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Eshelby's stress tensors in finite elastoplasticity

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Summary. This work examines critically the role that the Eshelby (energy-momentum) tensor or its degenerate form, the Mandel stress, should logically play as the driving force in an invariant formulation of the thermomechanics of finite-strain elasto-plasticity. Here the stress measure of which Mandel advocated the use in elastoplasticity, is shown to coincide, up to a sign, with the quasi-static Eshelby stress tensor expressed in the elastically released intermediate configuration. The various "constitutive" representations for the plastic rate are then discussed in terms of various thermodynamically conjugate pairs of "forces" and "velocities" for anisotropic materials.

1 Introduction

The identification of the *driving force* of anelasticity is central to a thermomechanical approach to this type of irreversible behavior. In classical small-strain plasticity, we all agree that this quantity is none other than the deviatoric part of the Cauchy stress; but this is some kind of degeneracy. The answer is much less clear cut and requires some *critical* evaluation in the case of finite-strain elastoplasticity. If we accept the viewpoint expressed by Epstein and Maugin [1], [2] in their theory of uniformity and homogeneity of materials, then anelasticity is one possible manifestation of a *local structural rearrangement*. Accordingly, the *Eshelby stress tensor* (originally called *energy-momentum tensor* by J. D. Eshelby – cf. Maugin [3], [4]) should be, in the appropriate form, the *driving* force behind finite-strain anelasticity because such a (material) tensor indeed is thermodynamical dual to the Noll-Epstein *uniformity mapping* (up to a sign). This viewpoint was implemented in Maugin [5] with a view to justifying the expression of path-integrals in elastoplasticity by means of the theory of so-called "material forces". But that does not solve the real problem which needs a closer critical look.

The present paper has indeed for purpose to re-examine the general finite-strain framework of elasto-plasticity in the light of the above-recalled conceptual vision, but also in critical comparison with some more classical approaches, in particular that of Mandel [6], and the general view presented in Cleja-Tigoiu and Soós [7]. That is, starting with the finite elasto-plasticity as now presented in textbooks and reviews by Lubliner [8], Maugin [9, Chapter 8], and Cleja-Tigoiu and Soós [7], and based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts due to Lee and Liu [10], with further elaboration by Mandel [6], we are led to emphasizing the role played by Eshelby's types of stress tensors. These are built either following Maugin [5] by duly accounting for the pseudo-dissipation function, or using the original definition of Epstein and Maugin [1] and identifying their *uniformity map* and the inverse plastic deformation (and therefore identifying the elastically-released configuration to the crystal reference of Epstein and Maugin). The connections between these various tensors are investigated as well as their invariance with respect to a change of frame in the actual configuration (objectivity). The relationship with Mandel's stress is established, and the thermodynamical admissibility of such "driving forces" is assessed. Some results in Maugin [5] only for small strains are thus generalized to the finite-strain framework (Eq. (21) in particular; such an equation will play a role in elastoplastic fracture). The paper concludes with a critical discussion of plastic-evolution equations in the light of the newly introduced conjugacies between pairs of stress tensors and plastic rates. Although some readers may think that is one more contribution to an endless discussion – or a settled matter (depending on the reader) –, we believe that it helps one to recognize the definite role played by the Mandel stress – or quasi-static Eshelby stress – in finite-strain elastoplasticity.

2 Pseudo-potential

The following notations, essentially employed in Maugin [5], will be used:

$\mathcal{K}_R$	– the reference configuration,
$\mathcal{K}_{\mathcal{R}}$	– the intermediate or relaxed (stress free) configuration,
$\mathbf{T} = J_{\mathbf{F}}\sigma\mathbf{F}^{-T}$	– the first Piola-Kirchhoff stress tensor in $\mathcal{K}_R$,
σ	– the symmetric Cauchy stress tensor,
$\mathbf{F} = \partial\chi/\partial\mathbf{X}$	– the gradient of the motion, χ with $J_{\mathbf{F}} = \det\mathbf{F} > 0$,
$\mathbf{v} = \partial\chi/\partial t$	– the velocity at the point $\mathbf{X}$ and at time t,
$\varrho_o(\mathbf{X})$	– the reference mass density,
$\mathbf{p}_R = \varrho_o(\mathbf{X})\mathbf{v}(\mathbf{X}, t)$	– the linear momentum per unit volume of $\mathcal{K}_R$,
$(A)_S = \frac{1}{2}(A + A^T)$	– the symmetric part of the tensor A,
$(A)_A = \frac{1}{2}(A - A^T)$	– the skew-symmetric part of the tensor A,
$\mathbf{I}_R, \mathbf{I}_{\mathcal{R}}$	– identity tensors,
Lin	– the set of all second-order tensors,
Sym	– the set of all symmetric tensors of Lin,
$\mathbf{A}\cdot\mathbf{B} := \text{tr}\,\mathbf{A}\mathbf{B}^T$	– the inner product of $\mathbf{A}, \mathbf{B} \in Lin$

In the presence of a dissipative mechanical behaviour, but in the absence of thermal conduction, there hold Cauchy's equations of motion of the body,

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{p}_R - \text{div}_R\mathbf{T} = 0\,, \\ &\mathbf{T}\mathbf{F}^T = \mathbf{F}\mathbf{T}^T, \end{aligned} \tag{1}$$

and the second law of thermodynamics written in the form of the Clausius-Duhem inequality

$$-(\dot{W} + N\dot{\theta}) + \text{tr}\,\mathbf{T}^T\dot{\mathbf{F}} \geq 0\,, \tag{2}$$

all of them being considered with respect to the reference configuration $\mathcal{K}_R$. Here N and W denote, respectively, the entropy and free energy densities per unit volume at $\mathcal{K}_R$.

In the subsequent developments we consider the framework of elastic-plastic materials with intermediate (or relaxed, or stress-free) configurations, $\mathcal{K}_{\mathcal{R}}$, and internal variables[1], considered to be a set of tensors and scalars (depending on the attached physical significance), denoted by $\bar{\alpha}$, and represented as a vector in $\mathbf{R}^n$.

We admit the *multiplicative decomposition* of $\mathbf{F}$ into elastic $\mathbf{F}^e$ and anelastic (or plastic) parts $\mathbf{F}^p$. Therefore

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p\,, \quad J_{\mathbf{F}} = J^e J^p\,, \quad J^e = \det \mathbf{F}^e\,, \quad J^p = \det \mathbf{F}^p\,, \quad J_{\mathbf{F}} = \det \mathbf{F} > 0\,. \tag{3}$$

Hypothesis: We accept *the existence* of the free energy density, ψ, per unit volume of $\mathcal{K}_{\mathcal{R}}$, as a function of $\mathbf{F}^e = \mathbf{F}(\mathbf{F}^p)^{-1}$, $\bar{\alpha}$, θ, which is a constitutive assumption (see Mandel [6], Teodosiu and Sidoroff [11], Cleja-Tigoiu and Soós [7], Maugin [9], and so on) in the mentioned framework. Consequently the following relationships:

$$\begin{aligned} W &= J^p\psi(\mathbf{F}^e, \bar{\alpha}, \theta; \mathbf{X}) \\ &\equiv \bar{W}(\mathbf{F}^e, \alpha, \theta; \mathbf{X}) = \tilde{\psi}(\mathbf{F}, (\mathbf{F}^p)^{-1}, \bar{\alpha}, \theta; \mathbf{X}) \end{aligned} \tag{4.1 – 2}$$

with $\alpha = (\bar{\alpha}, J^p)$ or $\alpha = (\bar{\alpha}, \varrho_{\mathcal{R}})$

hold, since the mass densities in $\mathcal{K}_{\mathcal{R}}$ and J^p are related by $\varrho_{\mathcal{R}} = \varrho_0 (J^p)^{-1}$. Obviously, when the plastic incompressibility is accepted, i.e., $J^p = 1$, then α and $\bar{\alpha}$ are equal.

The following consequences of (2):

$$N = -\frac{\partial \bar{W}}{\partial \theta}\,, \qquad \mathbf{T} = \left(\frac{\partial \bar{W}}{\partial \mathbf{F}^e}\right)(\mathbf{F}^p)^{-T} \tag{5}$$

and the reduced inequality

$$\mathrm{tr}\,(\mathbf{T}^T\mathbf{F}^e\dot{\mathbf{F}}^p) + \mathcal{A}\cdot\dot{\alpha} \geq 0\,, \quad \text{with} \quad \mathcal{A} = -\frac{\partial \bar{W}}{\partial \alpha} \tag{6}$$

were derived in Maugin [5].

It was further assumed *the existence of a pseudo-potential of dissipation* $\mathcal{D}_p$, such that $\mathcal{D}_p = \mathcal{D}_p(\dot{\mathbf{F}}^p, \dot{\alpha}; \mathbf{X})$, which satisfied the following relations:

$$(\mathbf{F}^e)^T\mathbf{T} = \frac{\partial \mathcal{D}_p}{\partial \dot{\mathbf{F}}^p}\,, \qquad \mathcal{A} = \frac{\partial \mathcal{D}_p}{\partial \dot{\alpha}}\,. \tag{7}$$

On account of (4.2) the equivalent relationships

$$(\mathbf{F}^e)^T\frac{\partial \bar{W}}{\partial \mathbf{F}^e} = \left(\frac{\partial \mathcal{D}_p}{\partial \dot{\mathbf{F}}^p}\right)(\mathbf{F}^p)^T\,, \qquad -\frac{\partial \bar{W}}{\partial \alpha} = \frac{\partial \mathcal{D}_p}{\partial \dot{\alpha}} \tag{8.1 – 2}$$

hold.

Remark 1. The relation (8.1) apparently imposes that $\bar{W}$ involves the dependence on $\mathbf{F}^p$(not only through the dependence on J^p, which was included in α), although $\bar{W}$ was considered to be a function of F^e and α, only.

In order to remedy this paradox we introduce here a *new* pseudo-potential of dissipation $\hat{\mathcal{D}}_p$, related to the old one by

$$\hat{\mathcal{D}}_p(\mathbf{L}^p, \dot{\alpha}; \mathbf{X}) = \mathcal{D}_p(\dot{\mathbf{F}}^p, \dot{\alpha}; \mathbf{X}) \quad \text{where} \quad \mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}\,. \tag{9}$$

[1] They are introduced with respect to the intermediate configurations.

The derivative chain rule applied in Eq. (9.1) with respect to $\dot{\mathbf{F}}^p$ leads to

$$\left(\frac{\partial \mathcal{D}_p}{\partial \mathbf{L}^p}\right)(\mathbf{F}^p)^{-T} = \frac{\partial \mathcal{D}_p}{\partial \dot{\mathbf{F}}^p}\,, \tag{10}$$

and consequently (8) becomes

$$(\mathbf{F}^e)^T \frac{\partial \bar{W}}{\partial \mathbf{F}^e} = \frac{\partial \hat{\mathcal{D}}_p}{\partial \mathbf{L}^p}\,, \qquad -\frac{\partial \bar{W}}{\partial \alpha} = \frac{\partial \hat{\mathcal{D}}_p}{\partial \dot{\alpha}}\,. \tag{11}$$

Hence, the explicit dependence of the pseudo-potential $\hat{\mathcal{D}}_p$ on $\mathbf{L}^p$ and $\dot{\alpha}$ can be derived in the form

$$\hat{\mathcal{D}}_p(\mathbf{L}^p, \dot{\alpha}; \mathbf{X}) = \operatorname{tr}\left\{\left(\frac{\partial \bar{W}}{\partial \mathbf{F}^e}\right)^T \mathbf{F}^e \mathbf{L}^p\right\} - \frac{\partial \bar{W}}{\partial \alpha} \cdot \dot{\alpha} + f(\mathbf{X})\,, \tag{12}$$

We add the physically reasonable hypothesis that $\hat{\mathcal{D}}_p(0, 0; \mathbf{X}) = 0$, i.e., the dissipation occurs only when in an elastico-plastic process there are nonzero rates of plastic deformation or of the internal variables. In such way we proved that

$$\mathcal{D}_p(\dot{\mathbf{F}}^p, \dot{\alpha}; \mathbf{X}) = \operatorname{tr}\left\{\left(\frac{\partial \bar{W}}{\partial \mathbf{F}^e}\right)^T \mathbf{F}^e \mathbf{L}^p\right\} - \frac{\partial \bar{W}}{\partial \alpha} \cdot \dot{\alpha}\,, \tag{13}$$

i.e. the pseudo-potential of dissipation $\mathcal{D}_p$ depends on $\dot{\mathbf{F}}^p$ through $\mathbf{L}^p = \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1}$ only.

Let us observe that another form of Eq. (13) can be derived as a consequence of the objectivity principle adopted in elasto-plasticity[2]:

Proposition 1. In the framework of elasto-plasticity there exists a pseudo-dissipation potential dependent on $\mathbf{L}^p$, $\dot{\alpha}$ as well as $\mathbf{C}^e$, α defined by

$$\mathcal{D}_p(\dot{\mathbf{F}}^p, \dot{\alpha}; \mathbf{X}) \equiv 2\operatorname{tr}\left\{\left(\frac{\partial \hat{W}}{\partial \mathbf{C}^e}\right)(\mathbf{C}^e \mathbf{L}^p)_s\right\} - \frac{\partial \bar{W}}{\partial \alpha} \cdot \dot{\alpha}\,,$$
$$\text{where} \quad \bar{W}(\mathbf{F}^e, \alpha; \mathbf{X}) = \hat{W}(\mathbf{C}^e, \alpha; \mathbf{X})\,, \qquad \mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e\,. \tag{14}$$

Remark 2. If we use the (Noll)-Epstein-Maugin [1] definition of the *material uniformity* we have the following two results in the case of *finite-strain elastoplasticity:*

Result 1. For a materially uniform body the energy potential per unit volume of the relaxed configuration is *necessarily a function* of $\mathbf{F}^e$ and α.

Proof. Let $W = \bar{W}(\mathbf{F}, \alpha; \mathbf{X})$ be the energy potential per unit volume of *the reference configuration K_R*. Then for a *materially uniform body* we can remove the $\mathbf{X}$-dependence by imposing that this dependence occurs only through the *uniformity mapping* $\mathbf{K}(\mathbf{X})$, such that (essentially a change of "material" frame)

$$\bar{W}(\mathbf{F}, \alpha; \mathbf{X}) = J_K{}^{-1} \tilde{W}(\mathbf{F}\mathbf{K}(\mathbf{X}), \alpha)\,. \tag{15}$$

But in the elastoplasticity of Mandel the *reference crystal* of Epstein and Maugin can be taken as the (preferred) relaxed configuration of Mandel (this has a special name; this is the *isoclinic configuration* related to a special director frame). Then we can select $\mathbf{K}$ as $(\mathbf{F}^p)^{-1}$. With $\mathbf{F}^e = \mathbf{F}(\mathbf{F}^p)^{-1}$, it follows from Eq. (15) the desired result:

$$\bar{W}(\mathbf{F}, \alpha; \mathbf{X}) = J^p \tilde{W}(\mathbf{F}^e, \alpha)\,. \tag{16}$$

[2] The objectivity assumption was explicitly formulated in Cleja-Tigoiu and Soós [7].

Result 2. For a materially uniform body the pseudo-potential of dissipation per unit volume of the relaxed configuration is *necessarily a function* of $\mathbf{L}^p$ and $\dot{\alpha}$.

Proof. The proof is similar to that of Result 1 because the condition of material uniformity for the scalar valued function $\mathcal{D}_p = \bar{\mathcal{D}}_p(\dot{\mathbf{F}}^p, \dot{\alpha}; \mathbf{X})$ – pseudo-dissipation potential in the reference configuration – reads

$$\bar{\mathcal{D}}_p(\dot{\mathbf{F}}^p, \dot{\alpha}; \mathbf{X}) = J_K{}^{-1}\tilde{\mathcal{D}}_p(\dot{\mathbf{F}}^p\mathbf{K}(\mathbf{X}), \dot{\alpha}) \,, \tag{17}$$

where $\mathbf{K}(\mathbf{X})$ is a uniformity map. In Mandel's theory we can select $\mathbf{K}$ as $(\mathbf{F}^p)^{-1}$. On account of the definition of $\mathbf{L}^p$, (17) yields at once

$$\bar{\mathcal{D}}_p(\dot{\mathbf{F}}^p, \dot{\alpha}; \mathbf{X}) = J^p\tilde{\mathcal{D}}_p(\dot{\mathbf{L}}^p, \dot{\alpha}) \,, \tag{18}$$

QED, where $\tilde{\mathcal{D}}_p$ is indeed the pseudo-dissipation potential in the intermediate configuration.

3 The effective Lagrangian function

In Maugin [5] the *unbalance* of pseudo-momentum in finite elasto-plasticity was established in the following form:

$$\frac{\partial}{\partial t}\mathcal{P} - div_R\hat{\mathbf{b}} = \hat{\mathbf{f}}^{inh} + \mathbf{f}^d \,, \tag{19}$$

where $\mathcal{P} = \varrho_o\mathbf{CV}$ – represents *the pseudo-momentum*, here $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and $\mathbf{v} + \mathbf{FV} = 0$. Here $\hat{\mathbf{b}}$ defines the *dynamic* Eshelby stress tensor with respect to the reference configuration $\mathcal{K}_R$, via

$$\hat{\mathcal{L}} = \mathcal{L} - \int_{t_o}^{t} \mathcal{D}_p \, dt, \qquad \mathcal{L} = \frac{1}{2}\varrho_o(\mathbf{X})\,\mathbf{v}^2 - \bar{W}(\mathbf{F}^e, \alpha; \mathbf{X}) \tag{20}$$

and

$$\hat{\mathbf{b}} \equiv -(\hat{\mathcal{L}}\mathbf{I}_R + \mathbf{F}^T\mathbf{T}) \,. \tag{21}$$

The "material forces" of true inhomogeneity $\hat{\mathbf{f}}^{inh}$ and plastic quasi-inhomogeneity $\mathbf{f}^d$ in the right-hand side of (19) play a role in fracture studies, but their exact expressions need not be recalled here for they are irrevelevant to our analysis.

Here we prove the following result:

$$\hat{\mathcal{L}} = \frac{1}{2}\varrho_o\mathbf{v}^2 - \int_{t_o}^{t} \mathrm{tr}\,\{\mathbf{T}^T\dot{\mathbf{F}}\}\, dt \,, \tag{22}$$

which is an extension to finite deformation of the result obtained by Maugin [5] (see formula (62)) in the small-strain approximation.

Taking into account the expression (13) for the pseudo-potential, and the following formula:

$$\dot{\mathbf{F}}(\mathbf{F}^p)^{-1} = \dot{\mathbf{F}}^e + \mathbf{F}^e\mathbf{L}^p \tag{23}$$

derived from (3.1), we can compute

$$\hat{\mathcal{L}} = \frac{1}{2}\varrho_o \mathbf{v}^2 - \int_{t_o}^{t} \left(\mathrm{tr}\left\{ \left(\frac{\partial \bar{W}}{\partial \mathbf{F}^e}\right)^T \dot{\mathbf{F}}^e \right\} + \frac{\partial \bar{W}}{\partial \alpha}\cdot \dot{\alpha} \right) dt - \int_{t_o}^{t} \mathcal{D}_p \, dt$$

$$= \frac{1}{2}\varrho_o \mathbf{v}^2 - \int_{t_o}^{t} \mathrm{tr}\left\{ (\mathbf{F}^p)^{-1} \left(\frac{\partial \bar{W}}{\partial \mathbf{F}^e}\right)^T \dot{\mathbf{F}} \right\} dt . \tag{24}$$

We substitute from (5.2) into (24), and (22) follows at once.

Proposition 2. In finite elasto-plasticity $\hat{\mathbf{b}}$ – the dynamic Eshelby stress tensor in $\mathcal{K}_R$, becomes

$$\hat{\mathbf{b}} = \left(\int_{t_o}^{t} \mathrm{tr}\,\{\mathbf{T}^T \dot{\mathbf{F}}\}\, dt - \frac{1}{2}\varrho_o \mathbf{v}^2 \right) \mathbf{I}_R - J^p (\mathbf{F}^p)^T (\mathbf{C}^e S_{\mathcal{R}}) (\mathbf{F}^p)^{-T} \tag{25}$$

since

$$\begin{aligned} &\mathbf{F}^T\mathbf{T} = J^p(\mathbf{F}^p)^T(\mathbf{C}^e S_{\mathcal{R}})(\mathbf{F}^p)^{-T}, \quad \text{where} \\ &S_{\mathcal{R}} = J^e(\mathbf{F}^e)^{-1}\sigma(\mathbf{F}^e)^{-T} \end{aligned} \tag{26}$$

denotes the symmetric Piola-Kirchhoff stress tensor with respect to the intermediate configuration, $\mathcal{K}_{\mathcal{R}}$. Formula (25) is an extension to finite deformation of formula (63) from Maugin [5], given in the case of *small deformation* only.

Remark 3. As the kinetic energy $\mathbf{v}^2/2$ is an invariant in a uniformity mapping, the Epstein-Maugin definition of the Eshelby stress tensor in terms of the uniformity maps $\mathbf{K}$ can be extended to the dynamical case by setting:

$$\mathbf{b} := \frac{\partial \mathcal{L}}{\partial \mathbf{K}} \mathbf{K}^T , \tag{27}$$

where $\mathcal{L}$ is the "Lagrangian" per unit volume of the reference configuration

$$\mathcal{L} = \frac{1}{2}\varrho_o(\mathbf{X})\, \mathbf{v}^2 - W . \tag{28}$$

4 Eshelby's stress tensors

Now the *quasi-static* Eshelby stress tensor $\mathbf{b}_q$ is introduced, as in Epstein and Maugin [1], [2] by identifying their uniformity mapping $\mathbf{K}$ and the inverse plastic deformation $(\mathbf{F}^p)^{-1}$, with respect to the intermediate configuration $\mathcal{K}_{\mathcal{R}}$, by

$$\mathbf{b}_q = -\frac{\partial \tilde{\psi}}{\partial (\mathbf{F}^p)^{-1}} . \tag{29}$$

As straightforward calculation which makes use of (4.3), (5.2) and (3.4) leads to its expression through $\hat{\mathbf{b}}_q$ – the *quasi-static* Eshelby's stress tensor with respect to $\mathcal{K}_R$,

$$\begin{aligned} &\hat{\mathbf{b}}_q = \mathbf{b}_q(\mathbf{F}^p)^T \quad \text{which} \\ &\hat{\mathbf{b}}_q = \begin{cases} W\mathbf{I}_R - \mathbf{F}^T\mathbf{T} & \text{when} \quad J^p \neq 1 \\ -\mathbf{F}^T\mathbf{T} & \text{when} \quad J^p = 1 . \end{cases} \end{aligned} \tag{30}$$

When the first Piola-Kirchhoff stress tensor $\mathbf{T}$ is replaced from (26.2) by $S_{\mathcal{R}}$ – the symmteric Piola-Kirchhoff stress tensor relative to $\mathcal{K}_{\mathcal{R}}$, then

$$\hat{\mathbf{b}}_q = W\mathbf{I}_R - J^p(\mathbf{F}^p)^T\left(\mathbf{C}^e S_{\mathcal{R}}\right)(\mathbf{F}^p)^{-T} \quad \text{when} \quad J^p \neq 1\,, \tag{31}$$

and the relation between the dynamical (21) and quasi-static (31) Eshelby stress tensors in $\mathcal{K}_R$ can be established in the form

$$\hat{\mathbf{b}} = \hat{\mathbf{b}}_q + \left(\int\limits_{t_o}^{t} \mathcal{D}_p\, dt - \frac{1}{2}\varrho_o \mathbf{v}^2\right)\mathbf{I}_R \quad \text{when} \quad J^p \neq 1\,. \tag{32}$$

The introduction of the *dynamic* Eshelby's stress tensor in the intermediate configuration, $\hat{\mathbf{b}}_{\mathcal{R}}$,

$$\hat{\mathbf{b}}_{\mathcal{R}} = (J^p)^{-1}(\mathbf{F}^p)^{-T}(\hat{\mathbf{b}})(\mathbf{F}^p)^T \tag{33}$$

with Eq. (25) leads to

$$\hat{\mathbf{b}}_{\mathcal{R}} + \frac{1}{2}\varrho_{\mathcal{R}}\mathbf{v}^2\mathbf{I}_{\mathcal{R}} = (J^p)^{-1}\left(\int\limits_{t_o}^{t} \mathcal{D}_p\, dt + \bar{W}(\mathbf{F}^e, \alpha, \mathbf{X})\right)\mathbf{I}_{\mathcal{R}} - \mathbf{C}^e S_{\mathcal{R}}$$

$$\equiv \left(\int\limits_{t_o}^{t} \tilde{\mathcal{D}}_p(\dot{\mathbf{F}}^p, \dot{\alpha}; \mathbf{X})\, dt + \psi(\mathbf{F}^e, \bar{\alpha}; \mathbf{X})\right)\mathbf{I}_{\mathcal{R}} - \mathbf{C}^e S_{\mathcal{R}} \tag{34}$$

in the case of a plastically compressible body. In the last equality of (34) appear the pseudo-dissipation potential and the energy potential (see (4) and (18) in connection with (14)), both of them being referred to the intermediate configuration.

Remark 4. Based on the objectivity assumption it follows the invariance (with respect to a change of frame in the actual configuration) of the term contained in the left hand side of the previous relation, in which (14) was used, too. As a consequence of (34) it follows the *invariance* for $\hat{\mathbf{b}}_{\mathcal{R}} + 1/2\varrho_{\mathcal{R}}\mathbf{v}^2\mathbf{I}_{\mathcal{R}}$ and *not* for the dynamic Eshelby's stress tensor $\hat{\mathbf{b}}_{\mathcal{R}}$. A similar result can be derived form (32) in terms of the Eshelby tensor $\hat{\mathbf{b}}$, i.e., $\hat{\mathbf{b}} + 1/2\varrho_o\mathbf{v}^2\mathbf{I}_R = \hat{\mathbf{b}}_q + \left(\int_{t_o}^{t} \mathcal{D}_p\, dt\right)\mathbf{I}_R$ is invariant with respect to a change of frame in the actual configuration.

Remark 5. In the plastic incompressible case, *the push-forward* to the intermediate configuration of the quasi-static Eshelby stress tensor with respect to the reference configuration, $\hat{\mathbf{b}}_q$, can be derived from (33) with (30) and (26). It follows that

$$(\hat{\mathbf{b}}_q)_{\mathcal{R}} = -\mathbf{C}^e S_{\mathcal{R}} \equiv -\Sigma\,. \tag{35}$$

That is, in this last case the quasi-static Eshelby stress tensor with respect to $\mathcal{K}_{\mathcal{R}}$, up to a sign, *coincides* with Σ, the Mandel stress measure, a non-symmetric tensor, which plays a central role in the finite elasto-plastic models elaborated in [6], [11]–[15].

5 Constitutive representations

The *dissipative* nature of elastoplastic materials was expressed by the Clausius-Duhem inequality, which leads to the following equivalent reduced inequalities written either with respect to $\mathcal{K}_{\mathcal{R}}$:

$$\begin{aligned} &\operatorname{tr}(\Sigma^T\mathbf{L}^p) + \mathcal{A}\cdot\dot{\alpha} \geq 0 \iff -\operatorname{tr}\{(\hat{\mathbf{b}}_{\mathcal{R}} + \hat{\mathcal{L}}_{\mathcal{R}}\mathbf{I}_{\mathcal{R}})^T\mathbf{L}^p\} + \mathcal{A}\cdot\dot{\alpha} \geq 0 \\ &\text{with} \quad \mathcal{A} = -(J^p)^{-1}\frac{\partial\bar{W}}{\partial\alpha}\,, \qquad \hat{\mathcal{L}}_R = (J^p)^{-1}\hat{\mathcal{L}}\,, \end{aligned} \tag{36}$$

$$\mathrm{tr}\,(S_{\mathcal{R}}\mathbf{D}_{\mathcal{R}}{}^{p}) + \mathcal{A}\cdot\dot{\alpha} \geq 0 \quad \text{with} \quad \mathbf{D}_{\mathcal{R}}{}^{p} = (\mathbf{C}^{e}\mathbf{L}^{p})_{S}\,, \tag{37}$$

or in $\mathcal{K}_t$ – the current configuration:

$$\begin{aligned} &\mathrm{tr}\,(\sigma\hat{\mathbf{D}}^{p}) + \mathcal{A}_t\cdot\dot{\alpha} \geq 0 \quad \text{where} \\ &\hat{\mathbf{D}}^{p} = \left(\mathbf{F}^{e}\mathbf{L}^{p}(\mathbf{F}^{e})^{-1}\right)_{S}, \qquad \mathcal{A}_t = -J_{F}{}^{-1}\frac{\partial\bar{W}}{\partial\alpha}\,. \end{aligned} \tag{38}$$

In (38) σ is a stress measure in the current configuration, while $\mathcal{A}_t$ as well as $\mathcal{A}$ are conjugated with $\dot{\alpha}$, both of them being internal variable sets in the relaxed configuration. The *push-forward* to the actual configuration of the internal tensorial variable $\mathcal{A}_t$[3] given by

$$\mathbf{a} = \mathbf{F}^{e}\mathcal{A}_t\mathbf{F}^{eT} \quad \text{or} \quad \mathcal{A}_t = (\mathbf{F}^{e})^{-1}\mathbf{a}\mathbf{F}^{e-T} \tag{39}$$

(while for the internal scalar variable the dependence of $\mathbf{F}^{e}$ appears through J^{e}, only) leads to an equivalent form of the reduced dissipation inequality (38), i.e.,

$$\mathrm{tr}\,(\sigma\hat{\mathbf{D}}^{p}) + \mathbf{a}\cdot(\mathbf{F}^{e})^{-T}\dot{\alpha}(\mathbf{F}^{e})^{-1} \geq 0\,. \tag{40}$$

Remark 6. From the *reduced* dissipation inequalities, written in $\mathcal{K}_{\mathcal{R}}$, it follows that $\Sigma = \mathbf{C}^{e}S_{\mathcal{R}}$, the Mandel stress measure, and $\mathbf{L}_p = \dot{\mathbf{F}}^{p}\mathbf{F}^{p-1}$ are *conjugate variables.*

On the other hand, when $\mathrm{tr}\,\mathbf{L}^{p} = 0$, i.e. the plastic incompressibility is considered, then the quasi-static Eshelby stress tensor referred to $\mathcal{K}_{\mathcal{R}}$, i.e. $(\hat{\mathbf{b}}_q)_{\mathcal{R}}$, is, up to a sign (see (29)), conjugate to the same plastic rate $\mathbf{L}^{p}$. But in the last case the quasi-static Eshelby stress tensor coincides with $-\Sigma$. This result is similar to those concerning the dynamic Eshelby stress tensor $\hat{\mathbf{b}}_{\mathcal{R}}$, established by Maugin [5].

Based on thermodynamic arguments, i.e. via reduced dissipation inequalities (involving the so-called intrinsic dissipation; cf. Maugin [9], for that notation) the following *thermodynamical dualities:*

$\Sigma := \mathbf{C}^{e}S_{\mathcal{R}}$;	$\mathbf{L}^{p} = \dot{\mathbf{F}}^{p}(\mathbf{F}^{p})^{-1}$	conjugate variables, from Eq. (36)
$(\hat{\mathbf{b}}_{\mathcal{R}}) + \hat{\mathcal{L}}_{\mathcal{R}}\mathbf{I}_{\mathcal{R}}$;	$\mathbf{L}^{p}$ when $\mathrm{tr}\,\mathbf{L}^{p} = 0$	conjugate variables
$(\hat{\mathbf{b}}_{\mathcal{R}})$ and $(\hat{\mathbf{b}}_q)_{\mathcal{R}}$;	$\mathbf{L}^{p}$ when $\mathrm{tr}\,\mathbf{L}^{p} = 0$	conjugate variables
$S_{\mathcal{R}}$;	$\mathbf{D}_{\mathcal{R}}{}^{p} = (\mathbf{C}^{e}\mathbf{L}^{p})_{S}$	conjugate variables, via Eq. (37)
σ;	$\mathbf{D}^{p} = \left(\mathbf{F}^{e}\mathbf{L}^{p}(\mathbf{F}^{e})^{-1}\right)_{S}$	are conjugate variables, via Eq. (38)

can be established between the different stress measures and their appropriate measures for the rate of plastic deformations, which effectively produce a dissipation. For instance, it was observed in Maugin [5], [9], that this kind of plastic spin $\mathbf{W}_{\mathcal{R}}{}^{p} = (\mathbf{C}^{e}\mathbf{L}^{p})_{A}$ does not produce any dissipation.

These sets will be considered here as a basis for the formulation of thermodynamically admissible evolution equations in *anisotropic* finite elasto-plasticity.

5.1 Elasto-plastic rate independent models

The following set of *plasticity-like* evolution equations for *the rate of plastic deformation* and for the set of internal variables can be considered in general forms, associated to the reduced inequalities, but bearing in mind their appropriate yield conditions. Finally, we shall complete the elastoplastic models with *the elastic type constitutive equations* derived from Eq. (5.2).

[3] In what follows we pay formally attention to the internal tensorial variables.

Model M1. In the first model we take Σ and $\mathbf{L}^p$ as *conjugate variables*, via (36.1). The *plasticity-like evolution equations*

$$\mathbf{L}_p = \mu_1 \frac{\partial f_1}{\partial \Sigma}(\Sigma, \mathcal{A}), \qquad \dot{\alpha} = \mu_1 \frac{\partial f_1}{\partial \mathcal{A}}(\Sigma, \mathcal{A}) \tag{41}$$

are associated to *the yield condition*

$$f_1(\Sigma, \mathcal{A}) = 0, \tag{42}$$

where

$$\mathcal{A} = -\frac{1}{J^p}\frac{\partial \hat{W}}{\partial \alpha}(\mathbf{C}^e, \alpha). \tag{43}$$

Here $\mu_1 \geq 0$ is the *plastic multiplier* obeying the conditions $f_1 \neq 0$, $\mu_1 f_1 = 0$. It will be determined from the consistency requirement $\mu_1 \dot{f}_1 = 0$.

The *elastic type* constitutive equation is given by

$$S_{\mathcal{R}} = \frac{2}{J^p}\frac{\partial \hat{W}}{\partial \mathbf{C}^e}(\mathbf{C}^e, \alpha) \tag{44}$$

as a consequence of (26.1) and (5.2), where $\hat{W}(\mathbf{C}^e, \alpha)$ replaces $\bar{W}$ via (14.2).

Taking into account the definition (35) for Σ and (44), the function $\hat{\Sigma}$ can be introduced by

$$\Sigma = \frac{2}{J^p}\hat{\Sigma}(\mathbf{C}^e, \alpha) := \frac{2}{J^p}\mathbf{C}^e\frac{\partial \hat{W}}{\partial \mathbf{C}^e}(\mathbf{C}^e, \alpha). \tag{45}$$

Hence the *symmetry condition* $\Sigma \mathbf{C}^e = \mathbf{C}^e \Sigma^T$ is fulfilled.

The *flow rule* (41.1) gives not only the *plastic stretching* $\mathbf{D}^p = (\mathbf{L}^p)_S$, but also the *plastic spin* $\mathbf{W}^p = (\mathbf{L}^p)_A$.

Model M2. In this model $S_{\mathcal{R}}$ and $\mathbf{D}_{\mathcal{R}}{}^p := (\mathbf{C}^e\mathbf{L}^p)_S$ are considered to be *conjugate variables*, referred also to the intermediate configuration.

We adopt the plasticity-like evolution equations (see Maugin [5])

$$\begin{aligned} &\mathbf{D}_{\mathcal{R}}{}^p = \mu_2 \frac{\partial f_2}{S_{\mathcal{R}}}(S_{\mathcal{R}}, \mathcal{A}) \quad \text{with} \quad \mathbf{D}_{\mathcal{R}}{}^p := (\mathbf{C}^e\mathbf{L}^p)_S \\ &\dot{\alpha} = \mu_2 \frac{\partial f_2}{\partial \mathcal{A}}(S_{\mathcal{R}}, \mathcal{A}) \end{aligned} \tag{46}$$

with $\mathcal{A}$ defined in Eq. (43) and μ_2 being the *plastic multiplier*.

In this case the *yield surface* is considered to be dependent on $S_{\mathcal{R}}$ and $\mathcal{A}$, i.e.,

$$f_2(S_{\mathcal{R}}, \mathcal{A}) = 0, \tag{47}$$

and the *elastic type constitutive equation* is prescribed by (44), as in the model M1.

Model M3. Now we formulate a possible model which involves the *conjugate variables* σ and $\mathbf{D}^p = \left(\mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}\right)_S$, referred to the actual configuration, via Eq. (38).

This time the *elastic type constitutive equation* is derived from (26), (5.2), (14.2) in the form

$$\sigma = \frac{2}{J^F}\mathbf{F}^e\frac{\partial \hat{W}}{\partial \mathbf{C}^e}(\mathbf{C}^e, \alpha)(\mathbf{F}^e)^T. \tag{48}$$

The internal tensorial variables "**a**", in the actual configuration, are expressed by

$$\mathbf{a} = -\frac{1}{J^F}\,\mathbf{F}^e\,\frac{\partial \hat{W}}{\partial \alpha}\,(\mathbf{C}^e, \alpha)\,(\mathbf{F}^e)^T \tag{49}$$

as it was mentioned in (39), along with (38.3).

In this case the *yield surface* is assumed to be dependent on σ and $\mathbf{a}$, i.e.,

$$f_3(\sigma, \mathbf{a}) = 0\,, \tag{50}$$

and the *flow rule* equations associated to (50) are derived in the form

$$\hat{\mathbf{D}}^p = \mu_3\,\frac{\partial f_3}{\partial \sigma}\,(\sigma, \mathbf{a})\,, \qquad (\mathbf{F}^e)^{-T}\dot{\alpha}(\mathbf{F}^e)^{-1} = \mu_3\,\frac{\partial f_3}{\partial \mathbf{a}}\,(\sigma, \mathbf{a}) \tag{51}$$

with the *plastic multiplier* μ_3 defined in an appropriate manner taking into account the consistency condition.

The form of the evolution equation (51.2) is a direct consequence of the definition (39), for the internal tensorial variables, which are conjugated (see (40)) with $(\mathbf{F}^e)^{-T}\dot{\alpha}(\mathbf{F}^e)^{-1}$.

The following *comments* are in order concerning the evolution equations (41), (46), (51):

(i) We remark that, as *a consequence of the objectivity principle* (see for instance Cleja-Tigoiu [16]), the first two models M1 and M2 are written in an invariant form with respect to a change of frame in the actual configuration, while the isotropy of the function $f_3(\sigma, \mathbf{a})$

$$f_3(\mathbf{Q}\sigma\mathbf{Q}^T, Q[\mathbf{a}]) = \mathbf{Q} f_3(\sigma, \mathbf{a})\,\mathbf{Q}^T$$

follows in the model M3, where $\mathbf{Q}$ is a proper orthogonal. When $\mathbf{a}$ represent internal scalar variables with respect to the actual configuration then $\mathbf{Q}[\mathbf{a}] = \mathbf{a}$, i.e., they are invariants, while for $\mathbf{a}$ a tensorial internal variable $\mathbf{Q}[\mathbf{a}] = \mathbf{Q}\mathbf{a}\mathbf{Q}^T$ holds.

(ii) By direct computation we get the following relationships between the *plastic rate* of deformations $\mathbf{D}_{\mathcal{R}}{}^p$ and $\hat{\mathbf{D}}^p$ defined in (37) and (38):

$$\mathbf{D}_{\mathcal{R}}{}^p = (\mathbf{F}^e)^T\,\hat{\mathbf{D}}^p\mathbf{F}^e\,, \tag{52}$$

and between their *appropriate plastic spins*

$$\begin{aligned} &\mathbf{W}_{\mathcal{R}}{}^p = (\mathbf{F}^e)^T\,\hat{\mathbf{W}}^p\mathbf{F}^e\,, \quad \text{where} \\ &\mathbf{W}_{\mathcal{R}}{}^p \equiv (\mathbf{C}^e\mathbf{L}^p)_A\,, \qquad \hat{\mathbf{W}}^p \equiv \left(\mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}\right)_A . \end{aligned} \tag{53}$$

Hence it follows the equivalence

$$\mathcal{W}_{\mathcal{R}}{}^p := (\mathbf{C}^e\mathbf{L}^p)_A = 0 \iff \hat{\mathbf{W}}^p := \left(\mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}\right)_A = 0\,. \tag{54}$$

(iii) The *local conservation law of mass in the intermediate configuration*, expressed by $\varrho_{\mathcal{R}} J^p = \varrho_o$, can be written as the equation of the continuity

$$\dot{\varrho}_{\mathcal{R}} + \varrho_{\mathcal{R}}\,\mathrm{tr}\,\mathbf{L}^p = 0 \tag{55}$$

in which $\mathrm{tr}\,\mathbf{L}^p$ is replaced by one of the formulae

$$\mathrm{tr}\,\mathbf{L}^p = \mathrm{tr}\,\hat{\mathbf{D}}^p = \mathrm{tr}\,(\mathbf{C}^e)^{-1}\,\mathbf{D}_{\mathcal{R}}{}^p\,, \tag{56}$$

in accordance with the constitutive assumptions made in the models M1 – M3.

Consequently, *the plastic incompressibility* can be characterized by one of the following equivalent conditions:

$$\operatorname{tr} \mathbf{L}^p = 0\,, \qquad \operatorname{tr} \hat{\mathbf{D}}^p = 0\,, \qquad \operatorname{tr}\left((\mathbf{C}^e)^{-1}\, \mathbf{D}_{\mathcal{R}}{}^p\right) = 0\,. \tag{57}$$

Now we pay attention to the determination of the plastic multiplier μ_j (with $j = 1, 2, 3$) from the consistency requirement $\mu_j \dot{f}_j = 0$, written on the yield surface $f_i = 0$, supposed to be a regular one.

Proposition 3. 1) In the elasto-plastic model M1, μ_1 is expressed by

$$\mu_1 H_1 = \mathbf{F}^e \left(\left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e} \right)^T \left[\frac{\partial f_1}{\partial \Sigma} \right] - \frac{1}{2} \frac{\partial^2 \hat{W}}{\partial \alpha \partial \mathbf{C}^e} \left[\frac{\partial f_1}{\partial \mathcal{A}} \right] \right) (\mathbf{F}^e)^T \cdot \mathbf{D}\,, \tag{58}$$

with the hardening parameter H_1

$$\begin{aligned} H_1 = &\left\{ \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e} \right)^T \left[\frac{\partial f_1}{\partial \Sigma} \right] - \frac{\partial^2 \hat{W}}{\partial \alpha \partial \mathbf{C}^e} \left[\frac{\partial f_1}{\partial \mathcal{A}} \right] \right\} \cdot \left(\mathbf{C}^e \frac{\partial f_1}{\partial \Sigma} \right)_S \\ &+ \frac{1}{4} \frac{\partial f_1}{\partial \mathcal{A}} \cdot \frac{\partial^2 \hat{W}}{\partial \alpha^2} \left[\frac{\partial f_1}{\partial \mathcal{A}} \right] + \frac{J^p}{4} \left(\operatorname{tr} \frac{\partial f_1}{\partial \Sigma} \right) \left(\Sigma \cdot \frac{\partial f_1}{\partial \Sigma} + \mathcal{A} \cdot \frac{\partial f_1}{\partial \mathcal{A}} \right). \end{aligned} \tag{59}$$

2) In (58), (59), $\left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e} \right)^T$ is the fourth-order tensor defined below:

$$\begin{aligned} &\left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e} \right)^T : Lin \to Sym, \quad \text{where} \\ &\left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e} \right)^T [\mathbf{B}] \cdot [\mathbf{A}] = \frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e} [\mathbf{A}] \cdot \mathbf{B} \equiv \left(\mathbf{A} \frac{\partial \hat{W}}{\partial \mathbf{C}^e} + \mathbf{C}^e \frac{\partial^2 \hat{W}}{\partial (\mathbf{C}^e)^2} [\mathbf{A}] \right) \cdot \mathbf{B} \end{aligned} \tag{60}$$

$\forall \mathbf{A} \in Sym, \mathbf{B} \in Lin.$

Proof. When we take into account the elastic type constitutive equation (44), the definitions (45) and (43) and evolution equations (41), as well as the relationships

$$\begin{aligned} &\dot{J}^p = J^p \operatorname{tr} \mathbf{L}^p\,, \\ &\dot{\mathbf{C}}^e = 2(\mathbf{F}^e)^T \mathbf{D} \mathbf{F}^e - 2(\mathbf{C}^e \mathbf{L}^p)_S\,, \qquad \text{where} \qquad \mathbf{D} = (\dot{\mathbf{F}} \mathbf{F}^{-1})_S\,, \end{aligned} \tag{61}$$

the derivative chain rule applied in (42) with respect to time t leads to (58) with (59).

Remark 7. Under the hypothesis that the *elastic function* from (44) is *independent of hardening*, i.e.,

$$\frac{\partial^2 \hat{W}}{\partial \alpha \partial \mathbf{C}^e} = 0\,, \tag{62}$$

the *geometrical interpretation* of $\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}$ follows at once. We pass to the yield function written in the *elastic strain space* (see Cleja-Tigoiu [16]) by the formula

$$f_1(\Sigma, \mathcal{A}) = f_1 \left(\frac{2}{J^p} \mathbf{C}^e \frac{\partial \hat{W}}{\partial \mathbf{C}^e} (\mathbf{C}^e, \alpha), \mathcal{A} \right) := \bar{f}_1(\mathbf{C}^e, \mathcal{A}, J^p)\,. \tag{63}$$

The *normal tensor* to the yield surface, prescribed by $\bar{f}_1$, has the representation

$$\mathbf{N}_1 := \frac{\partial \bar{f}_1}{\partial \mathbf{C}^e}(\mathbf{C}^e, \mathcal{A}, J^p) \equiv \frac{1}{J^p}\left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}\right)^T \left[\frac{\partial f_1}{\partial \Sigma}\right]. \tag{64}$$

Remark 8. Only the part of $\partial f_1/\partial \Sigma$ which does not belong to the kernel of $(\partial \hat{\Sigma}/\partial \mathbf{C}^e)^T$ enters the expression of the plastic multiplier.

On the other hand, $\mathbf{D}_{\mathcal{R}}{}^p$ from (41.1) is directed towards the outside of the yield surface $\bar{f}_1 = 0$ only when

$$\mathbf{D}_{\mathcal{R}}{}^p \cdot \frac{\partial \bar{f}_1}{\partial \mathbf{C}^e}(\mathbf{C}^e, \mathcal{A}, J^p) > 0 \Longleftrightarrow \left(\mathbf{C}^e \frac{\partial f_1}{\partial \Sigma}\right)_S \cdot \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}\right)^T \left[\frac{\partial f_1}{\partial \Sigma}\right] > 0\,. \tag{65}$$

Consequently, using the decomposition

$$\left(\mathbf{C}^e \frac{\partial f_1}{\partial \Sigma}\right)_S = \gamma \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}\right)^T \left[\frac{\partial f_1}{\partial \Sigma}\right] + \beta \mathbf{N}_{\mathcal{R}}{}^p \quad \text{where} \quad \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}\right)^T \left[\frac{\partial f_1}{\partial \Sigma}\right] \cdot \mathbf{N}_{\mathcal{R}}{}^p = 0\,, \tag{66}$$

with $\gamma > 0$, we get

$$\begin{aligned} H_1 = \gamma \left| \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}\right)^T \left[\frac{\partial f_1}{\partial \Sigma}\right] \right|^2 - \frac{\partial^2 \hat{W}}{\partial \alpha \partial \mathbf{C}^e}\left[\frac{\partial f_1}{\partial \mathcal{A}}\right] \cdot \left(\mathbf{C}^e \frac{\partial f_1}{\partial \Sigma}\right)_S \\ + \frac{1}{4}\frac{\partial f_1}{\partial \mathcal{A}} \cdot \frac{\partial^2 \hat{W}}{\partial \alpha^2}\left[\frac{\partial f_1}{\partial \mathcal{A}}\right] + \frac{J^p}{4}\left(\operatorname{tr} \frac{\partial f_1}{\partial \Sigma}\right)\left(\Sigma \cdot \frac{\partial f_1}{\partial \Sigma} + \mathcal{A} \cdot \frac{\partial f_1}{\partial \mathcal{A}}\right). \end{aligned} \tag{67}$$

Remark 9. In the *plastic incompressible case* (when $\operatorname{tr} \dfrac{\partial f_1}{\partial \Sigma} = 0$), under the hypothesis that the *elastic properties* are *independent of hardening*, i.e., (62) is fulfilled, the remarkable simplification in the expression of the plastic multiplier follows:

$$\begin{aligned} \mu_1 \bar{H}_1 &= \mathbf{F}^e \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}\right)^T \left[\frac{\partial f_1}{\partial \Sigma}\right] (\mathbf{F}^e)^T \cdot \mathbf{D} \\ \bar{H}_1 &= \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}\right)^T \left[\frac{\partial f_1}{\partial \Sigma}\right] \cdot \left(\mathbf{C}^e \frac{\partial f_1}{\partial \Sigma}\right)_S + \frac{1}{4}\frac{\partial f_1}{\partial \mathcal{A}} \cdot \frac{\partial^2 \hat{W}}{\partial \alpha^2}\left[\frac{\partial f_1}{\partial \mathcal{A}}\right] \end{aligned} \tag{68}$$

or

$$\bar{H}_1 = \gamma \left| \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e}\right)^T \left[\frac{\partial f_1}{\partial \Sigma}\right] \right|^2 + \frac{1}{4}\frac{\partial f_1}{\partial \mathcal{A}} \cdot \frac{\partial^2 \hat{W}}{\partial \alpha^2}\left[\frac{\partial f_1}{\partial \mathcal{A}}\right]. \tag{69}$$

Proposition 4. In model M2 the plastic multiplier μ_2 follows to be equal to

$$\mu_2 H_2 = \mathbf{F}^e \left(\frac{\partial^2 \hat{W}}{\partial (\mathbf{C}^e)^2}\left[\frac{\partial f_2}{\partial S_{\mathcal{R}}}\right] - \frac{1}{2}\frac{\partial^2 \hat{W}}{\partial \alpha \partial \mathbf{C}^e}\left[\frac{\partial f_2}{\partial \mathcal{A}}\right]\right)(\mathbf{F}^e)^T \cdot \mathbf{D}\,, \tag{70}$$

where

$$H_2 = \frac{J^p}{2}\left((\mathbf{C}^e)^{-1}\cdot\frac{\partial f_2}{\partial S_{\mathcal{R}}}\right)\left(S_{\mathcal{R}}\cdot\frac{\partial f_2}{\partial S_{\mathcal{R}}}+\mathcal{A}\cdot\frac{\partial f_2}{\partial \mathcal{A}}\right) + \left\{\frac{\partial^2\hat{W}}{\partial(\mathbf{C}^e)^2}\left[\frac{\partial f_2}{\partial S_{\mathcal{R}}}\right]-\frac{\partial^2\hat{W}}{\partial\alpha\partial\mathbf{C}^e}\left[\frac{\partial f_2}{\partial \mathcal{A}}\right]\right\}\cdot\frac{\partial f_2}{\partial S_{\mathcal{R}}}+\frac{1}{4}\frac{\partial f_2}{\partial \mathcal{A}}\cdot\frac{\partial^2\hat{W}}{\partial\alpha^2}\left[\frac{\partial f_1}{\partial \mathcal{A}}\right]. \tag{71}$$

The *proof* is similar to the previous one.

Remark 10. In the incompressible case (when $(\mathbf{C}^e)^{-1}\cdot\partial f_2/\partial S_{\mathcal{R}} = 0$) as it results from (57.3) with (46.2), under the supposition (62) the relation

$$\mu_2\bar{H}_2 = \mathbf{F}^e\left(\frac{\partial^2\hat{W}}{\partial(\mathbf{C}^e)^2}\right)^T\left[\frac{\partial f_2}{\partial S_{\mathcal{R}}}\right](\mathbf{F}^e)^T\cdot\mathbf{D} \tag{72}$$

holds. Here the hardening parameter is expressed by

$$\bar{H}_2 = \frac{\partial^2\hat{W}}{\partial(\mathbf{C}^e)^2}\left[\frac{\partial f_2}{\partial S_{\mathcal{R}}}\right]\cdot\frac{\partial f_2}{\partial S_{\mathcal{R}}}+\frac{1}{4}\frac{\partial f_2}{\partial \mathcal{A}}\cdot\frac{\partial^2\hat{W}}{\partial\alpha^2}\left[\frac{\partial f_1}{\partial \mathcal{A}}\right]. \tag{73}$$

5.2 Connections between models

To establish *a connection* between the model M2 and M3 we take into account (46) and (51), put together with (52) and (53).

We adopt the model M2, referred to the intermediate configuration, as a point of departure and we derive the model in the actual configuration.

The model M3 is associated to the yield function f_3 defined by

$$f_3 = (\sigma,\mathbf{a}) := f_2\big(J^e(\mathbf{F}^e)^{-1}\sigma(\mathbf{F}^e)^{-T},\quad J^e(\mathbf{F}^e)^{-T}\mathbf{a}(\mathbf{F}^e)^{-1}\big) \equiv f_2(S_{\mathcal{R}},\mathcal{A}), \tag{74}$$

due to the relationships between $(\sigma,\mathbf{a})$ and $(S_{\mathcal{R}},\mathcal{A})$

$$S_{\mathcal{R}} = J^e(\mathbf{F}^e)^{-1}\sigma(\mathbf{F}^e)^{-T}, \qquad \mathcal{A} = J^e(\mathbf{F}^e)^{-1}\mathbf{a}(\mathbf{F}^e)^{-T} \tag{75}$$

which yield, when we pass (via $\mathbf{F}^e$– the relative deformation gradient from $\mathcal{K}_{\mathcal{R}}$ to $\mathcal{K}_t$) from the intermediate configuration to the actual one. The relationships (75) are derived from (26) and (39), put together with (38.3) and (36.4).

Proposition 5. 1) The model M3, equivalently associated to M2, is characterized by the elastic constitutive equation (48) and the flow rule equations (51), associated to (74). Moreover, in the models

$$\begin{aligned}\frac{\partial f_3}{\partial\sigma}(\sigma,\mathbf{a}) &= J^e(\mathbf{F}^e)^{-T}\left(\frac{\partial f_2}{S_{\mathcal{R}}}(S_{\mathcal{R}},\mathcal{A})\right)(\mathbf{F}^e)^{-1},\\ \frac{\partial f_3}{\partial\mathbf{a}}(\sigma,\mathbf{a}) &= J^e(\mathbf{F}^e)^{-T}\left(\frac{\partial f_2}{\partial\mathcal{A}}(S_{\mathcal{R}},\mathcal{A})\right)(\mathbf{F}^e)^{-1}\end{aligned} \tag{76}$$

and $\mu_3 = (J^e)^{-1}\mu_2$.

2) The expression for μ_3 is specified by

$$\mu_3 H_3 = \left(\mathcal{L}\left[\frac{\partial f_3}{\partial\sigma}\right]-\frac{1}{2}\mathcal{M}\left[\frac{\partial f_3}{\partial\mathbf{a}}\right]\right)\cdot\mathbf{D}, \tag{77}$$

where

$$H_3 = \frac{J^F}{4}\left(\operatorname{tr}\frac{\partial f_3}{\partial \sigma}\right)\left(\sigma \cdot \frac{\partial f_3}{\partial \sigma} + \mathbf{a} \cdot \frac{\partial f_3}{\partial \mathbf{a}}\right)$$
$$+ \frac{\partial f_3}{\partial \sigma} \cdot \left(\mathcal{L}\left[\frac{\partial f_3}{\partial \sigma}\right] - \mathcal{M}\left[\frac{\partial f_3}{\partial \mathbf{a}}\right]\right) + \frac{1}{4}\frac{\partial f_3}{\partial \mathbf{a}} \cdot \mathcal{N}\left[\frac{\partial f_3}{\partial \mathbf{a}}\right]. \tag{78}$$

Here the fourth-order tensors are defined by

$$\mathcal{L}[\mathbf{A}] = \mathbf{F}^e\left(\frac{\partial^2 \hat{W}}{\partial (\mathbf{C}^e)^2}[(\mathbf{F}^e)^T \mathbf{A} \mathbf{F}^e]\right)(\mathbf{F}^e)^T,$$
$$\mathcal{M}[\mathbf{B}] = \mathbf{F}^e\left(\frac{\partial^2 \hat{W}}{\partial \alpha \partial \mathbf{C}^e}[(\mathbf{F}^e)^T \mathbf{B} \mathbf{F}^e]\right)(\mathbf{F}^e)^T, \tag{79}$$
$$\mathcal{N}[\mathbf{B}] = \mathbf{F}^e\left(\frac{\partial^2 \hat{W}}{\partial \alpha^2}[(\mathbf{F}^e)^T \mathbf{B} \mathbf{F}^e]\right)(\mathbf{F}^e)^T$$

$\forall \mathbf{A} \in Sym, \mathbf{B} \in R^n$ – the space to which the internal variables belong. All these tensorial fields are still dependent on the elastic deformation.

Proof. We differentiate in Eq. (74) with respect to σ and $\mathbf{a}$, and the equalities (76) follow at once. In Eq. (76) "$\mathbf{a}$" denotes a tensorial internal variable. When the scalar internal variables are considered in the models, $\mathbf{F}^e$ appears only through J^e.

On account of Eq. (52) and (53) the relation $\mu_3 = (J^e)^{-1}\mu_2$ yields, between μ_3 and μ_2 characterizing the model M3, associated to M1 and respectively, the model M1.

2) In order to specify the expression for μ_3, derived from the above one, we pass from the intermediate configuration to the actual one by using Eq. (76) in Eq. (70) and (71). A straitforward calculation leads to Eq. (77).

The connection between the models M1 and M3 is determined by the fact that Σ can be considered to determine the Cauchy stress σ through the local mapping $\mathbf{F}^e$, as a consequence of the relation

$$\Sigma = J^e(\mathbf{F}^e)^T \sigma (\mathbf{F}^e)^{-T}. \tag{80}$$

The *yield function* f_1 can be rewritten in an equivalent form in terms of the quantities referring to the actual configuration as

$$f_3(\sigma, \mathbf{a}) = f_1\left(J^e(\mathbf{F}^e)^T \sigma (\mathbf{F}^e)^{-T},\quad J^e(\mathbf{F}^e)^{-1}\mathbf{a}(\mathbf{F}^e)^{-T}\right) \equiv f_1(\Sigma, \mathcal{A}) \tag{81}$$

with the aid of the expressions (75) and (80).

Conversely, a yield function f_1 (in the relaxed configuration) can be associated with f_3 by the following representation:

$$f_1(\Sigma, \mathcal{A}) := f_3\left(\frac{1}{J^e}(\mathbf{F}^e)^{-T}\Sigma(\mathbf{F}^e)^T,\quad \frac{1}{J^e}\mathbf{F}^e \mathcal{A}(\mathbf{F}^e)^T\right) \equiv f_3(\sigma, \mathbf{a}). \tag{82}$$

Hence the following assertions can be proved.

Proposition 6. Let the model M1 be given.

1) The model M3, derived from M1, is described by the *elastic type constitutive equation* (48) and by the *flow rule equations*

$$\hat{\mathbf{D}}^p = \mu_3 \frac{\partial f_3}{\partial \sigma}(\sigma, \mathbf{a}), \qquad (\mathbf{F}^e)^{-T}\dot{\alpha}(\mathbf{F}^e)^{-1} = \mu_3 \frac{\partial f_3}{\partial \mathbf{a}}(\sigma, \mathbf{a}), \tag{83}$$

associated to the yield function introduced in (81), while the *nonzero plastic spin* $\hat{\mathbf{W}}^p$ is defined by

$$\hat{\mathbf{W}}^p = \mu_3 \left(\mathbf{F}^e \frac{\partial f_1}{\partial \Sigma}(\Sigma, \mathcal{A}) (\mathbf{F}^e)^{-1} \right)_A . \tag{84}$$

Here the arguments appearing in the right hand side of (84) are replaced by (80) and (75).

2) The *plastic multiplier* μ_3 is given by

$$\mu_3 H_3 = \frac{J^F}{2} \left[\left(\frac{\partial f_3}{\partial \sigma} \right) \sigma + \left(J^e \mathbf{F}^e \frac{\partial f_1}{\partial \Sigma} (\mathbf{F}^e)^{-1} \right)_A \sigma \right] \cdot \mathbf{D} + \left(\mathcal{L} \left[\frac{\partial f_3}{\partial \sigma} \right] - \frac{1}{2} \mathcal{M} \left[\frac{\partial f_3}{\partial \mathbf{a}} \right] \right) \cdot \mathbf{D} , \tag{85}$$

where

$$\begin{aligned} H_3 = {} & \frac{\partial f_3}{\partial \sigma} \cdot \left(\mathcal{L} \left[\frac{\partial f_3}{\partial \sigma} \right] - \mathcal{M} \left[\frac{\partial f_3}{\partial \mathbf{a}} \right] \right) + \frac{1}{4} \frac{\partial f_3}{\partial \mathbf{a}} \cdot \mathcal{N} \left[\frac{\partial f_3}{\partial \mathbf{a}} \right] \\ & + \frac{J^F}{2} \left[\frac{1}{2} \left(\operatorname{tr} \frac{\partial f_3}{\partial \sigma} \right) \left(\sigma \cdot \frac{\partial f_3}{\partial \sigma} + \mathbf{a} \cdot \frac{\partial f_3}{\partial \mathbf{a}} \right) + \frac{\partial f_3}{\partial \sigma} \sigma \cdot \frac{\partial f_3}{\partial \sigma} + \frac{\partial f_3}{\partial \sigma} \sigma \cdot J^e \left(\mathbf{F}^e \frac{\partial f_1}{\partial \Sigma} (\mathbf{F}^e)^{-1} \right)_A \right] . \end{aligned} \tag{86}$$

3) In the plastic *incompressible body*, under supposition (62), the expression for the *plastic multiplier* is still given by (85) and (86), in which

$$\mathcal{M} = 0 \qquad \text{and} \qquad \operatorname{tr} \frac{\partial f_3}{\partial \sigma} = 0 . \tag{87}$$

Proof. 1) In (81) we consider the differential with respect to σ – a symmetric tensor, hence

$$\frac{\partial f_3}{\partial \sigma} = J^e \left(\mathbf{F}^e \frac{\partial f_1}{\partial \Sigma} (\mathbf{F}^e)^{-1} \right)_S \tag{88}$$

holds. Using (52) and (53), put together with (41), as well as (88), we get (83.1) and (84). The evolution equation for internal variables, (83.2), follows from (46.2) (accepted in the model M1), put together with the formula

$$\frac{\partial f_3}{\partial \mathbf{a}} = J^e (\mathbf{F}^e)^{-T} \frac{\partial f_1}{\partial \mathcal{A}} (\mathbf{F}^e)^{-1} , \tag{89}$$

derived as a consequence of the derivative chain rule applied in (81) with respect to $\mathbf{a}$. On the other hand, the appropriate plastic multipliers in the models M3 (associated to M1) and, respectively, M1 are related by $\mu_3 = (J^e)^{-1} \mu_1$, as it follows from Eqs. (52) and (53), along with (41).

2) Then we invoke the relations

$$\begin{aligned} \left(\mathbf{C}^e \frac{\partial f_1}{\partial \Sigma} \right)_S &= (\mathbf{F}^e)^T \left(\mathbf{F}^e \frac{\partial f_1}{\partial \Sigma} (\mathbf{F}^e)^{-1} \right)_S \mathbf{F}^e , \\ \left(\mathbf{C}^e \frac{\partial f_1}{\partial \Sigma} \right)_A &= (\mathbf{F}^e)^T \left(\mathbf{F}^e \frac{\partial f_1}{\partial \Sigma} (\mathbf{F}^e)^{-1} \right)_A \mathbf{F}^e \end{aligned} \tag{90}$$

and the elastic type constitutive equation (48). Using the above relations, (60) can be equivalently rewritten in the form

$$\begin{aligned} \left(\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}^e} \right)^T \left[\frac{\partial f_1}{\partial \Sigma} \right] \cdot \left(\mathbf{C}^e \frac{\partial f_1}{\partial \Sigma} \right)_S = {} & \frac{1}{(J^e)^2} \frac{\partial f_3}{\partial \sigma} \cdot \mathcal{L} \left[\frac{\partial f_3}{\partial \sigma} \right] \\ & + \frac{J^F}{2(J^e)^2} \left[\frac{\partial f_3}{\partial \sigma} \sigma \cdot \frac{\partial f_3}{\partial \sigma} + \frac{\partial f_3}{\partial \sigma} \sigma \cdot J^e \left(\mathbf{F}^e \frac{\partial f_1}{\partial \Sigma} (\mathbf{F}^e)^{-1} \right)_A \right] . \end{aligned} \tag{91}$$

Taking into account the relations between $(\Sigma, \mathcal{A})$ and $(\sigma, \mathbf{a})$, as well as the formula (88), we obtain

$$\frac{J^p}{4}\operatorname{tr}\left(\frac{\partial f_1}{\partial \Sigma}\right)\left(\frac{\partial f_1}{\partial \Sigma}\cdot \Sigma+\frac{\partial f_1}{\partial \mathcal{A}}\cdot \mathcal{A}\right)=\frac{J^F}{4(J^e)^2}\left(\operatorname{tr}\frac{\partial f_3}{\partial \sigma}\right)\left(\sigma\cdot\frac{\partial f_3}{\partial \sigma}+\mathbf{a}\cdot\frac{\partial f_3}{\partial \mathbf{a}}\right). \tag{92}$$

By combining the preceding results, an algebraic manipulation leads to

$$H_1=\frac{1}{(J^e)^2}\,H_3$$

with H_3 given in (86), in terms of the quantities referring to the actual configuration.

Similarly, it can be proved that the expression which enters the right hand side of (58) is equal to

$$\frac{J^F}{2J^e}\left[\left(\frac{\partial f_3}{\partial \sigma}\right)\sigma+\left(J^e\mathbf{F}^e\frac{\partial f_1}{\partial \Sigma}(\mathbf{F}^e)^{-1}\right)_A\sigma\right]\cdot\mathbf{D}+\frac{1}{J^e}\left(\mathcal{L}\left[\frac{\partial f_3}{\partial \sigma}\right]-\frac{1}{2}\mathcal{M}\left[\frac{\partial f_3}{\partial \mathbf{a}}\right]\right)\cdot\mathbf{D}. \tag{93}$$

Consequently, the formula (85) follows when we fit together the above results, via the relations (58), (59).

3) In the plastic incompressible case it results that $\operatorname{tr}\,\partial f_3/\partial\sigma=0$, due to (57) and (88). From Eq. (62) with the definition of $\mathcal{M}$ (see (79)) it follows that $\mathcal{M}=0$. Hence the assertion in 3) results at once.

The following peculiar aspects can be put into evidence according to the *Proposition 6*:

Remark 11. In (85) and (86) the skew-symmetric part $(\mathbf{F}^e\,\partial f_1/\partial\Sigma\,(\mathbf{F}^e)^{-1})_A$ appears if and only if $(\partial f_3/\partial\sigma)\,\sigma$ is not a symmetric field, i.e., if $\partial f_3/\partial\sigma$ and σ are not permutable tensors (this happens in the kinematically hardening materials, for instance). Consequently, if $\hat{\mathbf{W}}^p\neq 0$ the *skew-symmetric tensor* which enters (85) influences the plastic stretching $\hat{\mathbf{D}}^p$ via the plastic multiplier.

Remark 12. $\hat{\mathbf{W}}^p=0$ if and only if $\left(\mathbf{F}^e\,\partial f_1/\partial\Sigma\,(\mathbf{F}^e)^{-1}\right)_A=0$, or equivalently (see (90.2)) if $\left(\mathbf{C}^e\,\partial f_1/\partial\Sigma\,(\Sigma,\mathcal{A})\right)_A=0$. We can make such an *additional supposition* in the model M3, and consequently in the model M1.

Hence, we emphasized that $\hat{\mathbf{W}}^p=0$ does not follow as a *direct consequence* of the general model M1, under the assumption that M3 is equivalently related to M1.

Remark 13. Conversely, we accept as a starting point a model M3 and we equivalently associate either a model M1 or a model M2 by the formulae involved in *Proposition 5.* and *6.* The derived models are not equivalent.

On the other hand when a model M3 is considered the expression of the plastic multiplier requires the knowledge of the *spin* $\dot{\mathbf{R}}^e(\mathbf{R}^e)^T$, and consequently the assumptions concerning the plastic spin $\mathbf{W}^p$, due to the kinematical relation existing between them in the framework of multiplicative decomposition (3.1).

To end the discussion of this paragraph we emphasize some remarkable consequences that follow:

– in the case of *small elastic strains*;

– in the case of *structurally isotropic materials* (see Loret [17], Dafalias [18], Cleja-Tigoiu, Soós [7]), i.e., by definition all constitutive and evolution functions, in the relaxed configuration, are *isotropic* with respect to their arguments.

Proposition 7. In the case of *small elastic strain*, but finite rotation and finite plastic deformation (see Mandel [6]), when $\mathbf{C}^e \simeq I + 2\varepsilon^e$ with $|\varepsilon^e| := \sqrt{\varepsilon^e \cdot \varepsilon^e} \ll 1$ the following estimates can be derived:

a) The polar decomposition of the elastic deformation $\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e$ leads to $\mathbf{U}^e \simeq I + \varepsilon^e$, $\mathbf{F}^e \simeq \mathbf{R}^e$, $J^e \simeq 1$.

b) $\Sigma \simeq S_{\mathcal{R}} \simeq (\mathbf{R}^e)^T \sigma \mathbf{R}^e$, $S_{\mathcal{R}} \simeq 4\dfrac{\partial W}{\partial \varepsilon^e}$, where $\hat{W}(\mathbf{C}^e, \alpha) = W(\varepsilon^e, \alpha)$,

c) $\mathcal{A}_t \simeq \mathcal{A}$, $\mathbf{a} \simeq (\mathbf{R}^e)^T \mathcal{A}\mathbf{R}^e$, $\mathbf{a} = -\dfrac{\partial W}{\partial \alpha}$,

d) $\mathbf{D}_{\mathcal{R}}{}^p \simeq \mathbf{D}^p = (\mathbf{L}^p)_S$, $\mathbf{W}_{\mathcal{R}}{}^p \simeq \mathbf{W}^p = (\mathbf{L}^p)_A$, $\hat{\mathbf{D}}^p \simeq (\mathbf{R}^e)^T \mathbf{D}^p \mathbf{R}^e$, and $\hat{\mathbf{W}}^p \simeq (\mathbf{R}^e)^T \mathbf{W}^p \mathbf{R}^e$,

e) $f_1(\Sigma, \mathcal{A}) \simeq f_2(S_{\mathcal{R}}, \mathcal{A}) \simeq f_2\big((\mathbf{R}^e)^T \sigma \mathbf{R}^e, (\mathbf{R}^e)^T \mathbf{a} \mathbf{R}^e\big)$.

Moreover, if and only if the function f_2 (in the relaxed configuration) is an *isotropic function* with respect to both arguments (for instance in the case of *isotropic structural materials*) the dependence of f_2 on $(\sigma, \mathbf{a})$ yields. Hence, the function $f_3(\sigma, \mathbf{a}) = f_2(S_{\mathcal{R}}, \mathcal{A})$.

Generally the function f_3, associated to f_2 by the procedure invoked in the *Proposition 5*, $f_3(\sigma, \mathbf{a}) := f_2(S_{\mathcal{R}}, \mathcal{A}) \simeq f_2\big((\mathbf{R}^e)^T \sigma \mathbf{R}^e, (\mathbf{R}^e)^T \mathbf{a}\big)$, remains *still* dependent on the elastic rotation (due to the fact that $\mathbf{R}^e$ represents the local deformation from the relaxed configuration to the actual one).

Proposition 8. We assume that:

1. The elasto-plastic model describes a *structurally isotropic material;*
2. The energy function involved in (4.2) allows the decomposition[4] $\psi(\mathbf{F}^e, \bar{\alpha}) = \psi_1(\mathbf{C}^e) + \psi_2(\bar{\alpha})$, with ψ_2 – the so-called stored energy, then

a) $\Sigma \in Sym$, $\Sigma = (\mathbf{R}^e)^T \bar{\Sigma}(\mathbf{R}^e)\mathbf{R}^e$ with

$$\bar{\Sigma}(\mathbf{B}^e) := 2\mathbf{B}^e \frac{\partial \psi_1}{\partial \mathbf{B}^e}(\mathbf{B}^e) \in Sym, \qquad \text{where} \qquad \mathbf{B}^e = \mathbf{F}^e(\mathbf{F}^e)^T \tag{94}$$

b) $\dfrac{\partial f_1}{\partial \Sigma} \in Sym$, $\mathbf{D}^p = \mu_1 \dfrac{\partial f_1}{\partial \Sigma}$ and $\mathbf{W}^p = 0$, with

$$\frac{\partial f_1}{\partial \Sigma} = \mathbf{R}^e \frac{\partial f_1}{\partial \bar{\Sigma}} (\mathbf{R}^e)^T, \quad \text{where} \qquad f_1(\Sigma, \mathcal{A}) = f_1(\bar{\Sigma}, \mathcal{B}) \quad \text{with} \quad \Sigma = \mathbf{R}^e \bar{\Sigma} (\mathbf{R}^e)^T, \qquad \mathcal{B} = \mathbf{R}^e \mathcal{A} (\mathbf{R}^e)^T \tag{95}$$

c) According to the adopted definitions the following formulae:

$$\mathbf{D}_{\mathcal{R}}{}^p = \mu_1 \mathbf{R}^e \left(\mathbf{B}^e \frac{\partial f_1}{\partial \bar{\Sigma}} \right)_S (\mathbf{R}^e)^T, \qquad \mathbf{W}_{\mathcal{R}}{}^p = \mu_1 \mathbf{R}^e \left(\mathbf{B}^e \frac{\partial f_1}{\partial \bar{\Sigma}} \right)_A (\mathbf{R}^e)^T \tag{96}$$

hold.

Moreover, $\mathbf{W}_{\mathcal{R}}{}^p = 0$ if and only if $\mathbf{B}^e$ and $\partial f_1 / \partial \bar{\Sigma}$, or equivalently if $\mathbf{C}^e$ and $\partial f_1 / \partial \Sigma$, are permutable tensors.

Obviously, this is the case when the yield function describes the isotropic hardening (the internal variables are scalar), while for a material which kinematically hardens this is not true.

Concluding remarks

(i) The associative evolution equations for $\hat{\mathbf{D}}^p$, (in our notation) of the particular form of Eq. (51) were derived in [19], [20], assuming a yield function dependent on $S_{\mathcal{R}}$ and $\mathcal{A}$ – internal scalar variable. They adopted the flow rule (46) (together with $\mathbf{W}_{\mathcal{R}}{}^p = 0$, since the skew-sym-

[4] The micromechanical motivation for such an assumption can be found in Teodosiu and Sidoroff [11].

metric of $\mathbf{C}^e\mathbf{L}^p$ has *no dual driving force* and does not contribute to the energy dissipation rate) as a point of departure.

Constitutive descriptions of the inelastic deformation processes are expressed in terms of $\left(\mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}\right)_S$ and σ, (i.e. with respect to the actual configuration) in Besseling and Van der Giessen [14], starting from an appropriate model M1, but involving some non-symmetric tensorial internal variables (similar to Σ). Just this $\hat{\mathbf{D}}^p$ – cf. Eq. (36.2) – was employed by Dafalias [18], in the case of small elastic deformations. The evolution equations for *the plastic spin* $\hat{\mathbf{W}}^p = \left(\mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}\right)_A$ were derived (independently on the dissipation principle), by using the representations theorems for anisotropic bodies, given in Liu [21].

(ii) The nine-dimensional flow rule was proposed by Mandel [6] (see also Halphen [12], Halphen and Son [13]) associated to the yield function dependent on Σ and $\mathcal{A}$, i.e. the evolution equations of the form (41). A discussion about this flow rule and also about equivalent evolution equations with respect to the reference and the intermediate configurations can be found in Lubliner [8], [15]. In Cleja-Tigoiu [22] some Σ – models were proposed for anisotropic finite elastoplasticity, when the yield condition is a function of Mandel's stress measure, which in general is not a symmetric tensor. The considered models are of the form Eq. (41) and (42). Some comments about the associated *plastic spins* $\{\mathbf{C}^e\mathbf{L}^p\}_A$ and $\{\mathbf{L}^p\}_A$ are also made.

(iii) Furthermore, based on the thermodynamic arguments, since Σ and $\mathbf{L}^p$ are conjugate variables, an evolution equation written in terms of $\mathbf{L}^p$ as dependent on Σ (i.e. on the quasi-static Eshelby stress tensor $(\hat{\mathbf{b}}_q)_{\mathcal{R}}$) and some proper internal variables, is equally justified. Only when the maximum dissipation postulate is considered a certain undeterminancy occurs as pointed out in Lubliner [15] and Cleja-Tigoiu [22].

(iv) The different elasto-plastic theories elaborated for *anisotropic* bodies are not equivalent, as we emphasized in our analysis concerning the connection between them.

(v) The elasto-plastic boundary value problems at large deformations can be correctly formulated and solved only if the evolution equations for the plastic spin are considered, too. It becomes evident when we look at the dynamic equations written for instance in the form (19), given by Maugin [5]. The dynamic Eshelby stress tensor, $\hat{\mathbf{b}}$, contains the full stress $\mathbf{T}$ and the motion gradient $\mathbf{F}$ in its *anisotropic* part and also the free energy function and pseudo-potential of dissipation. Let us remark that the pseudo-potential of dissipation is dependent on $\mathbf{L}^p$ and $\dot{\alpha}$, and on the other hand W is a function of $\mathbf{F}^e = \mathbf{F}(\mathbf{F}^p)^{-1}$. Hence, not only $\mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}$, but $(\mathbf{F}^p)^{-1}$ itself is necessary in order to have correctly defined functions.

(vi) When the evolution equations for plastic deformation and internal variables are predicted based on thermodynamic arguments, then $(\mathbf{C}^e\mathbf{L}^p)_S$ and $\dot{\alpha}$ are described. Although the *plastic spin* is irrelevant from the thermodynamical point of view, i.e., only $(\mathbf{C}^e\mathbf{L}^p)_S$ contributes to the energy dissipation rate, the problem of the determination of the evolution equations in finite elasto-plasticity remains still open. Therefore, additional hypotheses, independent on the dissipation principle, have to be introduced in order to postulate evolution equations for plastic deformations.

(vii) Finally, we note, on the one hand, that basing on the thermodynamic arguments, Σ and $\mathbf{L}^p$ being the conjugate variables, an evolution equation written in terms of $\mathbf{L}^p$ as dependent on Σ (i.e., on the quasi-static Eshelby stress tensor $(\hat{\mathbf{b}}_q)_{\mathcal{R}}$) and some proper internal variables is equally justified.

On the other hand, we remark that the Eshelby stress tensors (more precisely these invariant combinations containing Eshelby stress tensors) can be utilized in describing the evolution equations, written in invariant form, with respect to the intermediate configuration.

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