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MULTIPLICITY OF POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENT IN SOME CONTRACTIBLE DOMAINS.

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In this paper we prove that, for every positive integer k, there exists a contractible bounded domain Ω in $\mathbb{R}^N$ with $N \geq 3$, where the problem (*) (see Introduction) has at least k solutions.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 3$. In recent years there has been much interest in nonlinear elliptic equations of the form

$$(*) \qquad \begin{cases} \Delta u + u^{2^*-1} = 0 & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $2^* = \frac{2N}{N-2}$ is the critical exponent for the Sobolev imbedding $H_0^{1,2}(\Omega) \subseteq L^p(\Omega)$.

The problem $(*)$ is a simplified model of some variational problems in physics and geometry, whose common feature is a lack of compactness (see for example the Yamabe's problem in [1], [25]).

Indeed, the solutions of $(*)$ correspond to the critical points u of the functional

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx, \quad \text{with} \quad u > 0;$$

but this functional does not satisfy the classical Palais-Smale's condition, since the imbedding $H_0^{1,2}(\Omega) \subseteq L^{2^*}(\Omega)$ is not compact; therefore it is not possible to use the standard variational methods to find critical points.

A first contribution to problem $(*)$ is the following negative result due to Pohozaev.

Theorem (Pohozaev [21]). *If the bounded domain Ω is star-shaped, then $(*)$ has no solution.*

Nevertheless, more recently Brezis and Nirenberg have pointed out that lower-order perturbations of the nonlinear term in $(*)$ can reverse this situation, and the perturbed problem can have solution, as follows also from general bifurcation theory (see [22], [19]).

Among the other results, Brezis and Nirenberg obtain in [6] the following theorem.

Theorem (Brezis-Nirenberg [6]). *Let $\Omega \subseteq \mathbb{R}^N$ with $N \geq 3$ and λ_1 denote the first eigenvalue of $-\Delta$ in $H_0^{1,2}(\Omega)$. There exists λ^* in $[0, \lambda_1[$, such that, if $\lambda \in]\lambda^*, \lambda_1[$, then problem*

$$\begin{cases} \Delta u + u^{2^*-1} + \lambda u = 0 & in \quad \Omega \\ u > 0 & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

has at least one solution. Moreover $\lambda^* = 0$, *when* $N \geq 4$.

Note that in this theorem no special assumption is made about the bounded domain Ω.

On the other hand, one can exploit the shape of Ω in order to find solutions of the problem $(*)$: for example, if Ω is an annulus (i.e. $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$), it is easy to verify (see Kazdan-Warner [15]) that $(*)$ has a radial solution.
Therefore the existence of solutions of the problem $(*)$ seems to be connected with the shape of Ω.

The most remarkable result in this direction is the following theorem of Bahri-Coron.

Theorem (Bahri-Coron [2],[3]). *If the domain* Ω *has nontrivial topology (see* [2], [3]*), then the problem* $(*)$ *has at least one solution.*

In [4] H. Brezis has pointed out the question whether it is possible to replace in Pohozaev's theorem the assumption "Ω is star-shaped" by "Ω has trivial topology"; in other words, whether there exist domains Ω with trivial topology on which $(*)$ has solution.

An existence result for solutions of $(*)$ in some contractible bounded domains has been obtained by W. Ding (see also remark (29)):

Theorem (W. Ding [11]). *Assume* $N \geq 4$*; let* $\Omega = A_r \backslash C_\epsilon$*, where*

$A_r = \{x \in \mathbb{R}^N : 0 < r < |x| < 1\}$ and C_ϵ is the cylinder defined in notations (4).

There exists $r_0 \in (0,1)$, such that, for $r \in (0, r_0)$ there exists $\epsilon(r) > 0$ such that, if $r \in (0, r_0)$ and $\epsilon \in (0, \epsilon(r))$, then $(*)$ has a solution on $\Omega = A_r \backslash C_\epsilon$.

A weaker result is obtained by Ding when $N = 3$ (see [11]).

In this paper we obtain existence and multiplicity results for solutions of the problem $(*)$ in some contractible bounded domains (see theorem (8)).

The proof uses essentially some results of P.L. Lions and Struwe (see theorems (11) and (16)), which applie the "concentration-compactness principle" to analyse the minimizing sequences for the best Sobolev constant S (see definition (9)), and the Palais-Smale's sequences of the functional f.

At the end, we point out that the solutions of problem $(*)$, which we find in this paper, correspond to critical values in $]\frac{1}{N}S^{N/2}, \frac{2}{N}S^{N/2}[$ of the related functional f, while the critical values obtained in [6] (and also in [7], [8]) lie in the interval $]0, \frac{1}{N}S^{N/2}[$.

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Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 3$ and $2^* = \frac{2N}{N-2}$ be the critical Sobolev exponent for the imbedding $H_0^{1,2}(\Omega) \subseteq L^p(\Omega)$.

We are concerned with the following problem:

$$
(1) \qquad \begin{cases} \Delta u + u^{2^*-1} = 0 & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}
$$

We shall use the following functionals:

(2) Definition. Let $V = \{u \in H_0^{1,2}(\Omega) | \int_\Omega |u|^{2^*} dx = 1\}$; we define the functionals $f : H_0^{1,2}(\Omega) \to \mathbb{R}$; $\varphi : V \to \mathbb{R}$; $F : H_0^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ and $g : H_0^{1,2}(\Omega) \to \mathbb{R}$ in the following way:

$$
f(u) = \frac{1}{2} \int_\Omega |Du|^2 dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx \quad \forall u \in H_0^{1,2}(\Omega)
$$

$$
\varphi(u) = \int_\Omega |Du|^2 dx \quad \forall u \in V;
$$

$$
F(u) = \frac{1}{2} \int_{\mathbb{R}^N} |Du|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx \quad \forall u \in H_0^{1,2}(\mathbb{R}^N);
$$

$$
g(u) = \int_\Omega x_N |u(x)|^{2^*} dx \quad \forall u \in H_0^{1,2}(\Omega) \quad (\text{where } x = (x_1, ..., x_N)).
$$

(3) Remark. It is easy to verify that:

a) *a function u in $H_0^{1,2}(\Omega)$, $u \geq 0$ and $u \not\equiv 0$, solves problem (1) if and only if u is a critical point for the functional f;*

b) *a function u in $H_0^{1,2}(\Omega)$, $u \not\equiv 0$, is a critical point for f if and only if $\overline{u} = \frac{u}{\|u\|_{2^*}}$ is a critical point for φ and $u = \|D\overline{u}\|_2^{\frac{N-2}{2}} \overline{u}$.*

Then, problem (1) is equivalent to search nonnegative critical points of φ.

We introduce the following notations:

(4) Notations. Let

$$B(c,r) = \{x \in \mathbb{R}^N : |x-c| < r\} \quad \forall c \in \mathbb{R}^N, \quad r > 0;$$

$$C_\epsilon = \left\{x = (x_1, ..., x_N) \in \mathbb{R}^N | x_N \geq 0, \ \sum_{i=1}^{N-1} x_i^2 \leq \epsilon^2\right\} \quad \forall \epsilon > 0;$$

For every $k \in \mathbb{N}$, let T_k be the bounded cylinder

$$T_k = \left\{x = (x_1, ..., x_N) \in \mathbb{R}^N | -1 < x_N < k; \ \sum_{i=1}^{N-1} x_i^2 < 1\right\}.$$

Let $c_k = (0, ..., 0, h) \in \mathbb{R}^N$ for every $h = 0, 1, ..., k-1$; given k real numbers r_h, with $0 < r_h < \frac{1}{2}$ for $h = 0, 1, ..., k-1$, let $D_k = T_k \setminus \bigcup_{h=0}^{k-1} \overline{B(c_h, r_h)}$.
It is evident that the bounded domain $D_k \setminus C_\epsilon$ is contractible for every $\epsilon > 0$.

(5) Definition. Suppose that the domain Ω in $\mathbb{R}^N$ have radial symmetry with respect to the axis x_N, that is it results $T(\Omega) = \Omega$ for every transformation T in $\mathbb{R}^N$ of the type

$$\begin{cases} T(x_1, ..., x_N) = (0(x_1, ..., x_{N-1}), x_N) \\ \text{with 0 orthogonal transformation in } \mathbb{R}^{N-1}. \end{cases} \tag{6}$$

We call $H_s(\Omega)$ the space of all the functions of $H_0^{1,2}(\Omega)$ with the same symmetry, that is:

$$H_s(\Omega) = \{u \in H_0^{1,2}(\Omega) | u \circ T = u \ \ \forall \text{ transformation } T \text{ of type (6)}\}.$$

Note that the domains D_k and $D_k \backslash C_\epsilon$ (see notations (4)) have evidently radial symmetry with respect to the axis x_N.

(7) Remark. Suppose that Ω has radial symmetry with respect to the axis x_N. One can easily verify that $\Delta^{-1}(|u|^{2^*-2}u)$ lies in $H_s(\Omega)$ (see definition (5)) for every u in $H_s(\Omega)$; therefore we have that grad $f(u) \in H_s(\Omega)$ for every u in $H_s(\Omega)$ and grad $\varphi(\overline{u}) \in H_s(\Omega)$ for every $\overline{u}$ in $H_s(\Omega) \cap V$.
Consequently, every critical point for f [or for φ] on $H_s(\Omega)$ is a critical point for f [or for φ].

The main result in this paper is the following theorem:

(8) Theorem. *For every $k \in \mathbb{N}$, there exists a contractible bounded domain Ω in $\mathbb{R}^N$ with $N \geq 3$, such that the problem* (1) *has at least k solutions.*
Precisely, there exists $\overline{\epsilon} > 0$ such that, if $\Omega = D_k \backslash C_\epsilon$ (see notations (4)) *with $0 < \epsilon < \overline{\epsilon}$, then problem* (1) *has at least k solutions in $H_s(\Omega)$. Moreover, if we call $u_1, ..., u_k$ the solutions and set $\overline{u}_h = \frac{u_h}{\|u_h\|_{2^*}}$ for $h = 1, ..., k$, we have:*

$$\int_\Omega |D\overline{u}_h|^2 dx = \min \left\{ \int_\Omega |Du|^2 dx | u \in H_s(\Omega); \right.$$

$$\left. \|u\|_{2^*} = 1;\ h-1 < g(u) < h \right\}$$

for every $h = 1, ..., k$ (see definition (2)).

The proof, reported at the point (18), is based on the results which follow.

Let S be the best constant for the Sobolev imbedding $H_0^{1,2}(\Omega) \subseteq L^{2^*}(\Omega)$, defined as follows:

$$(9) \qquad S = \inf \left\{ \int_\Omega |Du|^2 dx | u \in H_0^{1,2}(\Omega); \quad \int_\Omega |u|^{2^*} dx = 1 \right\}.$$

It is well known that the constant S verifies the following properties (see [21], [6], [24], [20], [13], [14]);

(10) Proposition. *Let S be the best Sobolev constant (see (9)). Then:*

a) *S is indipendent of Ω; it depends only on N;*

b) *S is not achieved in any bounded domain Ω;*

c) *when $\Omega = \mathbb{R}^N$, then S is achieved by the function*

$$\overline{U} = \frac{U}{\|U\|_{2^*}} \quad \text{with} \quad U(x) = \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}} ;$$

moreover, all minimizers for S are of the form

$$\overline{U}_{\sigma,x_0} = \frac{U_{\sigma,x_0}}{\|U_{\sigma,x_0}\|_{2^*}} \quad \text{where} \quad U_{\sigma,x_0}(x) = U\left(\frac{x-x_0}{\sigma}\right) ,$$

with $\sigma > 0$ and $x_0 \in \mathbb{R}^N$.

d) *if u in $H_0^{1,2}(\mathbb{R}^N)$ is a critical point for the functional F (see definition (2)) and $u \geq 0$, $u \not\equiv 0$, then it results $\frac{u}{\|u\|_{2^*}} = \overline{U}_{\sigma,x_0}$ for suitable $\sigma > 0$ and x_0 in $\mathbb{R}^N$.*

We recall the following result of P.L. Lions:

(11) Theorem (P.L. Lions [17]). *Let $(u_n)_n$ be a minimizing sequence for (6), that is:*

$u_n \in H_0^{1,2}(\mathbb{R}^N)$; $\int_{\mathbb{R}^N} |u_n|^{2^*} dx = 1$ *for every $n \in \mathbb{N}$, and*

$\lim_{n\to\infty} \int_{\mathbb{R}^N} |Du_n|^2 dx = S.$

Then there exist two sequences $(y_n)_n$ *in* $\mathbb{R}^N$ *and* $(\sigma_n)_n$ *in* $\mathbb{R}$ *with* $\sigma_n > 0$*, such that the sequence* $(\tilde{u}_n)_n$ *in* $H_0^{1,2}(\mathbb{R}^N)$ *defined as follows:*

$$\tilde{u}_n(x) = \sigma_n^{-\frac{N}{2^*}} u_n\left(\frac{x+y_n}{\sigma_n}\right),$$

is relatively compact in $L^{2^*}(\mathbb{R}^N)$.
It is evident that $(\tilde{u}_n)_n$ *is also a minimizing sequence and, if* $\tilde{u}_n \to \tilde{u}$ *in* $L^{2^*}(\mathbb{R}^N)$*, then*

$$\int_{\mathbb{R}^N} |D\tilde{u}|^2 dx = S.$$

We deduce the following proposition:

(12) Proposition. *If the minimizing sequence* $(u_n)_n$ *of theorem* (11) *is in* $H_0^{1,2}(\Omega)$*, with* Ω *bounded domain of* $\mathbb{R}^N$*, then the sequence* $(\sigma_n)_n$ *(see theorem* (11)*) satisfies* $\lim_{n\to\infty} \sigma_n = +\infty$.
Consequently, we can find a point $\overline{x}$ *in* $\overline{\Omega}$ *and a subsequence* $(u_{n_i})_i$ *such that*

$$\lim_{i\to\infty} \int_\Omega v|u_{n_i}|^{2^*} dx = v(\overline{x}) \tag{13}$$

for every function v *which is continuous in* $\overline{\Omega}$.

Proof. Suppose, by contradiction, that $(\sigma_n)_n$ (or a subsequence) is bounded; then $(y_n)_n$ (see theorem (11)) is bounded too and therefore the sequence $(\tilde{u}_n)_n$ is in $H_0^{1,2}(B(0,\rho))$ for ρ sufficiently large; if we suppose that $(\tilde{u}_n)_n$ (or a subsequence) converges to $\tilde{u}$ in $L^{2^*}(\mathbb{R}^N)$,

then the infimum in (9) is obtained by the function $\tilde{u}$ in the bounded domain $\Omega = B(0,\rho)$, in contradiction with proposition (10) (point (b)).
Now we observe that

$$u_n(x) = \sigma_n^{\frac{N}{2^*}} \tilde{u}_n \left[\sigma_n(x - x_n)\right] \quad \text{where} \quad x_n = \frac{y_n}{\sigma_n};$$

it follows that, for every $\epsilon > 0$, $(u_n)_n$ (or a subsequence) verifies

$$\lim_{n\to\infty} \int_{B(x_n,\epsilon)} |u_n|^{2^*} dx = \lim_{n\to\infty} \int_{B(0,\sigma_n\epsilon)} |\tilde{u}_n|^{2^*} dx = \tag{14}$$

$$= \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} dx = 1.$$

Since $\int_\Omega |u_n|^{2^*} dx = 1$ for every $n \in \mathbb{N}$, (14) implies that the sequence $(x_n)_n$ is bounded and, if a subsequence $(x_{n_i})_i$ converges to $\overline{x}$, then $\overline{x}$ lies in $\overline{\Omega}$ and (13) holds.

The functionals f and φ defined in (2) verifie the following Palais-Smale's condition:

(15) Proposition. *Let $N \geq 3$ and $\Omega \subseteq \mathbb{R}^N$. Then we have:*

a) *every sequence $(u_n)_n$ in $H_0^{1,2}(\Omega)$ such that*

$$\begin{cases} \lim_{n\to\infty} f(u_n) \in \left] \frac{1}{N} S^{N/2},\ \frac{2}{N} S^{N/2} \right[\\ \lim_{n\to\infty} \|f'(u_n)\|_{H^{-1,2}(\Omega)} = 0 \ , \end{cases}$$

is relatively compact in $H_0^{1,2}(\Omega)$;

b) *every sequence* $(\overline{u}_n)_n$ *in* V *(see definition* 2)) *such that*

$$\begin{cases} \lim_{n\to\infty} \varphi(\overline{u}_n) \in \,]S, 2^{\frac{2}{N}} S[\\ \lim \|\varphi'(\overline{u}_n)\|_{H^{-1,2}} = 0 \ , \end{cases}$$

is relatively compact in $H_0^{1,2}(\Omega)$.

The proof, reported at the point (17), is based on the following result of Struwe:

(16) Theorem. (Struwe [23]). *Let* $N \geq 3$ *and* $\Omega \subseteq \mathbb{R}^N$. *Suppose that the sequence* $(u_n)_n$ *in* $H_0^{1,2}(\Omega)$ *satisfies* $\sup_{n\in\mathbb{N}} f(u_n) < +\infty$ *and* $\lim \|f'(u_n)\|_{H^{-1,2}(\Omega)} = 0$.
Then, there exist a number $\overline{k} \in N_0$, *a function* u_0 *in* $H_0^{1,2}(\Omega)$ *critical point for the functional* f, *and* $\overline{k}$ *functions* $u_1, ..., u_{\overline{k}}$ *in* $H_0^{1,2}(\mathbb{R}^N)$ *critical points for the functional* F *(see definition* (2)), *such that* $(u_n)_n$ *(or a subsequence) verifies:*

$$u_n \rightharpoonup u_0 \quad \textit{weakly in} \quad H_0^{1,2}(\Omega);$$

$$\lim_{n\to\infty} \|Du_n\|_2^2 = \sum_{J=0}^{\overline{k}} \|Du_J\|_2^2;$$

$$\lim_{n\to\infty} f(u_n) = f(u_0) + \sum_{J=1}^{\overline{k}} F(u_J).$$

(17) *Proof of Proposition* (15).

a) Let u_0 and $\overline{k}$ be the function and the number which appear in theorem (16); it suffices to prove that $\overline{k} = 0$.

First we observe that, if u_J has constant sign and $u_J \not\equiv 0$, then proposition (10) (point (d)) implies that $F(u_J) = \frac{1}{N}\ S^{N/2}$;
on the contrary, if the sign of u_J is not constant, we have $F(u_J) = F(u_J^+) + F(u_J^-)$, where $u_J^+ = \max\{u_J, 0\} \not\equiv 0$ and $u_J^- = \max\{-u_J, 0\} \not\equiv 0$; using the properties of the Sobolev constant S (see proposition (10)) one can easily verifies that $F(u_J^+) \geq \frac{1}{N}\ S^{N/2}$ and $F(u_J^-) \geq \frac{1}{N}\ S^{N/2}$; therefore it is evident that we cannot have $f(u_0) = 0$; on the other hand, if $u_0 \not\equiv 0$, then $f(u_0) > \frac{1}{N}\ S^{N/2}$ and so we obtain that $\overline{k} = 0$.

b) In our assumptions, there exists a sequence $(\lambda_n)_n$ of multipliers such that

$$\lim \|\overline{u}_n + \lambda_n \Delta^{-1}(|\overline{u}_n|^{2^*-2}\overline{u})\|_{H^{-1,2}(\Omega)} = 0\ ;$$

we can deduce that

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \int_\Omega |D\overline{u}_n|^2 dx \in \left]S,\ 2^{\frac{2}{N}} S\right[.$$

Now it is easy to verify that the sequence $u_n = \lambda_n^{\frac{N-2}{4}}\,\overline{u}_n$ satisfies the assumptions of point a) and this implies obviously the thesis.

(18) *Proof of theorem* (8).

We observe that the domain $\Omega = D_k \backslash C_\epsilon$ (see notations (4)) has radial symmetry with respect to the axis x_N and then (see remark (7)) we look for solutions u in $H_s(\Omega)$.

Step I: For every $h = 0, ..., k-1$, let

$$(19)\qquad S_h = \inf\left\{\int_{D_k} |Du|^2 dx | u \in H_s(D_k);\ \|u\|_{2^*} = 1;\ g(u) = h\right\}$$

(see notations in (2), (4), (5), (9)).
We prove that

$$(20) \qquad S_h > S \quad \text{for every} \quad h = 0, ..., k-1.$$

In fact, suppose by contradiction that there exists a sequence $(u_n)_n$ in $H_s(\Omega)$ such that $\|u_n\|_{2^*} = 1$; $g(u_n) = h$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \int_{D_k} |Du_n|^2 dx = S$.
Then, because of proposition (12) and symmetry of the functions u_n, we have that $(u_n)_n$ (or a subsequence) verifies

$$\lim_{n\to\infty} g(u_n) = \overline{x}_N \ \text{ with } \ 0 < r_h \leq |\overline{x}_N - h|$$

and it is impossible, since $g(u_n) = h$ for every $n \in \mathbb{N}$.
Thus (20) is proved.

Step II: we prove that there exist k functions $\overline{v}_1, ..., \overline{v}_k$ in $H_s(D_k)$ such that, for every $h = 1, ..., k$, it results:

$$(21) \qquad \begin{cases} \int_{D_k} |\overline{v}_h|^{2^*} dx = 1; \quad h-1 < g(\overline{v}_h) < h; \\ \int_{D_k} |D\overline{v}_h|^2 dx < \min\left\{S_h | h = 0, ..., k-1\right\} \quad (\text{see}(19)); \\ \int_{D_k} |D\overline{v}_h|^2 dx < 2^{\frac{2}{N}} S. \quad (\text{see}(9)) \end{cases}$$

In fact, (20) and the properties of the constant S (see [6], [24]) implie that for every $\delta > 0$ there exists a function $\overline{v}$ in $C_0^\infty(B(0,\delta))$, invariant for orthogonal tranformations of $\mathbb{R}^N$, such that:

$$\begin{cases} \int_{B(0,\delta)} |\overline{v}|^{2^*} dx = 1; \\ \int_{B(0,\delta)} |D\overline{v}|^2 dx < \min\left\{S_h | h = 0, ..., k-1\right\}; \\ \int_{B(0,\delta)} |D\overline{v}|^2 dx < 2^{\frac{2}{N}} S. \end{cases}$$

Now, since $r_h < \frac{1}{2}$ for every $h = 0, ..., k-1$, we can evidently choose $\delta > 0$ in such a way that, if we define $\overline{v}_h$ as

$$\overline{v}_h(x) = \begin{cases} \overline{v}(x - \overline{c}_h) & \text{for} \quad x \in B(\overline{c}_h, \delta) \\ 0 & \text{for} \quad x \in D_k \setminus B(\overline{c}_h, \delta), \end{cases}$$

where $\overline{c}_h = (0, 0, ..., 0, h - \frac{1}{2}) \in \mathbb{R}^N$, then the functions $\overline{v}_1, ..., \overline{v}_k$ lie in $H_s(D_k)$ and satisfie the conditions (21).

Step III: Let $\Omega = D_k \setminus C_\epsilon$; we verifie that, for $\epsilon > 0$ sufficiently small, there exist k functions $\overline{u}_{1,\epsilon}, ..., \overline{u}_{k,\epsilon}$ in $H_s(\Omega)$ such that

$$\text{(22)} \qquad \begin{cases} \int_\Omega |\overline{u}_{h,\epsilon}|^{2^*} dx = 1; \quad h - 1 < g(\overline{u}_{h,\epsilon}) < h; \\ \int_\Omega |D\overline{u}_{h,\epsilon}|^2 dx < \min\{S_h | h = 0, ..., k-1\}; \\ \int_\Omega |D\overline{u}_{h,\epsilon}|^2 dx < 2^{\frac{2}{N}} S. \end{cases}$$

In fact, let $\overline{v}_1, ..., \overline{v}_k$ be the functions in $H_s(D_k)$ obtained above, verifying the conditions (21).
It is easy to find a family of cut-off functions z_ϵ in $C_0^\infty(\Omega) \cap H_s(\Omega)$ such that, if we set $u_{h,\epsilon}(x) = z_\epsilon(x)\overline{v}_h(x)$, then $u_{h,\epsilon} \in H_s(\Omega)$ and

$$\text{(23)} \qquad \begin{cases} \lim_{\epsilon \to 0^+} \int_\Omega |u_{h,\epsilon}|^{2^*} dx = \int_{D_k} |\overline{v}_h|^{2^*} dx = 1; \\ \lim_{\epsilon \to 0^+} \int_\Omega |Du_{h,\epsilon}|^2 dx = \int_{D_k} |D\overline{v}_h|^2 dx. \end{cases}$$

Therefore, we deduce easily that the functions $\overline{u}_{h,\epsilon} = \frac{u_{h,\epsilon}}{\|u_{h,\epsilon}\|_{2^*}}$ (for $h = 1, ..., k$) satisfie the conditions (22) when $\epsilon > 0$ is sufficiently small.

Step IV: Let $\Omega = D_k \backslash C_\epsilon$; we prove that for every $\epsilon > 0$ it results:

$$(24) \qquad \inf\left\{\int_\Omega |Du|^2 dx | u \in H_s(\Omega);\ \|u\|_{2^*} = 1;\ g(u) \geq 0\right\} > S.$$

In fact, if by contradiction there exists a sequence $(u_n)_n$ in $H_s(\Omega)$ such that

$$\|u_n\|_{2^*} = 1; \quad g(u_n) \geq 0 \quad \text{for every} \quad n \in \mathbb{N} \quad \text{and}$$

$$\lim_{n\to\infty} \int_\Omega |Du_n|^2 dx = S,$$

then proposition (12) and the symmetry of the functions u_n implie that $(u_n)_n$ (or a subsequence) satisfies $\lim_{n\to\infty} g(u_n) = \overline{x}_N$ with $-1 \leq \overline{x}_N \leq -r_0 < 0$, while $g(u_n) \geq 0$ for every $n \in \mathbb{N}$; thus (24) is proved.

Step V: Let $\Omega = D_k \backslash C_\epsilon$; we prove that, when $\epsilon > 0$ is sufficiently small, there exists the following minimum, for every $h = 1, ..., k$:

$$(25) \quad \min\left\{\int_\Omega |Du|^2 dx | u \in H_s(\Omega);\ \|u\|_{2^*} = 1;\quad h - 1 < g(u) < h\right\}.$$

In fact, since

$$S_h \leq \inf\left\{\int_\Omega |Du|^2 dx | u \in H_s(\Omega);\ \|u\|_{2^*} = 1;\ g(u) = h\right\},$$

(22) and (24) implie that, with the notations of definition (2), we have:

$$\inf\left\{\varphi(u) | u \in V;\ h - 1 < g(u) < h\right\} <$$

$$< \inf\left\{\varphi(u) | u \in V;\ g(u) = h \text{ or } g(u) = h - 1\right\};$$

$$\inf\left\{\varphi(u)|u\in V;\ h-1<g(u)<h\right\}\in\left]S,2^{\frac{2}{N}}S\right[.$$

Thus, it suffices to use the Palais-Smale's condition proved in proposition (15) to obtain the existence of the minimum in (25).
Finally we observe that the minimum in (25) is achieved on a function $\overline{u}_h\geq 0$ (otherwise we replace $\overline{u}_h$ by $|\overline{u}_h|$); so, using remarks (3) and (7), we obtain k solutions $u_1,...,u_k$ of problem (1) for $\Omega = D_k\backslash C_\epsilon$ with $\epsilon>0$ sufficiently small, and this completes the proof.

(26) Remark. When $\Omega = D_k\backslash C_\epsilon$ (see notations (4)) it is very plausible that problem (1) has more than k solutions.
Indeed we can prove (the proof is contained in a paper to appear) that:

a) if some r_h for $h=0,...,k-1$ (see notations (4)) is sufficiently small, then we can choose $\epsilon>0$ sufficiently small in such a way that the problem (1) has at least $k+1$ solutions for $\Omega = D_k\backslash C_\epsilon$.

b) it is possible to choose the numbers r_h $(h=0,...,k-1)$ and $\epsilon>0$ sufficiently small, in such a way that the problem (1) has at least $2k$ solutions when $\Omega = D_k\backslash C_\epsilon$ (see notations (4)).

(27) Remark. One can observe that the domain $\Omega = D_k\backslash C_\epsilon$ (see notations (4)) does not have regular boundary, but it is evident that, using the same technique, we can find contractible bounded domains Ω in $\mathbb{R}^N$, with regular boundary and more general shape (provided with an axis of radial symmetry), where the existence and multiplicity result of theorem (8) holds.

(28) Remark. We note that the symmetry of the domain Ω plaies an important role in the proof of theorem (8); but we can prove an analogous existence and multiplicity result also in suitable contractible bounded domains Ω without any symmetry property (this result is contained in a paper to appear).

(29) Remark. The existence result of Ding reported in the introduction can be usefully compared with the following theorem, which can be proved evidently with the same technique used in theorem (8).

(30) Theorem. *Assume $N \geq 3$; let $\Omega = A_r \backslash C_\epsilon$ be defined as in Ding's theorem (see introduction).*
For every $r \in (0,1)$ there exists $\epsilon(r) > 0$ such that, if $r \in (0,1)$ and $\epsilon \in (0, \epsilon(r))$, then $(*)$ has a solution on $\Omega = A_r \backslash C_\epsilon$.

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