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SOME RESULTS ON THE OCCURRENCE OF COMPACT MINIMAL SUBMANIFOLDS

Gregory J. Galloway

Varying the situation considered in Myers theorem, we show, via standard index form techniques, that a complete Riemannian manifold which admits a compact minimal submanifold is necessarily compact, provided a suitable curvature object is positive on the average along the geodesics issuing orthogonally from the minimal submanifold. By slightly recasting this result, one establishes the nonexistence of compact minimal submanifolds (in particular, closed geodesics) in complete noncompact manifolds which obey an appropriate curvature condition. A generalization of a result of Tipler concerning the occurrence of zeros of solutions to the scalar Jacobi equation is also obtained.

1. Introduction

Gromoll and Meyer [7] and Cheeger and Gromoll [3] gave a penetrating analysis of the structure of complete, noncompact manifolds of positive and nonnegative curvature. One of the results obtained by Gromoll and Meyer ([7], p. 80) is that a complete noncompact Riemannian manifold with positive Ricci curvature is connected at infinity. Using index form techniques and a recent result of Tipler [10] on the conjugacy of the scalar Jacobi equation, Chicone and Ehrlich [4] improved this result by showing that it suffices to require that the integral of the Ricci curvature along complete geodesics be positive.

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It seems reasonable to consider what other results, which are known to hold for positively curved manifolds, hold also for manifolds in which the curvature is assumed to be positive only on the average (in some sense). Using the second variation of arc length, Gromoll and Meyer show ([7], p. 81) that a complete noncompact Riemannian manifold M with positive sectional curvature admits no compact immersed totally geodesic submanifolds; in particular, M contains no closed geodesics. (In fact their Theorem 4, p. 85, shows much more, namely, that complete geodesics must go to ∞).

In this paper the preceeding result is generalized in several different respects. The nonexistence of compact minimal submanifolds is established under much weaker curvature conditions. The (in general, weaker) curvature objects considered are required to be positive only on the average in some sense. The essential point, as Theorem 1 of the next section illustrates, is that a complete Riemannian manifold which admits a compact minimal submanifold is itself going to be compact if the curvature is sufficiently positive.

2. The main results

We take a moment to introduce some notation and terminology. Let $\langle\ ,\ \rangle$ denote the metric of a Riemannian manifold M. If $t \to \gamma(t)$ is a curve in M, denote its tangent vector field by $\gamma'(t)$. If $t \to X(t)$ is a vector field along γ, denote its covariant derivative by $X'(t)$. By convention, all curves considered are parameterized by their arc length. Let $R(X,Y)$ denote the Riemann curvature transformation, and let $K(X,Y)$ denote the sectional curvature of the plane spanned by the vectors X and Y.

If Π is an r-plane spanned by the orthonormal vectors $e_1, e_2, \ldots, e_r$ at some point p of M and X is a vector orthogonal to Π at p, define

$$K(\Pi,X) = \sum_{i=1}^{r} K(e_i,X). \tag{1}$$

It is easily checked that $K(\Pi,X)$ is independent of the choice of the orthonormal frame which spans Π. If $\operatorname{codim}\Pi = 1$ then $K(\Pi,X) = \operatorname{Ric}(X)$. If $\gamma\colon [0,\infty) \to M$ is a geodesic and Π_0 is a plane through $\gamma(0)$, let Π_t denote the parallel translate of Π_0 along γ through $\gamma(t)$.

Finally, the scalar Jacobi equation,

$$x'' + k(t)x = 0 \tag{2}$$

is said to be focal on the interval $[0,\infty)$ if there exists a solution to (2) on $[0,\infty)$ which satisfies the initial conditions

$$x(0) = 1, \quad x'(0) = 0 \tag{3}$$

and which has at least one zero on $(0,\infty)$.

Our first theorem is a variation of Myers theorem (and generalizations of it; see especially [1]).

Theorem 1. *Let M be a complete Riemannian manifold with dimension ≥ 2. Let V be a compact immersed minimal submanifold of dimension $r \geq 1$. If along each geodesic $\gamma\colon [0,\infty) \to M$ issuing orthogonally from V,*

$$\int_0^\infty K(\Pi_t,\gamma'(t))dt > 0 \tag{4}$$

holds, where Π_0 is the tangent space of V at $\gamma(0)$ then M is necessarily compact. For simplicity of notation, we have adopted in (4) the convention: $\int_0^\infty = \liminf_{t\to\infty} \int_0^t$, which will be employed throughout the paper whenever such integrals arise.

The proof of Theorem 1 is a variation of the technique considered in [6] and relies on the following two lemmas.

Lemma 1. *Let V be an r-dimensional immersed minimal submanifold of a Riemannian manifold M. Let $\gamma\colon [0,\infty) \to M$ be a geodesic issuing orthogonally from V at $p = \gamma(0)$. If (2) is focal, where $k(t) = K(\Pi_t,\gamma'(t))/r$ and Π_0 is the*

<u>tangent space of</u> V <u>at</u> p, <u>then there exists a focal point to</u> V <u>along</u> γ.

<u>Proof of Lemma 1</u>. The proof uses standard Morse Index Theory techniques. Since (2) is focal, there exists a solution $\phi: [0,\infty) \to R$ to (2) and (3) such that $\phi(t_0) = 0$ for some $t_0 > 0$.

Let L be the collection of smooth vector fields X along $\gamma|_{[0,t_0]}$ which are perpendicular to γ, tangent to V at $\gamma(0)$, and which satisfy $X(t_0) = 0$, $X'(0) = 0$. Introduce the index form I along $\gamma|_{[0,t_0]}$. For $X,Y \in L$ we have (see [2], p. 221),

$$I(X,Y) = -\int_0^{t_0} \langle X'' + R(X,\gamma')\gamma', Y\rangle dt + \langle b_T X(0), Y(0)\rangle , \tag{5}$$

where $T = \gamma'(0)$ and b_T is the second fundamental form of V at $\gamma(0)$.

Let $\{e_1, e_2, \ldots, e_r\}$ be an orthonormal basis of the tangent space of V at $\gamma(0)$. Extend the basis vectors to orthonormal vector fields along $\gamma|_{[0,t_0]}$ by parallel translation. For each $i = 1, \ldots, r$, define,

$$X_i(t) = \phi(t)e_i. \tag{6}$$

Note that each $X_i \in L$. Substitution of (6) into (5) gives,

$$I(X_i,X_i) = -\int_0^{t_0} (\phi'' + k(e_i,\gamma')\phi)\phi dt + \langle b_T e_i, e_i\rangle .$$

Then, using (1), we obtain upon summation,

$$\sum_{i=1}^{r} I(X_i,X_i) = -r\int_0^{t_0} (\phi'' + k(t)\phi)\phi dt + \operatorname{tr} b_T = 0,$$

since ϕ is a solution of (2) and V is minimal.

Therefore, for some i, $I(X_i,X_i) \leq 0$. However, by standard index form results (see, e.g. Theorem 4, p. 228 in [2]) we must have $I(X_i,X_i) > 0$ unless there is some point on $\gamma|[0,t_0]$ which is a focal point to V along $\gamma|[0,t_0]$.

The following result of Tipler and the geometric Lemma 1 provide sufficient conditions for the occurrence of focal points.

<u>Lemma 2 (Tipler [10])</u>. *If* $\int_0^\infty k(t)dt > 0$ *then* (1) *is focal on* $[0,\infty)$.

In the next section a more general condition on $k(t)$ is obtained. Because of the more technical nature of this generalized condition we find it preferable in this section to state our main results in terms of Tipler's condition.

We now prove Theorem 1.

<u>Proof of Theorem 1</u>. Suppose M is not compact. Then there exists a sequence of points $\{q_i\}$ such that $d(q_i,V) \to \infty$. (Here d is the metric distance function). Let $\gamma_i\colon [0,t_i] \to M$ be a geodesic from a point $p_i = \gamma_i(0) \in V$ to q_i whose length realizes the distance from q_i to V. $T_i = \gamma_i'(0)$ is necessarily orthogonal to V. Since the unit normal bundle of V is compact, there exists a subsequence $\{(p_j,T_j)\}$ which converges to a point (p,T) in the unit normal bundle. Let $\gamma\colon [0,\infty) \to M$ be the geodesic issuing from $p \in V$ with initial tangent T. It is not hard to see, using the continuous dependence of geodesics on the initial data, that for each t, the length of the segment $\gamma|[0,t]$ realizes the distance from $\gamma(t)$ to V. In particular, γ is focal point free. But (4) and Lemmas 1 and 2 imply, on the contrary, that there is a focal point along γ.

The following corollary to Theorem 1 singles out the codimension one case.

<u>Corollary 1</u>. *Let* M *be a complete Riemannian manifold and let* V *be a compact immersed minimal submanifold of codimension*

one. If along each geodesic $\gamma: [0,\infty) \to M$ *issuing orthogonally from* V, *the condition,*

$$\int_0^\infty \mathrm{Ric}(\gamma'(t))dt > 0$$

holds, in particular if $\mathrm{Ric} > 0$ *on* M, *then* M *is necessarily compact.*

Corollary 1 shows that in the proof of a theorem of Frankel ([5], p. 70) it is not necessary to consider the (more complicated) "M noncompact" case.

Theorem 1 can be recast to make a statement about the nonexistence of compact minimal submanifolds in complete noncompact manifolds. A *ray* emanating from a point p in M is a geodesic $\gamma: [0,\infty) \to M$ such that $\gamma(0) = p$ and the length of each segment of γ realizes the distance between its end points. It is well known that if M is complete and noncompact, there is at least one ray emanating from each of its points. Intuitively, a ray emanating from a point p is a minimal geodesic from p to ∞.

Theorem 2. Let M *be a complete noncompact Riemannian manifold. Suppose along each ray* $\gamma: [0,\infty) \to M$ *the condition*

$$\int_0^\infty K(\Pi_t,\gamma'(t))dt > 0 \tag{7}$$

holds for every r-*plane* Π_0 *at* $\gamma(0)$, *perpendicular to* $\gamma'(0)$. *Then* M *contains no compact immersed minimal submanifolds of dimension greater than or equal to* r.

Proof. If Π is an s-plane perpendicular to X, where $s \geq r$, then $K(\Pi,X)$ can be written as a sum of terms $K(P,X)$ where P is an r-plane perpendicular to X. Indeed, if Π is spanned by the orthonormal vectors $e_1, e_2, \dots, e_s$ then

$$K(\Pi,X) = \frac{1}{r}\sum_{i=1}^{s} K(\Pi^i,X) ,$$

where Π^i is the r-plane spanned by $e_{i \bmod r}, e_{(i+1) \bmod r},$

$\ldots, e_{(i+r-1)\bmod r}$. Thus, (7) holds for all Π_0 of dimension $s \geq r$.

Now, suppose V is a minimal submanifold of dimension $s \geq r$. The proof of Theorem 1 shows that there exists a ray $\gamma\colon [0,\infty) \to M$ issuing orthogonally from V which is focal point free. But this contradicts (7) and Lemmas 1 and 2.

The following two corollaries single out the dimension one and codimension one cases.

Corollary 2. *Let M be a complete noncompact Riemannian manifold. Suppose along each ray $\gamma\colon [0,\infty) \to M$ the sectional curvature condition,*

$$\int_0^\infty K(X(t),\gamma'(t))dt > 0$$

holds for each parallelly propagated vector field X along and orthogonal to γ. Then M contains no compact immersed minimal submanifolds of dimension ≥ 1. In particular, M contains no closed geodesics.

Corollary 3. *Let M be a complete noncompact Riemannian manifold. If along each ray the condition,*

$$\int_0^\infty \mathrm{Ric}(\gamma'(t))dt > 0$$

holds, in particular if $\mathrm{Ric} > 0$ on M, then M contains no compact immersed minimal submanifolds of codimension one.

3. A generalization of Lemma 2

In this section we modify Tipler's argument, using well known methods in the theory of conjugate ordinary differential equations (see [8],[9]) to obtain a generalization of Lemma 2. The following lemma extends Tipler's result to a slightly more general self-adjoint differential equation.

Lemma 3. *Consider the initial value problem,*

$$(8) \qquad \begin{aligned}(q(t)x')' + p(t)x &= 0,\\ x(0) &= x_0 > 0,\\ x'(0) &= x_0',\end{aligned}$$

where $q(t)$ *and* $p(t)$ *are continuous on* $[0,\infty)$ *and* $q(t) > 0$ *on* $[0,\infty)$. *If*

$$(9) \qquad \int_0^\infty \frac{1}{q(t)}\,dt = +\infty,$$

and

$$(10) \qquad \int_0^\infty p(t)dt > \frac{q_0x_0'}{x_0}$$

(*where* $q_0 = q(0)$) *then every solution to* (8) *has a zero on* $(0,\infty)$.

Proof. Let $x(t)$ be a solution to (8) on $[0,\infty)$ and suppose, on the contrary, that x does not have a zero on $(0,\infty)$. The quantity $z = -qx'/x$ obeys the differential equation

$$z' = \frac{z^2}{q} + p\ .$$

Integrating this equation from 0 to t gives,

$$(11) \qquad z(t) = \int_0^t z^2/q\ dt + \int_0^t p\ dt - \frac{q_0x_0'}{x_0}\ .$$

The inequality (10) implies that there exists a number $t_1 \geq 0$ and a number $c > 0$ such that

$$(12) \qquad \int_0^t p\,dt - \frac{q_0x_0'}{x_0} > c$$

for all $t \in [t_1,\infty)$. Thus (11) and (12) imply

$$z(t) > \int_0^t z^2/q\ dt + c$$

for all $t \in [t_1,\infty)$. Introduce the function $R(t)$ by,

$$R(t) = \int_0^t z^2/q \, dt + c.$$

$R > 0$ and $R' = z^2/q > R^2/q$ on $[t_1,\infty)$. Dividing by R^2 and integrating from t_1 to t we obtain,

$$-\frac{1}{R(t)} > \int_{t_1}^t \frac{1}{q} \, dt - \frac{1}{R(t_1)} \; . \tag{13}$$

Equation (9) implies that the right hand side of (13) tends to $+\infty$ as $t \to \infty$, which contradicts the fact that R is positive on $[t_1,\infty)$.

The following theorem can be used in lieu of Lemma 2 to obtain refinements of the results presented in Section 2.

Theorem 3. *Suppose for some* $a > 0$ *and some* $\lambda < 1$,

$$\int_0^\infty (a+t)^\lambda k(t)\,dt > \frac{\lambda^2}{4(1-\lambda)} \cdot \frac{1}{a^{1-\lambda}} \; . \tag{14}$$

Then (2) *is focal on* $[0,\infty)$.

Note that (14) reduces to the condition of Lemma 2 when $\lambda = 0$. However, it is easy to construct examples in which the condition of Lemma 2 is not satisfied but (14) is. For each positive integer n, define:

$$k_n(t) = \begin{cases} -2 \;, & 0 \le t \le \varepsilon \\ 1 \;, & n \le t \le n + \varepsilon \\ 0 \;, & \text{elsewhere on } [0,\infty). \end{cases}$$

Then, for all n, $\int_0^\infty k_n(t)dt < 0$. However, for *any* $a > 0$ and *any* λ, $0 < \lambda < 1$, (14) is satisfied (with $k(t) = k_n(t)$) for all n sufficiently large.

Also, note that (14) can be rewritten as follows,

$$\int_0^\infty (a+t)^\lambda [k(t) - \frac{\lambda^2}{4} \cdot \frac{1}{(a+t)^2}]\,dt > 0 \; .$$

Proof of Theorem 3. The argument is patterned after the proof of a theorem of R.A. Moore (see Theorem 2, p. 127 in [9]).

Let $g(t) = a + t$, and make the substitution,

$$x = g^{\lambda/2}(t)y.$$

Then (1) becomes

$$(q(t)y')' + p(t)y = 0,$$

where,

$$q(t) = g^{\lambda}(t) \quad \text{and} \quad p(t) = \frac{\lambda(2-\lambda)}{4} \frac{1}{g^{2-\lambda}} + g^{\lambda}k.$$

Now,

$$\int_0^\infty \frac{1}{q(t)}\,dt = \int_0^\infty \frac{1}{(t+a)^{\lambda}}\,dt = +\infty$$

and

$$\begin{aligned}\int_0^\infty p(t)dt &= -\frac{\lambda(2-\lambda)}{4(1-\lambda)} \frac{1}{a^{1-\lambda}} + \int_0^\infty (t+a)^{\lambda}k(t)dt \\ &> -\frac{\lambda}{2}\frac{1}{a^{1-\lambda}} \\ &= \frac{q(0)y'(0)}{y(0)}\ .\end{aligned}$$

Therefore, by Lemma 3, y has a zero on $(0,\infty)$ and, hence, so does x.

As a final remark, Theorem 3 can be used to refine Tipler's Theorem 2 in [10] which establishes the conjugacy of (1) under the assumption, $\int_{-\infty}^{\infty} k(t)dt > 0$. As a consequence, one obtains refinements of the geometric applications of Tipler's Theorem 2 considered in [4].

References

[1] Ambrose, W.: A theorem of Myers. Duke Math. J. 24, 345-348 (1957)

[2] Bishop, R.L. and Crittenden, R.J.: Geometry of manifolds. New York and London, Academic Press, 1964

[3] Cheeger, J. and Gromoll, D.: On the structure of complete manifolds of nonnegative curvature. Ann. Math. 96, 413-443 (1972)

[4] Chicone, C. and Ehrlich, P.: Line integration of Ricci curvature and conjugate points in Lorentzian and Riemannian manifolds. Man. Math. 31, 297-316 (1980)

[5] Frankel, T.: On the fundamental group of a compact minimal submanifold. Ann. Math. 83, 68-73 (1966)

[6] Galloway, G.J.: Compactness criteria for Riemannian manifolds. Preprint, Mathematics Department, University of Miami, Coral Gables, Fla

[7] Gromoll, D. and Meyer, W.: On complete open manifolds of positive Ricci curvature. Ann. Math. 90, 75-90 (1969)

[8] Leighton, W.: On self-adjoint differential equations of second order. J. London Math. Soc. 27, 37-47 (1952)

[9] Moore, R.A.: The behavior of solutions of a linear differential equation of second order. Pacific J. Math. 5, 125-145 (1955)

[10] Tipler, F.J.: General relativity and conjugate ordinary differential equations. J. Diff. Eq. 30, 165-174 (1978)

Gregory J. Galloway
Department of Mathematics
University of Miami
Coral Gables, FL 33124
U.S.A.

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