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ON THE GEOMETRY OF MODULI SPACES

DEDICATED TO KARL STEIN

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We construct a Kähler metric on the moduli spaces of compact complex manifolds with $c_1<0$ and of polarized compact Kähler manifolds with $c_1=0$, which is a generalization of the Petersson-Weil metric. It is induced by the variation of the Kähler-Einstein metrics on the fibers that exist according to the Calabi-Yau theorem. We compute the above metric on the moduli spaces of polarized tori and symplectic manifolds. It turns out to be the Maaß metric on the Siegel upper half space and the Bergmann metric on a symmetric space of type III resp. In particular it is Kähler-Einstein with negative curvature.

The relationship between deformation theory and Teichmüller theory of Riemann surfaces was investigated by Weil. He used the fact that the Kodaira-Spencer class is given by a quadratic differential, to define a metric on the Teichmüller space, which had been studied before in quite a different context by Petersson. Weil posed the problem about the possible Kähler property and the curvature of this metric. Ahlfors gave an affirmative answer to the first question and showed that the Ricci

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curvature and the holomorphic sectional curvature are strictly negative. However, the methods are restricted to Riemann surfaces.

For higher dimensional manifolds, coarse moduli spaces exist both in the algebraic and non-algebraic case. In the algebraic case, constructions were given by Matsusaka, Mumford, Fogarty, Popp and others. Apart from the class of canonically polarized manifolds one mostly considers polarized algebraic varieties - these are equipped with an embedding into a projective space. In analytic geometry one had to regard as "models" compact manifolds together with Kähler classes. The coarse moduli space of polarized Kähler manifolds with vanishing first Chern class was constructed in (SCH2) and in general for polarized non-ruled manifolds in (SCH3) and independently by Fujiki (FU).

In the present paper we construct an intrinsic metric first on local, universal families of manifolds with $c_1<0$ and $c_1=0$ and show its Kähler property. The metric is induced by the variation of the Kähler-Einstein metrics on the corresponding fibers according to the Calabi-Yau theorem: the norm on the tangent space is given by the norm of the harmonic representatives of the Kodaira-Spencer classes with respect to the Kähler-Einstein metric. A partial result by means of a different construction was given by Koiso (KO). He studies real deformations of Einstein metrics on fixed manifolds and constructs a metric for the classes of manifolds with $c_1<0$ and $c_1=0$, $b_2=1$. Both classes consist of projective manifolds. Since the volume is kept fixed in the latter case, all families are polarized by themselves.

We can show that the above generalized Petersson-Weil metrics can be pushed down to the moduli spaces in a unique way. Among Kähler-Einstein manifolds with non-positive curvature, there are classes, closed under small deformations, which consist just of algebraic varieties, namely those with $c_1<0$ and the unitary manifolds. As the corresponding moduli spaces can be constructed from Hilbert schemes, one might suppose a possible relation to Kähler (-Einstein) metrics on quasi-projective varieties. The chosen approach allows a computation in the remaining cases; namely for moduli spaces of polarized tori we obtain the Maaß metrics on Siegel upper half spaces, and the metric on the coarse moduli space of symplectic manifolds is induced by the Bergman metric on bounded symmetric domains of type III. In both cases we get Kähler-Einstein metrics with negative curvature.

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1. Families of polarized Kähler manifolds, moduli spaces

(1.1) The assignment of a fixed Kähler class to a compact Kähler manifold turned out to be the proper substitute for a model in the algebraic situation. Such objects, so-called "polarized Kähler manifolds" are the correct ones in regard of the construction and differential geometric investigation of moduli spaces, since the distinction of Kähler manifolds by polarizations is necessary for the Hausdorff property of moduli spaces.

DEFINITION: (i) Let X_o be a compact complex manifold and λ_o a Kähler class. Then (X_o, λ_o) is called a polarized manifold.

(ii) A family of polarized manifolds (X_s, λ_{X_s}), $s \in S$, parametrized by a reduced complex space S, is given by a proper, smooth holomorphic map $f: X \to S$ together with $\lambda_{X/S} \in (R^1 f_* \Omega^1_{X/S})(S)$ such that $f^{-1}(s) = X_s$ and $\lambda_{X/S} | X_s = \lambda_{X_s}$ for $s \in S$.

(iii) Let $(f: X \to S, \lambda_{X/S})$ and $(g: Y \to S, \lambda_{Y/S})$ be families of polarized manifolds, then an isomorphism is a biholomorphic map $h: X \to Y$ over S with $h^* \lambda_{Y/S} = \lambda_{X/S}$. In particular, an isomorphism of polarized manifolds has to be compatible with Kähler classes.

DEFINITION: Let $f:X\to S$ be a family of complex compact manifolds. Then a (strong) relative Kähler form $\omega_{X/S}$ is a relative (1,1)-form, which is given by $\omega_{X/S}|U_j = i\partial_{/S}\bar\partial_{/S}p_j$ with respect to a suitable open covering $\{U_j\}$ of X. The functions p_j have to be strictly plurisubharmonic on $U_j\cap X_s$, $s\in S$ and p_j-p_k are harmonic on $U_j\cap U_k$. A weak relative Kähler form is given in a similar way, where the p_j-p_k need only be harmonic on the fibers $X_s\cap U_j\cap U_k$. ($\partial_{/S}$ and $\bar\partial_{/S}$ denote derivatives in fiber direction.)

(1.2) PROPOSITION: Let $(X\to S, \lambda_{X/S})$ be a family of polarized manifolds. Then, locally with respect to S, $\lambda_{X/S}$ is represented by a (strong) relative Kähler form $\omega_{X/S}$. In particular, any relative Kähler class is induced by a global Kähler form on X for sufficiently small S.

PROOF: By variation of Hodge structures the real section $\lambda_{X/S}\varepsilon(R^1f_*\Omega^1_{X/S})(S)$ corresponds to a fixed $\lambda_o\varepsilon(R^2f_*\mathbb{R})(S)$ $\simeq H^2(X_o,\mathbb{R})$. Consider the short exact sequence

$$0\to\mathbb{R}_X\xrightarrow{\ \ell\ }\mathcal{O}_X\xrightarrow{\ \mathrm{Re}\ }\mathcal{H}_X\longrightarrow 0$$

where Re maps a holomorphic function onto its real part and ℓ is the multiplication by i. It induces

$$R^1f_*\mathcal{H}_X\longrightarrow R^2f_*\mathbb{R}_X\longrightarrow R^2f_*\mathcal{O}_X$$

By construction the image of λ_o in $(R^2f_*\mathcal{O}_X)(S)$ vanishes. On a neighborhood of s_o it is represented by a cocycle $\sigma_{jk}\varepsilon Z^1(\mathcal{U},\mathcal{H}_X)$, where $\mathcal{U}$ is a suitable open covering of X.

So $\sigma_{jk}=\tau_j-\tau_k, \tau_j$ real differentiable functions. Set $\chi=i\partial_{/S}\overline{\partial}_{/S}\tau_j$. On the other hand $\lambda_{X/S}$ induces a differentiable family of Kähler forms $\psi=i\partial_{/S}\overline{\partial}_{/S}q_j$, q_j-q_k harmonic on fibers, so $\chi=\psi+i\partial_{/S}\overline{\partial}_{/S}g$, with a real function g on X. So $\psi=i\partial_{/S}\partial_{/S}r_j$, $r_j=\tau_j-g$ and $r_j-r_k=\sigma_{jk}$.

(1.3) DEFINITION: Let (X_o, λ_o) be a polarized compact manifold. A deformation of (X_o, λ_o) over a reduced complex space S with a distinguished base point $s_o \varepsilon S$ is a family of polarized manifolds $(X \to S, \lambda_{X/S})$ together with an ismorphism $(X_o, \lambda_o) \xrightarrow{\sim} (X_{s_o}, \lambda_{X_{s_o}})$. The usual notions of deformation theory carry over literally.

(1.4) PROPOSITION([SCH2,3]): Any polarized Kähler manifold possesses a versal Kähler deformation.

(1.5) Compact complex manifolds X_o with negative first Chern class are polarized (in the original algebraic sense) by the (positive) canonical bundle. The ampleness is preserved under local deformations, the canonical bundle is homogeneous, and the automorphism group is finite. Moreover by Matsusaka's theorem, in a family of canonically polarized manifolds, one can choose a uniform power of the canonical bundles to be very ample. Thus the Hilbert scheme induces complete deformations. For polarized Kähler manifolds with vanishing first Chern class we have an analogous situation (for a more general result see [SCH3]):

(1.6) <u>PROPOSITION</u>: <u>Let</u> $(X \longrightarrow S, \lambda_{X/S})$ <u>be a family of polarized manifolds with</u> $c_1(X_s)_{\mathbb{R}}=0$. <u>Then</u> $\dim \operatorname{Aut}(X_s)=\text{const.}$ <u>In particular, any such manifold possesses a universal Kähler deformation</u>.

(1.7) In general, there may be still isomorphic fibers in a universal deformation. This situation is much like the situation with the Teichmüller family: it can be handled either by imposing an additional structure like a marking to distinguish between isomorphic fibers - this can be done with K3 surfaces - or rather by means of an identification of corresponding points of the base. If there actually exist such points, one can see easily that there does no more exist a family of manifolds over the quotient space. The base spaces of universal deformations of marked objects may patch together and yield universal families. In the other situation, the first question is for the analyticity (and Hausdorff property) of the quotient spaces, and the second is, to glue these quotients together. The resulting space is called a "coarse moduli space".

(1.8) <u>DEFINITION</u>: The <u>moduli functor</u> $\mathcal{M}$ with respect to a given collection $\mathcal{K}$ of polarized manifolds assigns to a complex space S the set $\mathcal{M}(S)$ of isomorphism classes of families of objects from $\mathcal{K}$ over S. The map $\mathcal{M}(S) \longrightarrow \mathcal{M}(R)$ induced by a holomorphic map $R \to S$ is defined by means of base change. A <u>coarse moduli space</u> M for the moduli functor $\mathcal{M}$ is characterized uniquely up to isomorphism by the following conditions:

(i) there exists a morphism of functors $\Phi:\mathcal{M} \to h^M$

(ii) if p denotes the reduced point, then $\Phi(p)$ is bijective

(iii) if $\Psi:\mathcal{M} \to h^N$ is another morphism, then there is a unique holomorphic mapping $F:M \to N$ with $\Psi = h(F)\circ\Phi$.

THEOREM ([SCH2]): There exists a coarse moduli space of polarized compact manifolds with $c_{1,\mathbb{R}}=0$.

A more general result was proved in [SCH3] and, independently by Fujiki [FU].

(1.9) Our aim is to give a differential geometric description of the coarse moduli space of compact polarized manifolds with $c_1<0$ and $c_1=0$. This will be done by constructing an intrinsic Kähler metric on the bases of universal deformations, which can be pushed down to the moduli space in a unique way. As one can construct the coarse moduli spaces by the Calabi-Yau theorem, it is reasonable to investigate the variation of Kähler-Einstein metrics on the fibers.

2. Families of polarized Kähler-Einstein manifolds with non-positive curvature - construction of the canonical metric

(2.1) Let X_o be a compact Kähler manifold. We use the following notations. Let $z=z_j=(z_j^1,\ldots,z_j^n)$ be local (holmorphic) coordinates. A Kähler form is given by

$$\omega_{X_o} = \frac{i}{2}\, g_{\alpha\overline{\beta}}(z)dz^{\alpha}\wedge d\overline{z^{\beta}}.$$

The first Chern class is represented by the Ricci form:

$$c_1(X_o)=\frac{i}{2\pi}\left[\mathrm{Ricci}(\omega_{X_o})\right]\varepsilon H^2(X_o,\mathbb{R}).$$

$$\mathrm{Ricci}(\omega_{X_o})=-i\partial\bar\partial\log\det(g_{\alpha\bar\beta}(z))=R_{\alpha\bar\beta}dz^\alpha\wedge dz^{\bar\beta}.$$

The Ricci tensor can also be written as a contraction of the curvature tensor. We use the convention

$$R^\alpha{}_{\gamma\sigma\bar\beta}=+\frac{\partial}{\partial z^{\bar\beta}}\Gamma^\alpha_{\gamma\sigma}\quad,\quad R_{\alpha\bar\beta}=+R^\gamma{}_{\alpha\bar\beta\gamma}.$$

(2.2) We recall Yau's theorem as far as it concerns Kähler-Einstein manifolds.

THEOREM(S.T. YAU): (i) _Let_ X_o _be a compact complex manifold with_ $c_1(X_o)<0$. _Then there exists a unique Kähler form_ ω_{X_o} _with_ Ricci $(\omega_{X_o})=-\omega_{X_o}$.

(ii) Let (X_o, λ_o) be a polarized manifold with $c_1(X_o)_{\mathbb{R}}=0$. Then there exists a unique Kähler form $\omega_{X_o}\varepsilon\lambda_o$, which is Ricci-flat: $\mathrm{Ricci}(\omega_{X_o})=0$.

(2.3) PROPOSITION: _Let_ $X\to S$ _be a family of compact manifolds with_ $c_1<0$, _or let_ $(X\to S,\lambda_{X/S})$ _be a family of polarized manifolds with_ $c_1=0$. _Then the solutions of the Calabi-problem yield a (strong) relative Kähler metric._

PROOF: Set $\lambda_{X/S}:=-2\pi c_1(X/S)\varepsilon(R^1f_*\Omega^1_{X/S})(S)$ in the first case. The Kähler-Einstein metrics on the fibers depend differentiably on the parameter and thus give rise to a weak relative Kähler metric $\omega_{X/S}$ by (KO, 10.1, 10.5) (see also (SCH1)). With respect to an open covering $\{U_j\}$ of X it is given by $i\partial_{/S}\bar\partial_{/S}q_j$ as in (1.1). On the other hand, by (1.2), it is represented by a strong Kähler form $\tilde\omega_{X/S}=i\partial_{/S}\bar\partial_{/S}r_j$. So $\omega_{X/S}=\tilde\omega_{X/S}+i\partial_{/S}\bar\partial_{/S}\phi$ with a real function ϕ. The $\psi_j:=q_j-r_j-\phi$ satisfy $\partial_{/S}\bar\partial_{/S}\psi_j=0$. Set $p_j:=q_j-\psi_j=r_j+\phi$, so $\omega_{X/S}|U_j=i\partial_{/S}\bar\partial_{/S}p_j$ and $p_j-p_k=r_j-r_k$ is harmonic on U_{jk}.

(2.4) Given a family of manifolds with a strong, relative Kähler form $\omega_{X/S}$, we use holomorphic coordinates (z,s) on X, with $z\epsilon U\subset\mathbb{C}^n$ and $s\epsilon S$, such that $f(z,s)=s$. Then

$$\omega_{X/S}=\frac{i}{2}g_{\alpha\bar\beta}(z,s)dz^\alpha\wedge dz^{\bar\beta}.$$

We will use holomorphic coordinates in the first place together with covariant derivatives in fiber directions, denoted by indices in classical notation. Problems will be local with respect to the base, so S can be thought of as a reduced subspace of an open subset in a complex number space. Thus total and partial derivatives by coordinate functions s^u, $s^{\bar v}$ do make sense.

<u>LEMMA</u>: <u>Let $\omega_{X/S}$ be a relative Kähler form as above. Then</u>

$$(1)\qquad \frac{\partial R_{\bar\beta\gamma}}{\partial s^u} = -g^{\alpha\bar\delta}\left(\frac{\partial g_{\bar\beta\gamma}}{\partial s^u}\right)_{;\alpha\bar\delta} + R^\kappa{}_\gamma\frac{\partial g_{\bar\beta\kappa}}{\partial s^u} - R^{\bar\lambda}{}_{\bar\beta\gamma}{}^\kappa\frac{\partial g_{\kappa\bar\lambda}}{\partial s^u}$$

<u>or</u>

$$(2)\qquad \frac{\partial R_{\bar\beta\gamma}}{\partial s^u}dz^{\bar\beta}\wedge dz^\gamma = \Box\left(\frac{\partial g_{\bar\beta\gamma}}{\partial s^u}dz^{\bar\beta}\wedge dz^\gamma\right).$$

<u>By contraction</u>:

$$(3)\qquad g^{\gamma\bar\beta}\frac{\partial R_{\gamma\bar\beta}}{\partial s^u} = -g^{\alpha\bar\delta}g^{\gamma\bar\beta}\left(\frac{\partial g_{\bar\beta\gamma}}{\partial s^u}\right)_{;\alpha\bar\delta} = \Delta\left(g^{\gamma\bar\beta}\frac{\partial g_{\bar\beta\gamma}}{\partial s^u}\right).$$

The infinitesimal Kähler-Einstein condition reads:

$$(4)\qquad \frac{\partial R_{\gamma\overline{\beta}}}{\partial s^u} = k\cdot \frac{\partial g_{\gamma\overline{\beta}}}{\partial s^u}\ ; \quad k\in\mathbb{R}\ ,\ \text{here } k=0,-1$$

or

$$(5)\qquad \Delta\left(\frac{1}{g}\frac{\partial g}{\partial s^u}\right) = k\cdot\frac{1}{g}\cdot\frac{\partial g}{\partial s^u}\ ;\ g = \det(g_{\alpha\overline{\beta}}).$$

Observe that in general all of the above derivatives are no global tensors; still we use the calculus of covariant derivatives locally.

(2.5) A derivative of a relative Kähler form by a tangent vector on the base can only be defined as follows: Given a proper, smooth holomorphic map $X \to S$ as above, $s_o\in S$ a distinguished point, $S\subset U\subset\mathbb{C}^n$ such that $s_o=0$, and $\omega_{X/S}$ a weak Kähler form. For a differentiable trivialisation $\Phi: X_o\times S \xrightarrow{\sim} X$ over S we define

$$\frac{d}{ds^u}\,\omega_{X/S}\Big|_{s=o} := (\Phi^{-1}|X_o)^*\left(\left(\frac{d}{ds^u}\,\Phi^*(\omega_{X/S})\right)\Big|_{s=o}\right).$$

PROPOSITION: Let $\omega_{X/S}$ be a weak relative Kähler form and $\Phi: X_o\times S \xrightarrow{\sim} X$ a diffeomorphism over S. Then

(i) $\frac{d}{ds^u}\,\omega_{X/S}\Big|_{s=o}$ is a closed complex 2-form.

(ii) $\left(\frac{d}{ds^u}\,\omega_{X/S}\Big|_{s=o}\right)\in H^2(X_o,\mathbb{C})$ does not depend on the choice of Φ.

(iii) If $\omega_{X/S}$ is a strong relative Kähler form, then

$$\left(\frac{d}{ds^u}\,\omega_{X/S}\Big|_{s=o}\right) = 0\ .$$

REMARK: One can realize any representative of the above cohomology class by differentiable coordinates. But these are no longer trivializing - not even over a complex one-dimensional base.

We choose an open covering $\{U_j\}$ of X with holomorphic coordinates (z_j, s); $f(z_j, s)=s$. Denote by z_{jk} transition functions:

$$(z_j, s)=(z_{jk}(z_k, s), s).$$

We may assume that $\Phi|X_o\times\{0\}=\mathrm{id}$. Thus we get an extension w_j of holomorphic coordinates on the central fiber to differentiable coordinates of the whole family.

$$w_j=w_j(z_j, s), \quad w_j(z_j, 0)=z_j$$

$$w_j=w_{jk}(w_k)=z_{jk}(w_k, 0)$$

And Φ^{-1} is given by $(z_j, s) \longrightarrow (w_j(z_j, s), s)$.

Now, because of the relative closedness of $\omega_{X/S}$, one can show in an elementary way:

$$(6) \quad \left.\frac{d\omega_{X/S}}{ds^u}\right|_{s=o} = \left(\left.\frac{\partial g_{\alpha\overline{\beta}}(z, s)}{\partial s^u}\right|_{s=o} + b_{ju\alpha;\overline{\beta}} + b_{ju\overline{\beta};\alpha}\right) dz_j^\alpha \wedge dz_j^{\overline{\beta}}$$

$$+ b_{ju\overline{\delta};\overline{\beta}} dz_j^{\overline{\beta}} \wedge dz_j^{\overline{\delta}} + b_{ju\alpha;\gamma} dz_j^\alpha \wedge dz_j^\gamma$$

where $$b_{ju}^\alpha = b_{ju}^\alpha(z_j) = \left.\frac{\partial z_j^\alpha(w_j, s)}{\partial s^u}\right|_{\substack{s=o\\ w=z}}$$

and $$b_{ju}^{\overline{\beta}} = b_{ju}^{\overline{\beta}}(z_j) = \left.\frac{\partial z_j^{\overline{\beta}}(w_j, s)}{\partial s^u}\right|_{\substack{s=o\\ w=z}}.$$

These quantities satisfy the following transformation rules:

$$(7) \quad b_{ju}^\alpha(z_j) = \frac{\partial z_{jk}^\alpha(z_k, 0)}{\partial z_k^\gamma} \cdot \left(b_{ku}^\gamma(z_k) - \frac{\partial z_{kj}^\gamma(z_j, 0)}{\partial s^u} \right)$$

$$(8)\quad b^{\overline{\beta}}_{ju}(z_j) = \frac{\partial z^{\overline{\beta}}_{jk}(z_k,0)}{\partial z^{\overline{\delta}}_k} \cdot b^{\overline{\delta}}_{ku}(z_k)$$

So $B^{\overline{\beta}}_u := b^{\overline{\beta}}_{ju}$ and $B^{\alpha}_{u\overline{\beta}} := b^{\alpha}_{ju;\overline{\beta}}$ are (global) tensors.

<u>REMARK:</u> Since locally $B^{\alpha}_{u\overline{\beta}} = b^{\alpha}_{u;\overline{\beta}}$, this tensor defines a $\overline{\partial}$-closed (0,1)-form with values in the tangent bundle on X_o. In fact, it represents the image

$$\rho\left(\frac{d}{ds^u}\right) \varepsilon H_1(X_o, \mathcal{T}_{X_o}) \qquad \text{of} \quad \frac{d}{ds^u} \varepsilon T_{s_o}(S)$$

under the Kodaira-Spencer map ρ of the given deformation (see [KO-SP]).

We prove the proposition: (i) follows directly from the local description of $g_{\alpha\overline{\beta}}(z,s)$ and (6). An infinitesimal variation of Φ with $\Phi|X_o=\mathrm{id}$ kept fixed is given by a set $\tilde{E}^{\alpha}_u$, $\tilde{E}^{\overline{\beta}}_u$ of vector fields on X_o, with a certain condition on commutativity. The effect on the total derivative of $\omega_{X/S}$ is given by

$$b_{ju\alpha} \longrightarrow b_{ju\alpha} + \tilde{E}_{u\alpha}\ ; \quad b_{ju\overline{\beta}} \longrightarrow b_{ju\overline{\beta}} + \tilde{E}_{u\overline{\beta}}.$$

This shows (ii). Now let $\omega_{X/S}$ be a strong Kähler form. Then $\Phi^*\omega_{X/S}$ is a differentiable family of closed differentiable forms on X_o with constant cohomology class in $H^2(X_o,\mathbb{C})$. So by standard harmonic theory we have

$$(\Phi^*\omega_{X/S})(s) = \omega_{X/S}|X_o + d\nu(s)$$

where $\nu(s)$ is a differentiable family of 1-forms. Taking derivatives $\frac{d}{ds^u}$ on either side gives (iii).

We get the following set of equations on X_o:

$$\text{(9a)}\quad \left.\frac{\partial g^{j}_{\alpha\overline{\beta}}(z_j,s)}{\partial s^u}\right|_{s=o} + b_{ju\overline{\beta};\alpha} + B_{u\alpha;\overline{\beta}} = \tilde{E}_{u\alpha;\overline{\beta}} - \tilde{E}_{u\overline{\beta};\alpha}$$

$$\text{(9b)}\quad B_{u\alpha;\gamma} - B_{u\gamma;\alpha} = \tilde{E}_{u\alpha;\gamma} - \tilde{E}_{u\gamma;\alpha}$$

$$\text{(9c)}\quad B_{u\overline{\beta}\overline{\delta}} - B_{u\overline{\delta}\overline{\beta}} = \tilde{E}_{u\overline{\beta};\overline{\delta}} - \tilde{E}_{u\overline{\delta};\overline{\beta}}$$

(2.6) From now on, $\omega_{X/S}$ will be a strong relative Kähler form, which is Kähler-Einstein on all fibers according to (2.3). In equations (9) the 1-forms $\tilde{E}_{u\alpha}dz^{\alpha}+\tilde{E}_{u\overline{\beta}}dz^{\overline{\beta}}$ are unique up to cocycles. So there is still the scope to apply such tensors.

LEMMA: Given the relative Kähler-Einstein situation with $k=0$ or $k=-1$, let

$$A^{\alpha}_{u\overline{\beta}}\frac{\partial}{\partial z^{\alpha}}\, dz^{\overline{\beta}}$$

be the harmonic representative of

$$\rho\left(\frac{d}{ds^u}\right) = \left[B^{\alpha}_{u\overline{\beta}}\,\frac{\partial}{\partial z^{\alpha}}\, dz^{\overline{\beta}}\right]$$

on X_o, i.e. $A^{\alpha}_{u\overline{\beta};\gamma}g^{\gamma\overline{\beta}}=0$, Then

$$A_{u\overline{\beta}\overline{\delta}} = A_{u\overline{\delta}\overline{\beta}} .$$

PROOF: The class $\rho\left(\frac{d}{ds^u}\right)$ is represented by $C^{\alpha}_{u\overline{\beta}}:=B^{\alpha}_{u\overline{\beta}}-\tilde{E}^{\alpha}_{u;\overline{\beta}}$, where $C_{u\overline{\beta}\overline{\delta}}=C_{u\overline{\delta}\overline{\beta}}$. Now set $A^{\alpha}_{u\overline{\beta}}=C^{\alpha}_{u\overline{\beta}}+F^{\alpha}_{u;\overline{\beta}}$. Then $A_{u\overline{\beta}\overline{\delta}}dz^{\overline{\beta}}\wedge dz^{\overline{\delta}}$ is a $\overline{\partial}$-boundary. So we have to show that $A_{u\overline{\beta}\overline{\delta}}dz^{\overline{\beta}}\wedge dz^{\overline{\delta}}$ is harmonic. Now

$$\overline{\partial}\overline{\partial}^{*}(A_{u\overline{\beta}\overline{\delta}}dz^{\overline{\beta}}\wedge dz^{\overline{\delta}})=\overline{\partial}((A_{u\overline{\beta}\overline{\delta};\alpha}-A_{u\overline{\delta}\overline{\beta};\alpha})g^{\alpha\overline{\beta}}dz^{\overline{\delta}})=\overline{\partial}(A^{\alpha}_{u\overline{\delta};\alpha}dz^{\overline{\delta}})+0$$

Set $L_{\overline{\delta}}=A^{\alpha}_{u\overline{\delta};\alpha}$, then, as <u>locally</u> $A^{\alpha}_{u\overline{\beta};\gamma}=a^{\alpha}_{u;\overline{\beta}\gamma}$, we get

$$L_{\overline{\delta}} = a^{\alpha}_{u;\alpha\overline{\delta}} + a^{\sigma}_{u} R^{\alpha}{}_{\sigma\overline{\delta}\alpha} = a^{\alpha}_{u;\alpha\overline{\delta}} + k\cdot a_{u\overline{\delta}}$$

and

$$L_{\overline{\delta};\overline{\beta}} - L_{\overline{\beta};\overline{\delta}} = k\cdot(a_{u\overline{\delta};\overline{\beta}} - a_{u\overline{\beta};\overline{\delta}}).$$

So

$$\square(A_{u\overline{\beta\delta}}dz^{\overline{\beta}}\wedge dz^{\overline{\delta}}) = k\cdot A_{u\overline{\beta\delta}}dz^{\overline{\beta}}\wedge dz^{\overline{\delta}}; \quad k\leq 0$$

Hence

$$\square(A_{u\overline{\beta\delta}}dz^{\overline{\beta}}\wedge dz^{\overline{\delta}}) = 0.$$

Observe that in the above situation $F_{u\overline{\beta}}dz^{\overline{\beta}}$ is $\overline{\partial}$-closed. Pick $F_{u\alpha}$, such that $F_{u}\alpha dz^{\alpha}+F_{u\overline{\beta}}dz^{\overline{\beta}}$ is d-closed and replace all tensors $\tilde{E}_u$ by $E_u := \tilde{E}_u - F_u$. Set

$$A_{u\alpha} := B_{u\alpha} - E_{u\alpha}.$$

The new equations (9) are:

$$(9a')\quad \left.\frac{\partial g^{j}_{\alpha\overline{\beta}}}{\partial s^u}\right|_{s=o} + b_{ju\overline{\beta};\alpha} + B_{u\alpha;\overline{\beta}} = E_{u\alpha;\overline{\beta}} - E_{u\overline{\beta};\alpha}$$

$$(9b')\quad B_{u\alpha;\gamma} - B_{u\gamma;\alpha} = E_{u\alpha;\gamma} - E_{u\gamma;\alpha}$$

$$(9c')\quad B_{u\overline{\beta\delta}} - B_{u\overline{\delta\beta}} = E_{u\overline{\beta};\overline{\delta}} - E_{u\overline{\delta};\overline{\beta}}$$

The terms $b_{ju\overline{\beta};\alpha}-E_{u\overline{\beta};\alpha}$ are of the form $a_{ju\overline{\beta};\alpha}$ with $A_{u\overline{\beta\delta}}\big|_{U_j} = a_{ju\overline{\beta};\alpha}$. We arrive at

$$(10)\quad \left.\frac{\partial g_{\alpha\overline{\beta}}}{\partial s^u}\right|_{s=o} + a_{ju\overline{\beta};\alpha} + A_{u\alpha;\overline{\beta}} = 0$$

$$(11)\quad A_{u\alpha;\gamma} - A_{u\gamma;\alpha} = 0$$

$$(12)\quad A_{u\overline{\beta\delta}} - A_{u\overline{\delta\beta}} = 0$$

The harmonicity of $A^{\alpha}_{u\overline{\beta}}\frac{\partial}{\partial z^{\alpha}}\, dz^{\overline{\beta}}$ means

$$(13) \qquad A^{\alpha}_{u\overline{\beta};\gamma}g^{\gamma\overline{\beta}} = 0$$

or

$$(14) \qquad A^{\alpha}_{u\overline{\beta};\alpha} = 0 \, .$$

<u>PROPOSITION</u>: $A_{u\alpha}dz^{\alpha}$ <u>is harmonic</u>:

$$(15) \qquad A_{u\alpha;\overline{\beta}} = 0.$$

Observe that all $B_{\overline{u}}:=\overline{B_u}$ and $A_{\overline{u}}:=\overline{A_u}$ tensors enjoy analogous properties.

<u>LEMMA</u>: $$\Delta(g^{\alpha\overline{\beta}}a_{ju\overline{\beta};\alpha}) = k\cdot g^{\alpha\overline{\beta}}a_{ju\overline{\beta};\alpha} \, .$$

The proposition follows from the lemma and (5):

$$\Delta(g^{\alpha\overline{\beta}}A_{u\alpha;\overline{\beta}}) = k\cdot g^{\alpha\overline{\beta}}A_{u\alpha;\overline{\beta}} \text{ with } k\leq 0.$$

(2.7) <u>DEFINITION</u>: Let $f:X \to S$ or $(f:X \to S,\lambda_{X/S})$ resp. be a holomorphic family of compact manifolds X_s with $c_1<0$ or a family of polarized compact manifolds with $c_1(X_s)_{\mathbb{R}}=0$. Then the corresponding Kähler-Einstein metrics $g_{\alpha\overline{\beta}}$ on the fibers (according to the Calabi-Yau theorem) give rise to a hermitian form on the tangent space of S:

$$(16) \qquad G_{u\overline{v}}(s) := G\left(\left.\frac{d}{ds^u}\right|_s, \left.\frac{d}{ds^{\overline{v}}}\right|_s\right)$$

$$:= \int_{X_s} A^{\alpha}_{u\overline{\beta}}(z,s)\cdot A^{\overline{\beta}}_{\overline{v}\alpha}(z,s)\cdot g(z,s)\cdot dv(z,s)$$

where $A^{\alpha}_{u\bar{\beta}}\frac{\partial}{\partial z^{\alpha}}dz^{\bar{\beta}}$ is the harmonic representative of $\rho\left(\frac{d}{ds^{u}}\bigg|_{s}\right)$, $g=\det(g_{\alpha\bar{\beta}})$ and $dv=\frac{i^{m}}{2^{m}}\, dz^{1}\wedge dz^{\bar{1}}\wedge\ldots\wedge dz^{m}\wedge dz^{\bar{m}}$.

The fact that $G_{u\bar{v}}(s)$ is a hermitian form on $T_{s}(S)$ follows from (6) and (10). From the symmetry (12) we get:

$$(17)\qquad G_{u\bar{v}}(s) = \int_{X_s} A^{\alpha}_{u\bar{\beta}}(z,s)\cdot A^{\bar{\delta}}_{\bar{v}\gamma}(z,s)\, g^{\bar{\beta}\gamma}\cdot g_{\alpha\bar{\delta}}\cdot g\cdot dv\ .$$

(2.8) Since automorphisms of polarized manifolds with $c_1<0$ or $c_1=0$ resp. are isometries with respect to the unique Kähler-Einstein metrics, the above hermitian form is intrinsicly defined.

PROPOSITION: Given a universal Kähler deformation of a polarized manifold with $c_1<0$ or $c_1=0$ according to (1.6), the hermitian form (15) is positive definite.

PROOF: Any $G_{u\bar{u}}$ is positive because of (17) and the fact that $A^{\alpha}_{u\bar{\beta}}$ is the harmonic representative of the non-vanishing Kodaira-Spencer class.

(2.9) If $f:X\to S$ is a (proper, flat) family of complex manifolds, then any section of the tangent bundle of S induces a section of $R^1f_*\mathcal{T}_{X/S}$. In particular, it assigns to a point s an element of $H^1(X_s,\mathcal{T}_{X_s})$. (See also [B-P-SCH]). However, it is not clear from the beginning that the harmonic representatives depend differentiably on s, unless one knows that $\dim H^1(X_s,\mathcal{T}_{X_s})$=const. In fact, we have so far to restrict ourselves to these points, whereas in the case of polarized manifolds with vanishing first Chern class we can overcome this difficulty.

(2.10) PROPOSITION: *Let* $(X \to S, \lambda_{X/S})$ *be a universal local family of polarized manifolds with* $c_1=0$ *(over a reduced space). Then the hermitian metric* (16) *depends differentiably on* s.

PROOF: We use the structure theorem for such manifolds (see [B]). There is an unbranched finite covering $\pi: Y \to X$ such that the fibers Y_s have trivial canonical bundles. So $h^1(Y_s, \mathcal{T}_{Y_s})$=const. These manifolds become polarized by $\lambda_{Y/S}=\pi^* \lambda_{X/S}$ and the pull-backs of the Kähler-Einstein forms on X_s solve the Calabi problem on Y_s. Furthermore, one can pull back trivializing coordinates and show that there exist pull-backs of the $A^{\alpha}_{u\bar{\beta}}$, which stay harmonic with respect to the Kähler-Einstein forms on Y_s. But this means (see (17)) that the $G_{u\bar{v}}(s)$ can be evaluated on Y_s as well, where they are known to depend differentiably on s.

(2.11) Starting from real analytic deformation theory of Einstein metrics on a fixed differentiable manifold, Koiso produced a canonical Kähler metric on the base of a universal family of Kähler-Einstein manifolds with $c_1=0$ and $b_2=1$. Since the volume is left fixed in this setting, these manifolds form a polarized family from themselves. Furthermore, all of these manifolds are algebraic. We begin with a universal Kähler deformation as constructed in [SCH1], see (1.6). Since we are interested in the family rather than in the central fiber, we speak about "local universal families".

<u>THEOREM</u>: <u>Let</u> $(X \to S, \lambda_{X/S})$ <u>be a local universal family of polarized Kähler manifolds with</u> $c_1=0$. <u>Then the variation of the Kähler-Einstein structures on the fibers induces a Kähler metric on</u> S, <u>which is given by</u> (16).

(2.12) The coarse moduli space of polarized Kähler manifolds with $c_1=0$ was patched together from quotients of bases of universal local families by finite groups of automorphisms. Namely, given such a family, this group G consists of automorphisms ϕ of S, which can be extended to the family.

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\phi} & S \end{array}$$

As all $\Phi|X_s:(X_s,\lambda_{X_s}) \xrightarrow{\sim} (X_{\phi(s)},\lambda_{X_{\phi(s)}})$ are isometries, ϕ is an isometry of S. So the Kähler metric on S can be pushed down to S/G. In singular points one applies the Riemann extension theorem for plurisubharmonic functions as proved by Grauert and Remmert [G-R].

<u>THEOREM</u>: <u>The coarse moduli space of compact polarized manifolds with</u> $c_1=0$ <u>is a (reduced) Kähler space. The Kähler structure reflects the variation of the corresponding Kähler-Einstein structures.</u>

(2.13) The analogue of (2.10) is not clear for canonically polarized manifolds. Koiso gives a canonical metric, but leaves this point to the reader. One can probably not argue by a vanishing of the second tangential cohomology groups $H^2(X_s, \mathcal{T}_{X_s})$. We restrict ourselves here to the smooth points of bases of universal deformations and show the Kähler property - a different construction was given by Koiso in (K).

THEOREM: Given a local universal family of compact complex manifolds with $c_1<0$, then (16) defines a Kähler structure on the base.

THEOREM: The variation of Kähler-Einstein metrics on compact complex manifolds with $c_1<0$ induces a Kähler structure on a Zariski open subspace of the corresponding coarse moduli space, which contains all points with $H^2(X_s, \mathcal{T}_{X_s})=0$.

3. Proof of the Kähler property

(3.1) In order to show the Kähler property of the hermitian metric (16), we have to introduce simultaneous trivializing coordinates. In this paragraph, $X \to S$ is always provided with a strong relative Kähler form $\omega_{X/S}$ of Kähler-Einstein metric on the fibers with k=0 or -1. As everything is intrinsicly defined, one can work at singularities of the base either with the tangent space of S and its sections or with a desingularization of S and pull-backs. Although we lose the positive definiteness, this is sufficient to show the closedness of the (1,1)-form in question.

Any differentiable trivialization of $X \to S$ is induced by a set of commuting vector fields on X, which project down to $\frac{\partial}{\partial s^1}, \frac{\partial}{\partial s^{\bar{1}}}, \ldots, \frac{\partial}{\partial s^{\ell}}, \frac{\partial}{\partial s^{\bar{\ell}}}$. Thus we have a 2ℓ-parameter family Φ of diffeomorphisms of X, which project down to translations. So Φ_{s-t} induces a diffeomorphism $X_t \to X_s$, which we denote by $z(w,s,t)$. For fixed t this induces a diffeomorphism of the trivial family $X_s \times S$ to the original family with parameter t, which fits into the arguments of section 2.

(3.2) The derivative of the generalized Petersson-Weil metric (16) is given by

$$(18)\quad \frac{d}{ds^r} G_{u\bar{v}}(s) =$$

$$\int_{X_s} \frac{d}{dt^r}\bigg|_{\substack{t=s\\ w=z}} \left\{ A^{\alpha}_{u\bar{\beta}}(z(w,s,t),t) A^{\bar{\beta}}_{\bar{v}\alpha}(z(w,s,t),t) g(z(w,s,t),t) \cdot \cdot dv(z(w,s,t),t) \right\}$$

(3.3) We can now define the quantities of equations (9) to (15) with parameters:

$$(19)\qquad b^{\alpha}_{jr}(z_j,s) := \frac{\partial z^{\alpha}_j(w_j,s,t)}{\partial t^r}\bigg|_{\substack{t=s\\ w=z}}$$

and get

$$(20)\qquad A^{\alpha}_{r\overline{\beta}}(z,s) = B^{\alpha}_{r\overline{\beta}}(z,s) - E^{\alpha}_{r;\overline{\beta}}(z,s)$$

where the $E^{\alpha}_{r}(z,s)$ on X_s depend differentiably on s.

LEMMA:

$$(21)\qquad \frac{d}{ds^r}\Big[g(z(w,s,t)\cdot dv(z(w,s,t))\Big]\Big|_{\substack{t=s\\ w=z}} =$$

$$\Big[(E^{\gamma}_{r;\gamma} + E^{\overline{\delta}}_{r,\overline{\delta}})\Big] g(z,s)\cdot dv(z,s)$$

PROOF: First:

$$\frac{d}{ds^r}g(z(w,t,s),s)\Big|_{\substack{t=s\\ w=z}} =$$

$$g\cdot g^{\overline{\beta}\alpha}\cdot\left(\frac{\partial g_{\alpha\overline{\beta}}}{\partial s^r}(z,s) + \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{\gamma}}\, b^{\gamma}_{r}(z,s) + \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^{\overline{\delta}}}\, b^{\overline{\delta}}_{r}(z,s)\right)$$

second

$$\frac{d}{ds^r}dv(z(w,t,s))\Big|_{\substack{t=s\\ w=z}} =$$

$$\left(\frac{\partial b^{\gamma}_{r}(z,s)}{\partial z^{\gamma}} + \frac{b^{\overline{\delta}}_{r}(z,s)}{\partial z^{\overline{\delta}}}\right)\cdot g(z,s)\cdot dv(z,s).$$

The claim follows from (9a').

(3.4) A straightforward calculation of (18) yields terms, which are no global tensors. So we introduce total derivatives of corresponding differential forms with values in the holomorphic tangent bundle.

DEFINITION:
$$\frac{d\rho_u}{ds^r} := \left.\frac{d\rho_u}{ds^r}\right|_{(-1,1)} =$$

$$\left(\frac{\partial A^{\alpha}_{u\overline{\beta}}(z,s)}{\partial s^r} + A^{\alpha}_{u\overline{\beta};\gamma} b^{\gamma}_{r} + A^{\alpha}_{u\overline{\beta};\overline{\delta}} B^{\overline{\delta}}_{r} - A^{\gamma}_{u\overline{\beta}} b^{\alpha}_{r;\gamma} + A^{\alpha}_{u\overline{\delta}} B^{\overline{\delta}}_{r;\overline{\beta}}\right)\frac{\partial}{\partial z^{\alpha}} dz^{\overline{\beta}}$$

$$\frac{d\rho_{\overline{v}}}{ds^r} := \left.\frac{d\rho_{\overline{v}}}{ds^r}\right|(1,-1) =$$

$$\left(\frac{\partial A^{\overline{\beta}}_{\overline{v}\alpha}}{\partial s^r} + A^{\overline{\beta}}_{\overline{v}\alpha;\gamma} b^{\gamma}_{r} + A^{\overline{\beta}}_{\overline{v}\alpha;\overline{\delta}} B^{\overline{\delta}}_{r} - A^{\overline{\delta}}_{\overline{v}\alpha} B^{\overline{\beta}}_{r;\overline{\delta}} + A^{\overline{\beta}}_{\overline{v}\gamma} b^{\gamma}_{r\alpha}\right)\frac{\partial}{\partial z^{\overline{\beta}}}\, dz^{\alpha}.$$

REMARK: If the canonical pairing is denoted by a dot, then

(22) $$\frac{d}{ds^r} G_{u\overline{v}}(s) =$$

$$\int_{X_s}\left(\left\{\frac{d}{ds^r}\rho_u\right\}\cdot\rho_{\overline{v}} + \rho_u\cdot\left\{\frac{d}{ds^r}\rho_{\overline{v}}\right\} + (E^{\gamma}_{r;\gamma} + E^{\overline{\delta}}_{r;\overline{\delta}})\rho_u\cdot\rho_{\overline{v}}\right) g\, dv$$

(3.5) The proof of the Kähler property is now organized as follows: The partials of the harmonic representatives of the Kodaira-Spencer classes are hard to control, whereas the partials of the representatives $B^{\alpha}_{\overline{\beta}}(z,s)\partial/\partial z^{\alpha}\, dz^{\overline{\beta}}$ enjoy much symmetry. The difference is given by a family of coboundaries. Furthermore, the integral of the product of a $\overline{\partial}$-boundary and a harmonic form is zero. However, the $\overline{\partial}$-operator and the total derivative with respect to a parameter do not commute. But we can still eliminate partial derivatives with respect to the parameter. So finally $\dfrac{\partial G_{u\bar{v}}(s)}{\partial s^r} - \dfrac{\partial G_{r\bar{v}}(s)}{\partial s^u}$ can be computed as an integral of just terms A and E and their covariant derivatives.

<u>DEFINITION</u>: We set $\theta_u = B^{\alpha}_{u\bar{\beta}}(z,s)\,\dfrac{\partial}{\partial z^{\alpha}}\,dz^{\bar{\beta}}$

and $\varepsilon_u = E^{\alpha}_u \dfrac{\partial}{\partial z^{\alpha}}$; $\rho_u = \theta_u - \bar{\partial}\varepsilon_u$.

By the construction of z(w,s,t) we can see easily that the vector fields

$$B_u(z,s) = \frac{\partial}{\partial s^u} + b^{\alpha}_u(z,s)\frac{\partial}{\partial z^{\alpha}} + B^{\bar{\beta}}_u(z,s)\frac{\partial}{\partial z^{\bar{\beta}}}$$

and

$$B_{\bar{v}}(z,s) = \frac{\partial}{\partial s^{\bar{v}}} + B^{\alpha}_{\bar{v}}(z,s)\frac{\partial}{\partial z^{\alpha}} + b^{\bar{\beta}}_{\bar{v}}(z,s)\frac{\partial}{\partial z^{\bar{\beta}}}$$

commute: $[B_u, B_r] = 0$; $[B_u, B_{\bar{v}}] = 0$. From this fact we can deduce:

<u>LEMMA</u>:

(23) $$\left.\frac{d\theta_u}{ds^r}\right|(-1.1) = \left.\frac{d\theta_r}{ds^u}\right|(-1,1)$$

(24) $$\left.\frac{d\theta_{\bar{v}}}{ds^r}\right|(1,-1) = \left.\frac{d\theta_r}{ds^{\bar{v}}}\right|(1,-1) .$$

Here $\left.\dfrac{d\theta_r}{ds^{\bar{v}}}\right|(1,-1)$ denotes the (1,-1)-component of the total derivative of the (1,-1)-component of θ_r, i.e.

$$\left(\frac{\partial B^{\bar{\beta}}_{r;\alpha}}{\partial s^{\bar{v}}} + B^{\bar{\beta}}_{r;\alpha\gamma}B^{\gamma}_{\bar{v}} + B^{\bar{\beta}}_{r;\alpha\bar{\delta}}b^{\bar{\delta}}_{\bar{v}} - B^{\bar{\delta}}_{r;\alpha}b^{\bar{\beta}}_{\bar{v};\bar{\delta}} + B^{\bar{\beta}}_{r;\gamma}B^{\gamma}_{\bar{v};\alpha}\right)\frac{\partial}{\partial z^{\bar{\beta}}}\,dz^{\alpha}$$

COROLLARY:
$$\frac{d\rho_u}{ds^r} - \frac{d\rho_r}{ds^u} = - \frac{d}{ds^r}(\bar{\partial}\varepsilon_u) + \frac{d}{ds^u}(\bar{\partial}\varepsilon_r)$$

(3.6) REMARK: Let λ be a (1,0)-vector field on X_o. Then

$$\int_{X_o} (\bar{\partial}\lambda)\cdot\rho_{\bar{v}}\ g\ dv = 0\ .$$

The proof follows directly from the harmonicity of $\rho_{\bar{v}}$.

LEMMA:

(25) $$\bar{\partial}\left(\frac{d}{ds^r}\varepsilon_u\right) - \frac{d}{ds^r}\left(\bar{\partial}\varepsilon_u\right) = - E^\gamma_u B^\alpha_{r\bar{\beta};\gamma} + E^\alpha_{u;\gamma} B^\gamma_{r\bar{\beta}}$$

COROLLARY:

(26) $$\int_{X_o}\left(\frac{d}{ds^r}\rho_u - \frac{d}{ds^u}\rho_r\right)\cdot\rho_{\bar{v}}\ g\ dv =$$

$$\int_{X_o}(-E^\gamma_u B^\alpha_{r\bar{\beta}\gamma} + E^\alpha_{u\gamma}B^\gamma_{r\bar{\beta}} + E^\gamma_r B^\alpha_{u\bar{\beta}\gamma} - E^\alpha_{r\gamma}B^\gamma_{u\bar{\beta}})\ A^{\bar{\beta}}_{\bar{v}\alpha}\cdot g\cdot dv$$

(3.7) As $\theta_r\Big|_{(1,-1)} = \partial\tilde{\theta}_r$, where $\tilde{\theta}_r = B^{\bar{\beta}}_r \dfrac{\partial}{\partial z^{\bar{\beta}}}$, we get

LEMMA:

(27) $$\frac{d}{ds^{\bar{v}}}\theta_r = \partial\left(\frac{d}{ds^{\bar{v}}}\tilde{\theta}_r\right) + (-B^{\bar{\beta}}_{r\bar{\delta}}B^{\bar{\delta}}_{\bar{v}\alpha} + B^{\bar{\delta}}_r B^{\bar{\beta}}_{\bar{v}\alpha\bar{\delta}})\ \frac{\partial}{\partial z^{\bar{\beta}}}\ dz^\alpha$$

COROLLARY:

(28) $$\int\rho_u\cdot\frac{d}{ds^r}(\theta_{\bar{v}})\ g\ dv = -\int A^\alpha_{u\bar{\beta}}(B^{\bar{\beta}}_{r\bar{\delta}}B^{\bar{\delta}}_{\bar{v}\alpha} - B^{\bar{\delta}}_r B^{\bar{\beta}}_{\bar{v}\alpha\bar{\delta}})\ g\ dv$$

Finally:

LEMMA:

(29) $$\frac{d}{ds^r}(\partial\varepsilon_{\bar{v}})\Big|_{(1,-1)} = \partial\left(\frac{d}{ds^r}\varepsilon_{\bar{v}}\right)\Big|_{(1,-1)} + \left(E^{\bar{\beta}}_{\bar{v}\bar{\delta}}B^{\bar{\delta}}_{r\alpha} - E^{\bar{\delta}}_{\bar{v}}B^{\bar{\beta}}_{r\alpha\bar{\delta}}\right)\cdot\frac{\partial}{\partial z^{\bar{\beta}}}\,dz^{\alpha}$$

Altogether:

PROPOSITION:

(30) $$\frac{d}{ds^r}G_{u\bar{v}}(s) - \frac{d}{ds^u}G_{r\bar{v}}(s) =$$

$$\int_{X_s}\Bigg[\Big(A^{\alpha}_{u\bar{\beta}}(-B^{\bar{\beta}}_{r\bar{\delta}}B^{\bar{\delta}}_{\bar{v}\alpha} + B^{\bar{\delta}}_{r}B^{\bar{\beta}}_{\bar{v}\alpha\bar{\delta}} + B^{\bar{\delta}}_{r\alpha}E^{\bar{\beta}}_{\bar{v}\bar{\delta}} - B^{\bar{\beta}}_{r\alpha\bar{\delta}}E^{\bar{\delta}}_{\bar{v}})$$

$$- \text{(Terms with u and r exchanged)}$$

$$+ (-E^{\gamma}_{u}B^{\alpha}_{r\bar{\beta}\gamma} + E^{\alpha}_{u\gamma}B^{\gamma}_{r\bar{\beta}} + E^{\gamma}_{r}B^{\alpha}_{u\bar{\beta}\gamma} - E^{\alpha}_{r\gamma}B^{\gamma}_{u\bar{\beta}})\cdot A^{\bar{\beta}}_{\bar{v}\alpha}$$

$$+ A^{\alpha}_{u\bar{\beta}}\cdot(E^{\gamma}_{r\gamma} + E^{\bar{\delta}}_{r\bar{\delta}})\cdot A^{\bar{\beta}}_{\bar{v}\alpha} - (E^{\gamma}_{u\gamma} + E^{\bar{\delta}}_{u\bar{\delta}})\cdot A^{\alpha}_{r\bar{\beta}}\,A^{\bar{\beta}}_{\bar{v}\alpha}\Bigg]g\,dv$$

The above terms are generated from integrals with an intrinsic meaning and should be written down in some invariant form. But this does not seem to make the remaining calculations simpler.

(3.8) The rest of the proof is the computation of (30). We have to make extensive use of the harmonic and symmetric properties of ρ_u and $\rho_{\bar{v}}$, i.e. equations (10) to (15). The first step is to reduce everything to A- and E- tensors. In detail we get the following summands:

$$\int A^{\alpha}_{u\beta} A^{\bar{\delta}}_{r}\, A^{\bar{\beta}}_{v\alpha\bar{\delta}} g\cdot dv = 0 \qquad - \int (\text{r and u exchanged}) = 0$$

$$+\int (-E^{\gamma}_{u} A^{\alpha}_{r\bar{\beta}\gamma} A^{\bar{\beta}}_{v\alpha} + E^{\alpha}_{u\gamma} A^{\gamma}_{r\bar{\beta}} A^{\bar{\beta}}_{v\alpha} - E^{\gamma}_{u\gamma} A^{\alpha}_{r\bar{\beta}} A^{\bar{\beta}}_{v\alpha})\cdot g\cdot dv =$$

$$- \int (E^{\gamma}_{u} A^{\alpha}_{r\bar{\beta}} A^{\bar{\beta}}_{v\alpha})_{\gamma} g\, dv = 0$$

$$- \int (E^{\bar{\delta}}_{u} A^{\alpha}_{r\bar{\beta}} A^{\bar{\beta}}_{v\alpha})_{\bar{\delta}}\, g\, dv = 0 \qquad \pm \int (\text{r and u exchanged})$$

$$+\int (A^{\alpha}_{u\bar{\beta}} A^{\bar{\delta}}_{r} E^{\bar{\beta}}_{v\alpha\bar{\delta}} + A^{\alpha}_{u\bar{\beta}} A^{\bar{\delta}}_{r\alpha} E^{\bar{\beta}}_{v\bar{\delta}} - A^{\alpha}_{u\bar{\beta}} A^{\bar{\beta}}_{r\alpha\bar{\delta}} E^{\bar{\delta}}_{v} - A^{\bar{\delta}}_{u} A^{\alpha}_{r\bar{\beta}} E^{\bar{\beta}}_{v\alpha\bar{\delta}}$$

$$-A^{\bar{\delta}}_{u\alpha} A^{\alpha}_{r\bar{\beta}} E^{\bar{\beta}}_{v\bar{\delta}} + A^{\bar{\beta}}_{u\alpha\bar{\delta}} A^{\alpha}_{r\bar{\beta}} E^{\bar{\delta}}_{v})\, g\, dv = \ldots = 0$$

$$+\int (A^{\alpha}_{u\bar{\beta}} E^{\bar{\delta}}_{r\alpha}\, E^{\bar{\beta}}_{v\bar{\delta}} - A^{\alpha}_{u\bar{\beta}} E^{\bar{\beta}}_{r\bar{\delta}} E^{\bar{\delta}}_{v\alpha})\cdot g\cdot dv = 0$$

$$\pm\int (\text{r and u exchanged})$$

$$+\int (-E^{\gamma}_{u} E^{\alpha}_{r\bar{\beta}\gamma} + E^{\alpha}_{u\gamma} E^{\gamma}_{r\bar{\beta}} + E^{\alpha}_{u\bar{\beta}\gamma} E^{\gamma}_{r} - E^{\gamma}_{u\bar{\beta}} E^{\alpha}_{r\gamma})\cdot A^{\bar{\beta}}_{v\alpha}\cdot g\cdot dv = 0$$

4. Computation of the generalized Petersson-Weil metric – moduli spaces of polarized tori and symplectic manifolds

(4.1) There are few examples of compact, complex (at least two dimensional) manifolds, say with $c_1=0$, where a Kähler-Einstein (i.e. Ricci-flat) metric has actually been constructed. These are more or less tori and Kummer surfaces together with possible generalizations to higher dimensions. However, we will compute the variation of the Kähler-Einstein metric, i.e. the generalized Petersson-Weil metric in the most interesting cases, where it will be

Kähler-Einstein with negative curvature. The remaining classes consist of projective varieties.

(4.2) We first quote the classification theorem of compact Kähler manifolds with vanishing first Chern class, which is by itself a theorem on Ricci-flat manifolds. It is based on results of Berger and de Rham.

THEOREM ([B]): Let Z be a compact Kähler manifold with $c_1=0$. Then it possesses a finite unramified covering, which decomposes into a product of tori and irreducible, simply connected, symplectic and unitary manifolds.

The latter manifolds are characterized by the following properties: The canonical bundle is trivial; for unitary manifolds $H^o(Y,\Omega^p_Y)=0$ for $o<p<\dim(Y)$, $\dim(Y)\geq 3$, and for symplectic manifolds the dimension $\dim(X)=2r$ is even, and there exists a nowhere degenerate holomorphic 2-form ϕ with $H^o(X,\Omega_X^{2q}) = \mathbb{C}\phi^q$ for $0\leq q\leq r$ and in all other cases $H^o(X,\Omega^p_X)=0$.

(4.3) In a forthcoming note [SCH4] , we show, how to compute the coarse moduli space of all polarized manifolds with $c_1=0$ from the moduli spaces of polarized manifolds of the above types. As unitary manifolds are algebraic by Kodaira's theorem, one should study these in the algebraic context first, although the investigation of general Kähler classes cannot be reduced to the theory of ample line bundles.

(4.4) THEOREM: The Petersson-Weil metric on the fine moduli space of compact, polarized, marked, complex tori coincides with the Maaß metric on the Siegel upper half space. In particular it is Kähler-Einstein with negative curvature.

PROOF: We use classical notations. Let $x_1,\dots,x_{2n}$ be differentiable coordinates, E the (n×n)-identity matrix, $Z\in\mathfrak{H}=\{Z\in\mathbb{C}^{n^2};\ Z=Z^t,\ \mathrm{Im}(Z)>0\}$ and $z^\alpha=\Omega^\alpha_j x^j$, $\Omega=(E,Z)$. The torus corresponding to a modular point Z has holomorphic coordinates z^α. (Ω spans the lattice only up to a fixed coordinate transformation of $\mathbb{R}^{2n}$.)

$$(31)\quad z^\alpha=\frac{1}{2}(\tilde{w}^\alpha+\overline{\tilde{w}^\alpha})+Z^\alpha_{\ \gamma}\frac{1}{2i}(\tilde{w}^\gamma-\overline{\tilde{w}^\gamma}),\quad \tilde{w}^\alpha=x^\alpha+ix^{n+\alpha},\quad \overline{\tilde{w}^\alpha}=\tilde{w}^{\overline{\alpha}}$$

As in the construction of the canonical metric, given a modular point Z_o, we need trivializing coordinates w for the universal family, centered at Z_o, i.e. w=z for $Z=Z_o$; one can show:

$$(32)\quad z=\frac{i}{2}\left((\overline{Z}_o-Z)\cdot\mathrm{Im}(Z_o)^{-1}\cdot w+(Z-Z_o)\cdot\mathrm{Im}(Z_o)^{-1}\cdot\overline{w}\right).$$

As harmonic forms have constant coefficients with respect to parallizing coordinates, the terms of (17) become

$$(33)\quad A^\alpha_{jk\overline{\beta}}=\left.\frac{\partial^2 z^\alpha}{\partial w^{\overline{\beta}}\partial Z^j_{\ k}}\right|_{\substack{Z=Z_o\\ w=z}},$$

where the holomorphic parameter is $t_u=t_{jk}=Z^j_{\ k}$. So

$$A^\alpha_{jk\overline{\beta}}=\frac{\partial}{\partial Z^j_{\ k}}\left(\frac{i}{2}(Z-\overline{Z}_o)\cdot\mathrm{Im}(Z_o)^{-1}\right)^\alpha_{\ \overline{\beta}}=\frac{i}{2}\delta^\alpha_{\ j}(\mathrm{Im}(Z_o)^{-1})^k_{\ \overline{\beta}}$$

And

$$(34)\qquad A^{\alpha}_{jk\bar\beta}\,\overline{A^{\bar\beta}_{lm\alpha}} = \frac{1}{4}(\mathrm{Im}(Z_o)^{-1})^{k}{}_{\bar l}\cdot(\mathrm{Im}(Z_o)^{-1})^{\bar m}{}_{j}$$

Observe that in this context, as we are dealing with flat metrics, the tensors A and B of sections 2 and 3 agree, which is typical of flat metrics. In particular the functions in (34) are constant. So the canonical metric on $\mathcal{H}$ is

$$(35)\qquad G = \frac{1}{4}\,\mathrm{trace}\left(\mathrm{Im}(Z)^{-1}\cdot dZ\cdot \mathrm{Im}(Z)^{-1}\cdot d\bar Z\right).$$

This is exactly the Maaß metric from [MA].

(4.5) We come to the discussion of irreducible symplectic manifolds. This class is stable under small deformations by [BE, prop. 9].

THEOREM: The Petersson-Weil metric on the moduli space of polarized symplectic manifolds is a Kähler-Einstein metric with negative curvature. By the period map it is related to bounded symmetric domains of type III according to É. Cartan, i.e. to $SO_o(2,b-3)/SO(b-3)$.

Symplectic manifolds possess universal Kähler deformations. By Bogomolov's theorem, the base is smooth. Beauville develops the theory of irreducible symplectic manifolds from the period map ([BE]). We quote his results:

Let $f: X \to S$ be a local universal family of irreducible symplectic manifolds. Provide it with a marking, i.e. an isomorphism $R^2 f_* \mathbb{Z} \to S \times H^2(X_o, \mathbb{Z})$, where X_o is a fiber of f. Then the period map $p: S \longrightarrow \mathbb{P}(H^2(X_o, \mathbb{C})$ is given by a section $\phi\varepsilon f_*\Omega^2_{X/S}(S)$ in a way that $p(s)$ corresponds to the complex line in $H^2(X_o,\mathbb{C})$ spanned by $\phi(s)$. Furthermore $H^2(X_o,\mathbb{Z})$ is equippped with a quadratic form with signature $(3,b_2-3)$, whose extension to $H^2(X_o,\mathbb{C})$ can be computed by integration: $\alpha,\beta\varepsilon H^2(X_o\mathbb{C})$ then

$$(36)\qquad q(\alpha,\beta) = \frac{1}{2}\Big(r\int_{X_o}(\phi\bar{\phi})^r\cdot\int(\phi\bar{\phi})^{r-1}\alpha\beta +$$

$$(1-r)\int\phi^{r-1}\bar{\phi}^r\alpha\cdot\int\phi^r\bar{\phi}^{r-1}\beta + (1-r)\int\phi^{r-1}\bar{\phi}^r\beta\cdot\int\phi^r\bar{\phi}^{r-1}\alpha\Big) .$$

<u>LOCAL TORELLI THEOREM</u>: <u>Let</u> $\tilde{\Omega}\subset\mathbb{P}_{b-1}$, $b=b_2$, <u>be the open subset of a smooth quadric defined by</u>

$$\tilde{\Omega} = \{\psi\varepsilon\mathbb{P}_{b-1};\ q(\psi,\psi)=0,\ q(\psi,\bar{\psi})>0\}$$

<u>Then the period map</u> $p:S\to\tilde{\Omega}$ <u>is a local isomorphism</u>.

(4.6) We consider a local universal family of marked <u>polarized</u> symplectic manifolds $(X\to S,\lambda_{X/S})$. Then $\lambda_{X/S}$ corresponds to a fixed class $\lambda\varepsilon H^{1,1}(X_o,\mathbb{R})\subset H^2(X_o,\mathbb{C})$, with $q(\lambda,\bar{\lambda})>0$. And we can see easily by the local Torelli theorem that the base S corresponds exactly to an open subset of

$$\Omega = \tilde{\Omega}^\lambda = \{\psi\varepsilon\tilde{\Omega};\ q(\psi,\lambda)=0\} .$$

The signature of q, restricted to the orthogonal complement of λ is $(2,b-3)$.

For K3 surfaces one has a stronger result (see(SCH2)), based upon the strong Torelli theorem [BU-RA, LO-PE]. Namely there exists a fine moduli space of marked polarized K3 surfaces, which is an open subset of some $\Omega=\tilde{\Omega}^{\lambda}$. In the missing points s, $\lambda\varepsilon H^{1,1}(X_s,\mathbb{R})$ is no longer positive definite. The coarse moduli space is a quotient of the fine moduli space by the group of all automorphisms of the lattice $(H^2(X_o,\mathbb{Z}),q)$, which actually (and of course according to the general theory of moduli spaces) acts in a proper, discontinous way on Ω. At least in the algebraic case one can interpret the gaps of the moduli space in Ω as type I degenerations. The volume forms of the Ricci-flat metrics can be extended to these points, but not the metrics.

(4.7) As in the Andreotti-Weil setting of the period map of K3 surfaces, one has a close relationship between periods of symplectic manifolds and the Kodaira-Spencer map (see [BE]). The evaluation of differential forms on tangent vectors yields an isomorphism

$$(37)\quad H^1(X_s,\Omega^{1\sim}_{X_s})\otimes H^o(X_s,\Omega^2_{X_s}) \tilde{\to} H^1(X_s,\Omega^1_{X_s})\subset H^2(X_s,\mathbb{C})\simeq H^2(X_o,\mathbb{C}).$$

where the image vector space is in a canonical way isomorphic to the tangent space of $\tilde{\Omega}$ at the period point p(s). However, we will need a more precise description of the tangent spaces of Ω and $\tilde{\Omega}$ resp.

(4.8) We prove the main theorem in the following form:

THEOREM: Let $(X\to S,\lambda_{X/S})$ be a local universal family of polarized irreducible symplectic manifolds. If it is *provided with a marking, then the period map is an isometry of S with the generalized Petersson-Weil metric and an open subset of the period domain* $\Omega \simeq SO_0(2,b-3)/SO(b-3)$ *equipped with the Bergmann metric.*

REMARK: The canonical metric on Ω resembles much the Fubini-Study metric on $\mathbb{P}_n$ - just replace the canonical hermitian form on $\mathbb{C}^{n+1}$ by $-q$. For a period point ϕ and a tangent vector ϕ' at this point it is given by

$$(38)\quad G = G_\phi(\phi') = \frac{-q(\phi',\overline{\phi}')\, q(\phi,\overline{\phi}) + q(\phi',\overline{\phi})\, q(\phi,\overline{\phi}')}{q(\phi,\overline{\phi})^2}$$

(4.9) We compute the Petersson-Weil metric explicitly. Let $\phi(s)=F_{\alpha\gamma}(z,s)dz^\alpha \wedge dz^\gamma$ be a nowhere degenerate, relative holomorphic 2-form. Since we are interested in the norm of a tangent vector, we can restrict ourselves to a smooth one-dimensional base. Let $s_0=0$ be the distinguished base point. Then $\phi=\phi(0)=F_{\alpha\gamma}(0)dz^\alpha \wedge dz^\gamma$ induces the period point and $\phi' = \frac{d}{ds}\phi\Big|_{s=0}$ is the total derivative with respect to a differntial trivialization in the sense of section 2. We compute (38). By arguments used before, ϕ' is d-closed, since ϕ is not only $\overline{\partial}$-closed, but also d-closed. The coefficients of a holmorphic differential form are parallel according to Bochner's theorem ($R \geq 0$). The value of (38) only depends upon d-cohomology classes of ϕ and ϕ' resp. So we use

$$(39)\quad \phi' = \left(\frac{\partial F_{\alpha\gamma}}{\partial s}(z,0)+\frac{\partial F_{\alpha\gamma}}{\partial z^{\sigma}}b^{\sigma}+\frac{\partial F_{\alpha\gamma}}{\partial \overline{z^{\tau}}}\overline{B^{\tau}}+F_{\alpha\sigma}b^{\sigma}{}_{\gamma}+F_{\lambda\gamma}b^{\lambda}{}_{\alpha}\right)dz^{\alpha}\wedge dz^{\gamma}$$

$$+\ 2F_{\alpha\gamma}B^{\gamma}{}_{\overline{\beta}}dz^{\alpha}\wedge dz^{\overline{\beta}}$$

$$\sim\ \left(\frac{\partial F_{\alpha\gamma}}{\partial s}(z,0)+\frac{\partial F_{\alpha\gamma}}{\partial z^{\sigma}}b^{\sigma}+\frac{\partial F_{\alpha\gamma}}{\partial \overline{z^{\tau}}}\overline{B^{\tau}}+F_{\alpha\sigma}a^{\sigma}{}_{\gamma}+F_{\lambda\gamma}a^{\lambda}{}_{\alpha}\right)dz^{\alpha}\wedge dz^{\gamma}$$

$$+\ 2F_{\alpha\gamma}A^{\gamma}{}_{\overline{\beta}}dz^{\alpha}\wedge dz^{\overline{\beta}}$$

$$=:\ \phi'_{(2,o)}+\phi'_{(1,1)}\ .$$

The difference equals $d(F_{\alpha\sigma}E^{\sigma}dz^{\alpha})$, see(9)-(15). We will eliminate $\phi'_{(2,0)}$ by the harmonicity of $A^{\alpha}{}_{\overline{\beta}}\partial/\partial z^{\alpha}\ dz^{\overline{\beta}}$. A mere projection to the (1,1)-component as in the proof of the local Torelli theorem is not sufficient. Observe that $\phi'_{(1,1)}$ represents the cup-product of $\rho\left(\frac{d}{ds}\right)$ and ϕ in the view of (37).

(4.10) We use harmonic representatives and come to:

LEMMA:

$$\frac{1}{2}\,\phi'_{(1,1)} = F_{\alpha\gamma}A^{\gamma}{}_{\overline{\beta}}dz^{\alpha}\wedge dz^{\overline{\beta}}\quad \underline{\text{is harmonic}}.$$

The proof follows from (14) and the fact that $F_{\alpha\gamma}$ is parallel.

Further calculations yield:

$$(40)\qquad q(\sigma,\phi) = q(\sigma_{(0,2)},\phi)\ \text{for any 2-form}\ \sigma.$$

$$(41)\qquad q(\alpha_{(1,1)},\beta_{(2,0)})=0\ ;\ q(\alpha_{(1,1)},\gamma_{(0,2)})=0$$

for any (2,0)- and (0,2)-forms β and γ resp. In particular:

$$(42)\qquad q(\phi',\overline{\phi}') = q(\phi'_{(1,1)},\overline{\phi}'_{(1,1)}) + q(\phi'_{(2,0)},\overline{\phi}'_{(0,2)}).$$

<u>LEMMA</u>:

$$\phi'_{(2,0)}\cdot\phi^{r-1} = c\cdot\phi^{r},\quad c\in\mathbb{C}$$

The proof follows from the fact that $\phi'_{(1,1)}$ is harmonic and ϕ' is d-closed.

<u>COROLLARY</u>:

$$(43)\qquad q(\phi'_{(2,0)},\overline{\phi}'_{(0,2)})\ q(\phi,\overline{\phi}) = q(\phi'_{(2,0)},\overline{\phi})\ q(\phi,\overline{\phi}'_{(0,2)})$$

<u>PROPOSITION</u>:

$$(44)\qquad G = \frac{-q(\phi'_{(1,1)},\overline{\phi}'_{(1,1)})}{q(\phi,\overline{\phi})}$$

<u>for harmonic representatives as above</u>.

<u>LEMMA</u>:

$$(45)\qquad \phi^{r-1}\cdot\overline{\phi}^{r-1}\cdot\phi'_{(1,1)}\cdot\overline{\phi}'_{(1,1)} = -\frac{1}{4r^2}A^{\sigma}_{\overline{\tau}}\cdot\overline{A^{\tau}}_{\sigma}\cdot\phi^{r}\cdot\overline{\phi}^{r}.$$

<u>PROOF</u>: We use coordinates such that in a given point

$$\phi = \sum_{j=1}^{r} f_j\ dz^j\wedge dz^{j+r}$$

$$\phi^{r-1} = (r-1)!\sum_j f_1\cdot\ldots\cdot\hat{f}_j\cdot\ldots\cdot f_r\, dz^1\wedge dz^{r+1}\wedge\ldots\wedge\widehat{dz^j}\wedge\widehat{dz^{j+r}}\wedge\ldots\wedge dz^r\wedge dz^{2r}$$

Now

$$\phi'_{(1,1)}\cdot\overline{\phi}'_{(1,1)} = F_{\alpha\sigma}A^{\sigma}_{\overline{\beta}}\overline{F}_{\overline{\delta\tau}}\overline{A}^{\overline{\tau}}_{\gamma}dz^{\alpha}\wedge dz^{\overline{\beta}}\wedge dz^{\overline{\delta}}\wedge dz^{\gamma}\ .$$

One can show that the coefficient of $dz^{j}\wedge dz^{j+r}\wedge dz^{\overline{k}}\wedge dz^{\overline{k+j}}$ in $\phi'_{(1,1)}\cdot\overline{\phi}'_{(1,1)}$ equals

$$-f_j\overline{f}_k(A^{j}_{\overline{k}}\overline{A}^{\overline{k}}_{j} + A^{j}_{\overline{k+r}}\overline{A}^{\overline{k+r}}_{j} + A^{j+r}_{\overline{k}}\overline{A}^{\overline{k}}_{j+r} + A^{j+r}_{\overline{k+r}}\overline{A}^{\overline{k+r}}_{j+r})$$

$$=: -f_j\overline{f}_k\ H_{jk}$$

and

$$\phi^{r-1}\overline{\phi}^{r-1}\phi'_{(1,1)}\overline{\phi}'_{(1,1)} =$$

$$-((r-1)!)^2\left(\sum_{\substack{j=1\\k=1}}^{r}|f_1|^2\cdot\ldots\cdot|f_r|^2\cdot H_{jk}\right)\cdot dz^1\wedge dz^{r+1}\wedge\ldots\wedge dz^r\wedge dz^{2r}$$
$$\wedge dz^{\overline{1}}\wedge dz^{\overline{r+1}}\wedge\ldots\wedge dz^{\overline{r}}\wedge dz^{\overline{2r}}$$

$$= -\frac{1}{r^2}A^{\alpha}_{\overline{\beta}}\overline{A}^{\overline{\beta}}_{\alpha}\phi^r\overline{\phi}^r$$

<u>PROOF OF THE THEOREM</u>: By (45)

$$G = \frac{1}{r}\frac{\int A^{\alpha}_{\overline{\beta}}\overline{A}^{\overline{\beta}}_{\alpha}\ \phi^r\overline{\phi}^r}{\int\phi^r\overline{\phi}^r}$$

Observe that $\phi^r\overline{\phi}^r$ is a volume form of the type $|h|^2dv$, where h is a holomorphic function. By the Calabi-Yau theorem it equals $c\cdot g\cdot dv$, $c>0$, since its curvature form vanishes. So

$$G = \frac{1}{r}\frac{\int A^{\alpha}_{\overline{\beta}}\overline{A}^{\overline{\beta}}_{\alpha}g\ dv}{\int g\ dv}$$

which is up to a positive constant exactly the Petersson-Weil metric.

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