Because of the boundary conditions (14), it follows that

$$g_j(\pm b) = 0; \quad \left(\frac{d g_j}{dy}\right)_{\pm b} = 0 \tag{28}$$

and hence the integrated terms in (27) vanish. Thus equ. (26) may be written as

$$\begin{aligned} &\left[\int_{-b}^{+b} g^2{}_j\, dy\right] \frac{d^4 f_i}{dx^4} - \left[2 \int_{-b}^{+b} \left(\frac{d g_j}{dy}\right)^2 dy\right] \frac{d^2 f_i}{dx^2} + \\ &\qquad + \left[\int_{-b}^{+b} \left(\frac{d^2 g_j}{dy^2}\right)^2 dy\right] f_i = \frac{1}{D} \int_{-b}^{+b} q\, g_j\, dy. \end{aligned} \tag{29}$$

By a similar argument, equ. (11) may be rewritten as

$$\begin{aligned} &\left[\int_{-a}^{+a} f_i^2\, dx\right] \frac{d^4 g_j}{dy^4} - \left[2 \int_{-a}^{+a} \left(\frac{d f_i}{dx}\right)^2 dx\right] \frac{d^2 g_j}{dy^2} + \\ &\qquad + \left[\int_{-a}^{+a} \left(\frac{d^2 f_i}{dx^2}\right)^2 dx\right] g_j = \frac{1}{D} \int_{-a}^{+a} q\, f_i\, dx. \end{aligned} \tag{30}$$

Comparing the differential equations (29) and (30) and noting that the coefficients are positive it can be concluded that the obtained $f_i(x)$ and $g_j(y)$ will be of the same form [see equ. (17)], and that the final form of the generated solution will be independent of the initial choice of g_0. Hence the final form of w_{ij} is fixed in advance and the iteration procedure determines merely the parameters which appear in the solution.

It can be easily shown that the iterative procedure, using (9) and (11) [or the equivalent equations (29) and (30)], reduces to the following recurrence formulae with $n = 1$ as starting index:

$$\begin{aligned} w_{nn} = g_{n0}\Big[&K_{1n} \sinh\left(\alpha_n \frac{x}{a}\right) \sin\left(\beta_n \frac{x}{a}\right) + K_{2n} \cosh\left(\alpha_n \frac{x}{a}\right) \cos\left(\beta_n \frac{x}{a}\right) + K_{0n}\Big] \cdot \\ \cdot \Big[&K'_{1n} \sinh\left(\alpha'_n \frac{y}{b}\right) \sin\left(\beta'_n \frac{y}{b}\right) + K'_{2n} \cosh\left(\alpha'_n \frac{y}{b}\right) \cos\left(\beta'_n \frac{y}{b}\right) + K'_{0n}\Big] \end{aligned} \tag{31}$$

$$w_{(n+1)n} = f_{(n+1)0}\left[K_{1(n+1)} \sinh\left(\alpha_{n+1}\frac{x}{a}\right)\sin\left(\beta_{n+1}\frac{x}{a}\right) + \right.$$
$$\left. + K_{2(n+1)}\cosh\left(\alpha_{n+1}\frac{x}{a}\right)\cos\left(\beta_{n+1}\frac{x}{a}\right) + K_{0(n+1)}\right]\cdot$$
$$\cdot\left[K'_{1n}\sinh\left(\alpha'_n\frac{y}{b}\right)\sin\left(\beta'_n\frac{y}{b}\right) + \right.$$
$$\left. + K'_{2n}\cosh\left(\alpha'_n\frac{y}{b}\right)\cos\left(\beta'_n\frac{y}{b}\right) + K'_{0n}\right] \tag{32}$$

where α_1 and β_1 are given in (18) and

$$\left.\begin{aligned} \left.\begin{matrix}\alpha'_n\\ \beta'_n\end{matrix}\right\} &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{C_n}{A_n}} \pm \frac{B_n}{A_n}\right)^{\frac{1}{2}}\frac{b}{a} \\ \left.\begin{matrix}\alpha_{n+1}\\ \beta_{n+1}\end{matrix}\right\} &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{C'_n}{A'_n}} \pm \frac{B'_n}{A'_n}\right)^{\frac{1}{2}}\frac{a}{b}\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} \frac{g_{n0}\,D}{q\,a^4} &= \frac{E_n}{C_n\,K'_{0n}} \\ \frac{f_{n0}\,D}{q\,b^4} &= \frac{E'_n}{C_n'\,K_{0n}}\end{aligned}\right\}. \tag{34}$$

The expressions for A_n, B_n, C_n, and E_n are

$$\left.\begin{aligned} A_n &= K^2_{1n}\,I_1 + 2\,K_{1n}\,K_{2n}\,I_2 + 2\,K_{1n}\,K_{0n}\,I_3 + \\ &\qquad + K^2_{2n}\,I_4 + 2\,K_{2n}\,K_{0n}\,I_5 + 2\,K^2_{0n} \\ B_n &= \gamma_n\,K_{1n}\,I_1 + (\lambda_n\,K_{1n} + \gamma_n\,K_{2n})\,I_2 + \\ &\qquad + \gamma_n\,K_{0n}\,I_3 + \lambda_n\,K_{2n}\,I_4 + \lambda_n\,K_{0n}\,I_5 \\ C_n &= \varphi_n\,K_{1n}\,I_1 + (\psi_n\,K_{1n} + \varphi_n\,K_{2n})\,I_2 + \\ &\qquad + \varphi_n\,K_{0n}\,I_3 + \psi_n\,K_{2n}\,I_4 + \psi_n\,K_{0n}\,I_5 \\ E_n &= K_{1n}\,I_3 + K_{2n}\,I_5 + 2\,K_{0n}\end{aligned}\right\} \tag{35}$$

where

$$\left.\begin{aligned} K_{0n} &= +\,\alpha_n \sin\beta_n\cos\beta_n + \beta_n\sinh\alpha_n\cosh\alpha_n \\ K_{1n} &= +\,\alpha_n\sinh\alpha_n\cos\beta_n - \beta_n\cosh\alpha_n\sin\beta_n \\ K_{2n} &= -\,\alpha_n\cosh\alpha_n\sin\beta_n - \beta_n\sinh\alpha_n\cos\beta_n\end{aligned}\right\} \tag{36}$$

$$\left.\begin{aligned} \gamma_n &= \omega_n\,K_{1n} + 2\,\varkappa_n\,K_{2n} \\ \lambda_n &= \omega_n\,K_{2n} - 2\,\varkappa_n\,K_{1n}\end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned} \varphi_n &= (\omega_n{}^2 - 4\,\varkappa_n{}^2)\,K_{1n} + 4\,\omega_n\,\varkappa_n\,K_{2n} \\ \psi_n &= (\omega_n{}^2 - 4\,\varkappa_n{}^2)\,K_{2n} - 4\,\omega_n\,\varkappa_n\,K_{1n}\end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} \omega_n &= \beta_n{}^2 - \alpha_n{}^2 \\ \varkappa_n &= \alpha_n\,\beta_n\end{aligned}\right\} \tag{39}$$

and

$$\left.\begin{aligned}
I_1 &= \frac{1}{4}\Big\{-2 + (1/\beta_n)\sin(2\beta_n) + (1/\alpha_n)\sinh(2\alpha_n) - \\
&\quad - \frac{1}{\alpha_n^2+\beta_n^2}[\beta_n\cosh(2\alpha_n)\sin(2\beta_n) + \alpha_n\sinh(2\alpha_n)\cos(2\beta_n)]\Big\} \\
I_2 &= \frac{1}{4(\alpha_n^2+\beta_n^2)}[\alpha_n\cosh(2\alpha_n)\sin(2\beta_n) - \beta_n\sinh(2\alpha_n)\cos(2\beta_n)] \\
I_3 &= \frac{2}{\alpha_n^2+\beta_n^2}(\alpha_n\cosh\alpha_n\sin\beta_n - \beta_n\sinh\alpha_n\cos\beta_n) \\
I_4 &= \frac{1}{4}\Big\{2 + (1/\beta_n)\sin(2\beta_n) + (1/\alpha_n)\sinh(2\alpha_n) + \\
&\quad + \frac{1}{\alpha_n^2+\beta_n^2}[\beta_n\cosh(2\alpha_n)\sin(2\beta_n) + \alpha_n\sinh(2\alpha_n)\cos(2\beta_n)]\Big\} \\
I_5 &= \frac{2}{\alpha_n^2+\beta_n^2}(\alpha_n\sinh\alpha_n\cos\beta_n + \beta_n\cosh\alpha_n\sin\beta_n)
\end{aligned}\right\} . \quad (40)$$

The primed terms A'_n, B'_n, C'_n, E'_n and K' are obtained by replacing the unprimed $(\alpha_n, \beta_n, a/b)$ by $(\alpha'_n, \beta'_n, b/a)$.

From equation (31) and (32) it can be seen that the convergence of the generated solution will depend upon the convergence of the parameters α_n, β_n, α'_n, β'_n, f_{n0}, and g_{n0}. In order to study the convergence of the iterative process these parameters were evaluated for $a/b = 1.0$, 1.5, 2.0, 3.0, 5.0, 10.0 and the values for $a/b = 1.0$, 2.0 and 5.0 obtained after each iteration are presented in the following tables.

It can be seen that α_n, β_n, α_n', β_n', f_{n0}, and g_{n0} do converge for each of the ratios a/b to specific values and that in each case, even for long plates, the convergence is extremely rapid.

From the above it can be concluded that the final form of the one term approximation is

$$\begin{aligned}
w_\infty(x, y) = f\frac{q\,b^4}{D}\,&[K_1\sinh(\alpha\,x/a)\sin(\beta\,x/a) + \\
&+ K_2\cosh(\alpha\,x/a)\cos(\beta\,x/a) + K_0]\cdot \\
&\cdot [K'_1\sinh(\alpha'\,y/b)\sin(\beta'\,y/b) + \\
&+ K'_2\cosh(\alpha'\,y/b)\cos(\beta'\,y/b) + K'_0]
\end{aligned} \quad (41)$$

with the final values of the parameters for various ratios a/b, given in the following table.

Discussion of Results

In order to check the accuracy of the generated solution the deflection expression, equ. (41), was evaluated numerically along the x and y axes for various ratios a/b. These results are compared with those of other investigators [6], [7] in Fig. 2 and Fig. 3. The exact solution for the

$a/b = 1.0$

n	KANTOROVICH Method 0	1	2	3	4
α_n'	—	2.07912	2.07913	2.07913	2.07913
β_n'	—	1.20622	1.20620	1.20620	1.20620
$g_{n0}\, D/(q\, a^4)$	—	$0.20668 \cdot 10^{-3}$	$0.18715 \cdot 10^{-3}$	$0.18715 \cdot 10^{-3}$	$0.18715 \cdot 10^{-3}$
α_{n+1}	2.07515	2.07913	2.07913	2.07913	2.07913
β_{n+1}	1.14291	1.20620	1.20620	1.20620	1.20620
$f_{(n+1)0}\, D/(q\, b^4)$	$0.22035 \cdot 10^{-2}$	$0.18716 \cdot 10^{-3}$	$0.18715 \cdot 10^{-3}$	$0.18715 \cdot 10^{-3}$	$0.18715 \cdot 10^{-3}$

$a/b = 2.0$

n	KANTOROVICH Method 0	1	2	3	4
α_n'	—	1.14598	1.14662	1.14661	1.14662
β_n'	—	0.80076	0.80156	0.80155	0.80156
$g_{n0}\, D/(q\, a^4)$	—	$0.35100 \cdot 10^{-5}$	$0.34692 \cdot 10^{-5}$	$0.34692 \cdot 10^{-5}$	$0.34692 \cdot 10^{-5}$
α_{n+1}	4.15030	4.14784	4.14784	4.14782	4.14782
β_{n+1}	2.28582	2.30824	2.30826	2.30826	2.30826
$f_{(n+1)0}\, D/(q\, b^4)$	$0.58002 \cdot 10^{-3}$	$0.55749 \cdot 10^{-4}$	$0.55749 \cdot 10^{-4}$	$0.55749 \cdot 10^{-4}$	$0.55749 \cdot 10^{-4}$

$a/b = 5.0$

n	KANTOROVICH Method 0	1	2	3	4
α_n'	—	0.75539	0.75567	0.75568	0.75568
β_n'	—	0.63407	0.63431	0.63432	0.63432
$g_{n0}\, D/(q\, a^4)$	—	$0.11712 \cdot 10^{-11}$	$0.12151 \cdot 10^{-11}$	$0.11708 \cdot 10^{-11}$	$0.11708 \cdot 10^{-11}$
α_{n+1}	10.37575	10.37330	10.37350	10.37350	10.37350
β_{n+1}	5.71455	5.72845	5.72835	5.72835	5.72835
$f_{(n+1)0}\, D/(q\, b^4)$	$0.88606 \cdot 10^{-7}$	$0.75769 \cdot 10^{-9}$	$0.74019 \cdot 10^{-9}$	$0.74019 \cdot 10^{-9}$	$0.74019 \cdot 10^{-9}$

a/b	1.0	1.5	2.0	3.0	5.0	10.0
α	2.07913	3.11061	4.14782	6.22320	10.37350	20.73500
β	1.20620	1.74642	2.30826	3.44538	5.72835	11.44160
α'	2.07913	1.42408	1.14661	0.92511	0.75608	0.60170
β'	1.20620	0.90272	0.80155	0.72060	0.63432	0.53565
f	$0.18715 \cdot 10^{-3}$	$0.18213 \cdot 10^{-3}$	$0.55507 \cdot 10^{-4}$	$0.17202 \cdot 10^{-5}$	$0.73172 \cdot 10^{-9}$	$0.12446 \cdot 10^{-17}$
K_0	$0.19975 \cdot 10^{2}$	$0.21921 \cdot 10^{3}$	$0.23098 \cdot 10^{4}$	$0.21907 \cdot 10^{6}$	$0.14665 \cdot 10^{10}$	$0.29282 \cdot 10^{19}$
K_1	$-0.16586 \cdot 10$	$-0.25412 \cdot 10^{2}$	$-0.14233 \cdot 10^{3}$	$-0.12375 \cdot 10^{4}$	$0.18936 \cdot 10^{6}$	$0.97470 \cdot 10^{10}$
K_2	$-0.95819 \cdot 10$	$-0.31008 \cdot 10^{2}$	$-0.48076 \cdot 10^{2}$	$0.12984 \cdot 10^{4}$	$0.95325 \cdot 10^{4}$	$0.69666 \cdot 10^{10}$
K'_0	$0.19975 \cdot 10^{2}$	$0.45738 \cdot 10$	$0.25380 \cdot 10$	$0.15764 \cdot 10$	$0.10446 \cdot 10$	0.67002
K'_1	$-0.16586 \cdot 10$	0.16894	0.13073	$0.44930 \cdot 10^{-1}$	$0.16556 \cdot 10^{-1}$	$0.60655 \cdot 10^{-2}$
K'_2	$-0.95819 \cdot 10$	$-0.35507 \cdot 10$	$-0.22162 \cdot 10$	$-0.14663 \cdot 10$	$-0.10058 \cdot 10$	-0.65859

clamped infinitive strip [8]

$$w(y) = \frac{q\,b^4}{24\,D}\left(1 - 2\,\frac{y^2}{b^2} + \frac{y^4}{b^4}\right) \tag{42}$$

was also numerically evaluated and the corresponding graph is shown in Fig. 3. It can be seen that within the accuracy of the used scale, there

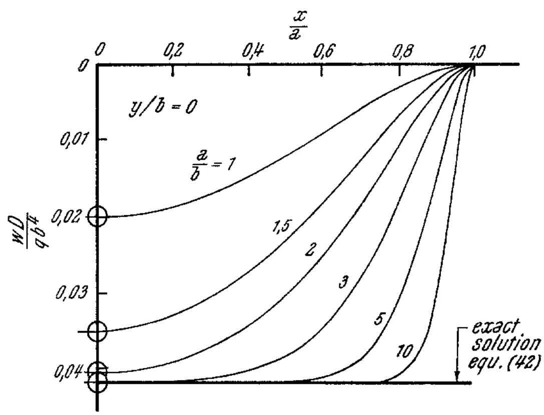

Fig. 2. Deflections. ——— equ. (41), ⊕ CZERNY [6] and EVANS [7]

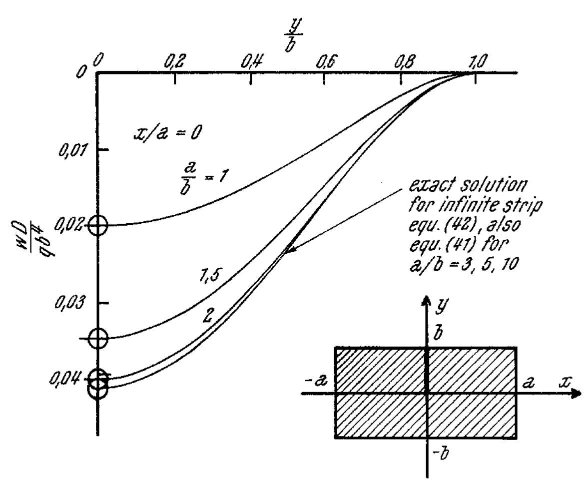

Fig. 3. Deflections. ——— equ. (41), ⊕ CZERNY [6] and EVANS [7]

is complete agreement, throughout the entire plate region, of the generated solution with those of other investigators.

A more severe test is the comparison of the moments and shearing forces which are obtained as higher derivatives of the deflection expression as follows:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right); \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right) \tag{43}$$

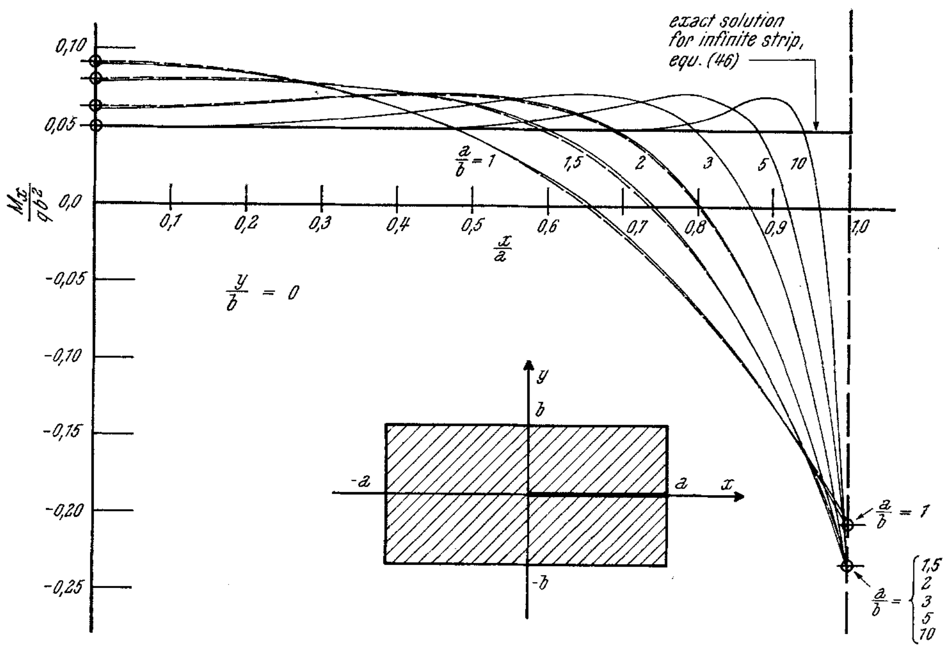

Fig. 4. Bending Moments. ——— Obtained from equ. (41), ... CZERNY [6], ⊕ EVANS [7]

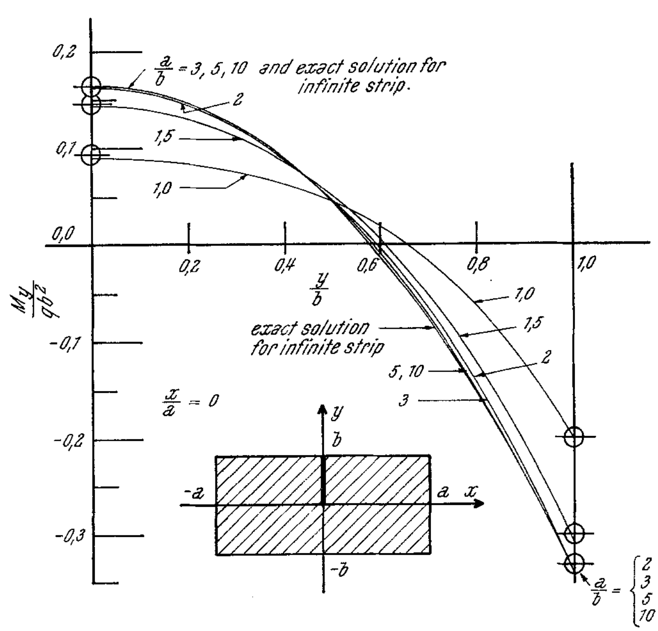

Fig. 5. Bending Moments. ——— Obtained from equ. (41), ⊕ EVANS [7]

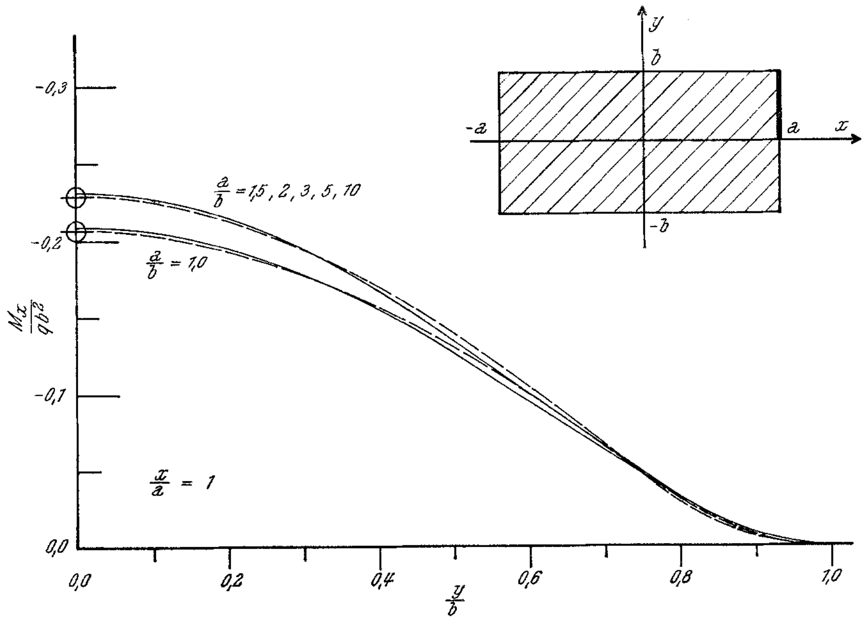

Fig. 6. Bending Moments. ——— Obtained from equ. (41), ... CZERNY [6], ⊕ EVANS [7]

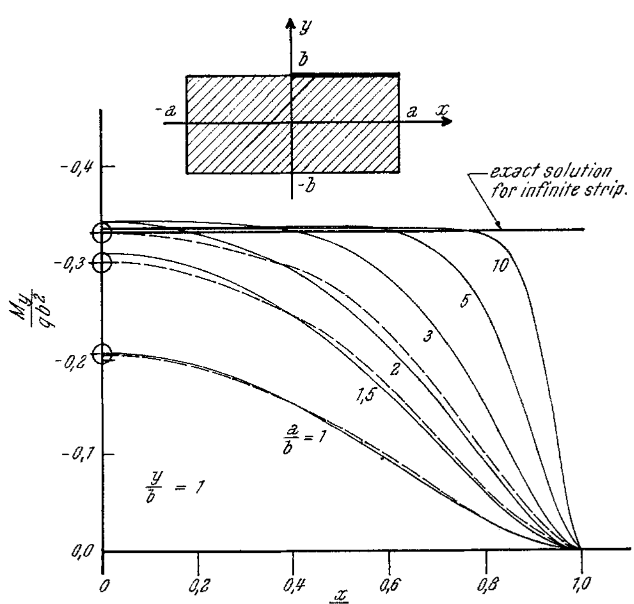

Fig. 7. Bending Moments. ——— Obtained from equ. (41), ... CZERNY [6], ⊕ EVANS [7]

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y} \tag{44}$$

$$Q_x = -D\frac{\partial}{\partial x}(\nabla^2 w); \quad Q_y = -D\frac{\partial}{\partial y}(\nabla^2 w). \tag{45}$$

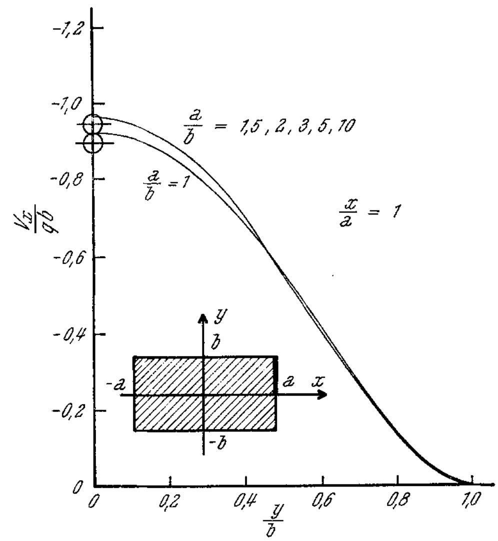

Fig. 8. Shearing forces. ------ Obtained from equ. (41), ⊕ CZERNY [6]

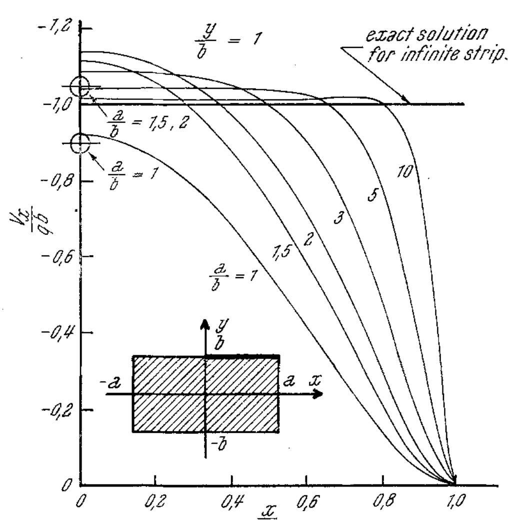

Fig. 9. Shearing forces. ——— Obtained from equ. (41), ⊕ CZERNY [6]

The expressions for the bending moments M_x and M_y were numerically evaluated for various ratios a/b along the x axis, along the y axis, as well as along the clamped boundary. These results are compared in Figs. 4 through 7 with some relevant results available in the literature [6—8], as well as with the exact bending moments for the clamped infinite strip

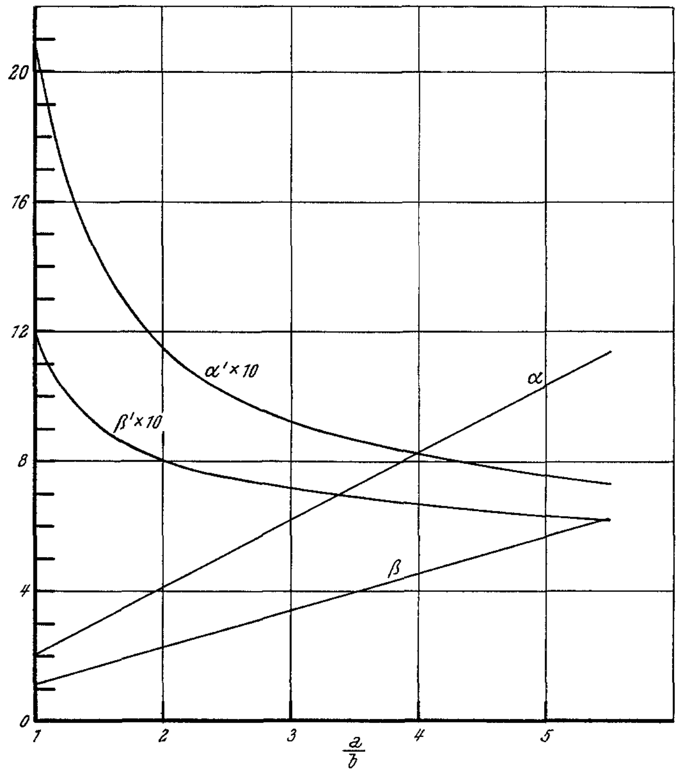
20
16
12
8
4
0
1
2
3
4
5
a/b
α'×10
β'×10
α
β

Fig. 10

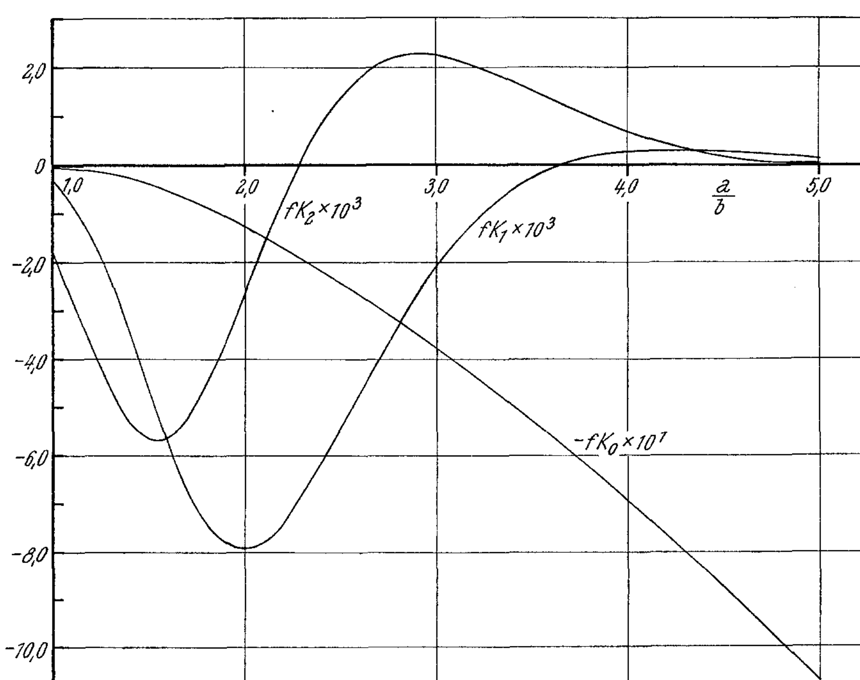
2,0
0
-2,0
-4,0
-6,0
-8,0
-10,0
1,0
2,0
3,0
4,0
a/b
5,0
fK2×10^3
fK1×10^3
-fK0×10^7

Fig. 11

obtained from equ. (42)

$$\left.\begin{aligned} M_x &= \frac{q\,b^2}{6}\,\nu\left(1 - 3\,\frac{y^2}{b^2}\right) \\ M_y &= \frac{q\,b^2}{6}\left(1 - 3\,\frac{y^2}{b^2}\right) \end{aligned}\right\}. \tag{46}$$

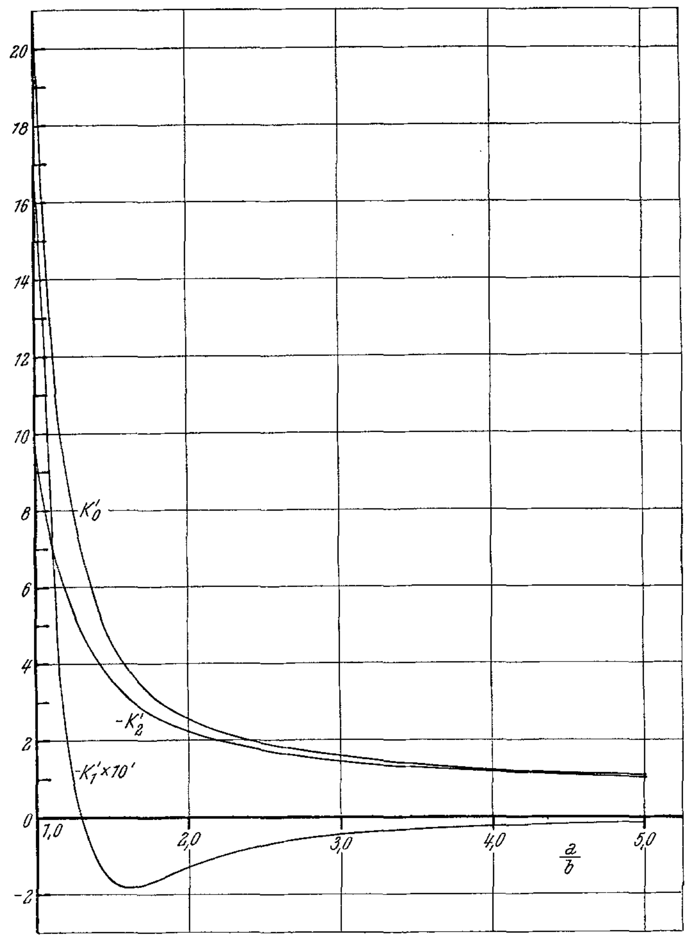

Fig. 12

Considering Fig. 4 it can be seen that for $\frac{a}{b} > 3$, the obtained bending moments approach, in the inner region of the plate, the exact value for the infinite strip $M_x/(q\,b^2) = 0.05$. It is of interest to note that the generated solution exhibits for increasingly long plates, let us say for $a/b > 5$, a distinct boundary layer phenomenon in the vicinity of $x = a$, as one could expect from an intuitive point of view.

In Fig. 5 it may be seen that with increasing a/b the generated solution approaches the exact solution for the infinite clamped strip. As expected, for long plates, these solutions coincide.

It can be seen that the agreement of the obtained moments with those obtained by other methods is, in general, close. Regarding the noticeable deviations in these graphs, it should be kept in mind that the results of the other investigators are also approximations. The same remark applies also to the deviations exhibited in Figures 8 and 9 where the shearing forces, which involve the third derivatives, are presented.

Because of the importance of the clamped plate in engineering practice, the lack of a simple exact solution in the technical literature, and the good agreement of the obtained numerical results with those obtained by other methods, it seems, that the simple closed form expression for $w(x, y)$ given in (41) is suitable, for most practical purposes, for the analysis of the deflections and stresses in a clamped rectangular plate subjected to a uniform lateral load.

Conclusions

The presented analysis shows that the final form of the generated solution is independent of the initial choice, and that the convergence of the suggested iterative procedure is very rapid.

It was found that the generated one term solution agrees very closely, throughout the plate, with corresponding results of other investigators. The closeness of the bending moments and shearing forces, which involve higher derivatives of the generated solution, with results of other investigators seems to be sufficient for most applications in engineering practice. In view of the lack of a closed form exact solution in the technical literature the coefficients occurring in the generated deflection expression (41) were evaluated for various plate side ratios and are presented in Fig. 10, Fig. 11, and Fig. 12 in order to simplify the utilization of equ. (41) in engineering practice.

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