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IRREDUCIBLE FILTERS AND SOBER SPACES

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A filter $\underline{F}$ on a convergence space is called irreducible, iff the set conv $\underline{F}$ of convergence points of $\underline{F}$ belongs to $\underline{F}$. A space is sober, iff for every irreducible filter $\underline{F}$ there is a unique point x with conv $\underline{F}$ = conv $\dot{x}$. The category Sob-Conv of sober convergence spaces is a full productive, but not a reflective subcategory of the category Conv of convergence spaces and continuous maps. For a topological space (X,t) the following are equivalent: (i) for every irreducible filter $\underline{F}$ on (X,t) there is a point x with $\underline{F}=\dot{x}$ (ii) (X,t) is both sober and T_D (iii) every subspace of (X,t) is sober (iv) every topological space finer than (X,t) is sober (v) whenever (Y,s) is a T_o-space whose lattice $\underline{O}$(Y,s) of open sets is isomorphic to $\underline{O}$(X,t), then (Y,s) ≈ (X,t). The category Sob-T_1 of sober T_1-spaces is the greatest epi-reflective subcategory of Top consisting of sober spaces, moreover Sob-T_1 is a "disconnectedness" in the sense of Preuß- Arhangel'skiĭ-Wiegandt (generated by all irreducible spaces), hence Sob-T_1 is (extremal epi)-reflective in Top. It is strictly between T_1 and T_2 and different from various sorts of weak Hausdorffness discussed in the literature.

Sober topological spaces were introduced by the Grothendieck school ([2] IV 4.2.1) and independently by T.Blanksma [6]. H.Herrlich has introduced them as the reflective hull of the Sierpinski space in the category Top of topological spaces and continuous maps [13] 1.3.2(e). Another approach to sober spaces is due to L.Skula [38] who introduced them as b-closed subspaces of powers of the Sierpinski space. In [13] and [5] Sob serves as a counterexample to a conjecture of J.F.Kennison (Sob is reflective in Top, but not epi-reflective in Top or T_2).

Later L.D.Nel and R.G.Wilson [31] , L.D.Nel [3o], S.S.Hong [2o], and R.-E.Hoffmann [16] studied sober spaces;

they identified the above mentioned approaches. Furthermore, sober spaces are involved in [19,21] and related papers. More recent investigations are due to R.-E.Hoffmann [18] who establishes a general theory of "topological functors admitting (generalized) Cauchy-completions" emphasizing the analogy between separated Cauchy-completion of a uniform space (resp. a quasi-metric space, resp. a "quasi-normed" vector space) and the sobrification of a topological space, in particular with regard to the Weil extension theorem.

H.Herrlich [13] and S.S.Hong [2o] gave a description of sober spaces in terms of certain open filters, namely proper filters $\underline{F}$ in the lattice $\underline{O}(X)$ of open subsets of a topological space X (ordered by inclusion) enjoying the following property: Every element of $\underline{F}$ contains a limit point of $\underline{F}$, i.e. $\underline{F}$ is a union of open neighborhood filters. These open filters are precisely what H.J.Kowalsky [23] has called a funnel ("Trichter") in a complete lattice L : A proper filter $\underline{F}$ in L is a funnel, iff for every $A \subseteq L$ $\sup A \in \underline{F}$ implies $a \in \underline{F}$ for some $a \in A$. A T_0-space X is sober, iff for every funnel $\underline{F}$ in $\underline{O}(X)$ holds $\bigcap \underline{F} \neq \emptyset$ ([13]), i.e. iff every funnel is an open neighborhood filter ([2o]). Already in 196o Kowalsky characterizes those complete lattices L which are isomorphic to lattices $\underline{O}(X)$ for some space X in terms of funnels ("funnels separate elements of L"); to such an L he associates a "standard space" ${}^{s}L$, i.e. the set of all funnels of L with open sets $O_a = \{\underline{F} \in {}^{s}L \mid a \in \underline{F}\}$ $(a \in L)$. ${}^{s}X := {}^{s}\underline{O}(X)$ is the reflection of X into $\underline{\text{Sob}}$ [2] .- A characterization of the dual lattices, the lattices of closed subsets of a space, in terms of irreducible (= weakly irreducible) elements and a construction of ${}^{s}X$ (as in [2]) was earlier given by S.Papert [33] in 1959 and - hidden in a more general framework - by J.R.Büchi [8] (prop.11,13) in 1952. (N.Funayama's paper [11] 195o is inaccessible to me.)

The present paper gives a definition of "sober space" in terms of filters (surprising to see that this definition is more "natural" than any other topological approach to sober spaces). The concept of "irreducible filter" may be considered as the fundamental concept of the paper. The conditions we investigate concern the elementary behaviour of these filters:

(I1) for every irreducible filter $\underline{F}$ on (X,t) there is a unique $x \in X$ with conv $\underline{F}$ = conv $\dot{x}$ ("sober");

(I2) for every irreducible filter $\underline{F}$ on (X,t) there is a (unique) $x \in X$ with $\underline{F} = \dot{x}$ ("sober + T_D");

(I3) for every irreducible filter on (X,t) there is exactly one convergence point ("sober + T_1").

In section 1 we study (I1) for convergence spaces characterizing in the case of topological spaces the so-called sober spaces. As a "natural" strengthening of (I1) condition (I2) for topological spaces is shown to be equivalent to "sober + T_D". These spaces are shown to be precisely those topologies all of whose subspaces are sober, and, resp., all of whose finer topologies are sober. They are those T_o-spaces which are uniquely determined by their lattice of open sets as discussed by H.J.Kowalsky [23] 5.1. Topological (I3)-spaces, i.e. sober T_1-spaces, are the greatest epireflective subcategory of Top consisting of sober spaces.

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A convergence space (X,q) [22] is a set X and a map q from X into the power set of the set of all proper filters on X subject to the following requirements:

(F1) If $\underline{F},\underline{G}$ are filters on X with $\underline{F} \subseteq G$, $\underline{F} \in q(x)$ for some $x \in X$, then $\underline{G} \in q(x)$.

(F2) For $x \in X$ we designate by $\dot{x}$ the ultrafilter of all subsets of X containing x: $\dot{x} \in q(x)$ for every $x \in X$.

(C) If $\underline{F} \in q(x)$, $x \in X$, then $\underline{F} \cap \dot{x} \in q(x)$. *)

For a filter $\underline{F}$ on (X,q) we define conv $\underline{F}$: $= \{x \in X | \underline{F} \in q(x)\}$.

Recall that a topological space $\mathfrak{X}$ is called "sober" [2], resp. a "p(oint) c(losure)-space" [31], resp. a "spectral space" [19] resp. a "primal space" [21], iff every irreducible, closed, non-empty subset A of X has a unique generic point x, i.e. $A = cl\{x\}$. (Irreducible means that whenever $O_i \cap A \neq \emptyset$ for open sets O_i, i=1,2, then $A \cap O_1 \cap O_2 \neq \emptyset$; Bourbaki [7], in addition, assumes that $A \neq \emptyset$.) "Sober" is strictly between T_o and T_2, it does not imply nor is it implied by T_1.

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*) Condition C is used in the proof that (I2) $\Rightarrow$ (I1) for convergence spaces - see 2.2 proof (a) $\Rightarrow$ (b)(ii).
If $\{x\} = \text{conv}\underline{F} \in \underline{F}$, then $\underline{F} = \dot{x}$, thus (I3) $\Rightarrow$ (I2).

1.1 LEMMA

A subset M of a topological space (X,q) is irreducible and non-empty, iff there is a filter $\underline{F}$ on X with $M \in \underline{F}$ and $M \subseteq \text{conv } \underline{F}$.

Proof:

Let $M \in \underline{F}$ (hence $M \neq \emptyset$) and $M \subseteq \text{conv } \underline{F}$, and let $O_i \cap M \neq \emptyset$ with O_i open in X, i = 1,2, then $O_i \cap \text{conv } \underline{F} \neq \emptyset$, hence $O_i \in \underline{F}$; in consequence $O_1 \cap O_2 \cap M \in \underline{F}$, in particular $O_1 \cap O_2 \cap M \neq \emptyset$

If M is an irreducible, non-empty subspace, then M and the open sets O in X with $O \cap M \neq \emptyset$ generate a filter $\underline{F}_M$ on X. Obviously $M \subseteq \text{conv } \underline{F}_M$.

1.2 LEMMA:

Let $\underline{F}$ be a filter on the topological space (X,q), then conv $\underline{F}$ is closed.

1.3 PROPOSITION:

Let (X,q) be a topological space:

(i) If $\text{conv } \underline{F} \in \underline{F}$ for some filter $\underline{F}$ on X, then conv $\underline{F}$ is an irreducible, closed, non-empty subspace of X.

(ii) If A is an irreducible, closed, nonempty subset of X, then $\text{conv } \underline{F}_A = A$ (cf 1.1 proof).

For the sake of convenience we introduce the following terminology.

1.4 DEFINITION:

A filter $\underline{F}$ on a convergence space (X,conv) is called irreducible, iff $\text{conv } \underline{F} \in \underline{F}$.

We observe that, if $g: (X,q) \to (Y,v)$ is continuous and if $\underline{F}$ is an irreducible filter on (X,q), then $g(\underline{F})$ is irreducible on (Y,v). If $\underline{F}$ is an irreducible filter on (X,q), then so is every refinement of $\underline{F}$.

1.5 THEOREM:

A topological space (X,t) is sober, iff for every irreducible filter $\underline{F}$ on (X,t) there is a unique point $x \in X$ with

$$\text{conv } \underline{F} = \text{conv } \dot{x} .$$

For convergence spaces we use this statement as the d e f i n i t i o n of "sober".

$\underline{G} \subset \underline{O}(X)$ for a topological space X is a funnel, iff $\underline{G}$ consists of the open sets of some $\underline{F}_A$ with $A \subseteq X$ as in 1.3(ii), i.e. iff $\underline{G}$ consists of the open sets of some irreducible filter $\underline{F}$ on X (O.Wyler). - A filter $\underline{F}$ on a topological space X is "weakly irreducible" (2.11 II) iff its open sets form a funnel in $\underline{O}(X)$.

1.6 PROPOSITION:

Let (X_i, q_i) be sober convergence spaces ($i \in I$), then the product space $(\prod_I X_i, \prod_I q_i)$ is sober.

Proof:

Let $\underline{F}$ be an irreducible filter on $(\prod_I X_i, \prod_I q_i)$, then $p_i(\underline{F})$ is irreducible, hence conv $p_i(\underline{F})$ = conv $\dot{x}_i$ for a unique $x_i \in X_i$. Consequently conv $\underline{F}$ = conv $\dot{x}$ with $p_i(x) = x_i$ and x is uniquely determined.

1.7 LEMMA:

Let (X,q) be a sober convergence space, and let $M \subseteq X$. Let q' denote the induced convergence structure on M: If M is closed in (X,q), then (M,q') is sober.

However, other than for topological spaces, there is no universal convergence-sobrification of every convergence space, i.e. the full subcategory Sob-Conv of Conv consisting of all sober convergence spaces is not reflective in Conv ; Sob-Conv is not closed under difference kernels in Conv:

1.8 EXAMPLE:

Let S be the set $\{0,1,2\}$; the only filters on S are $\dot{0}, \dot{1}, \dot{2}, \dot{0} \cap \dot{1}, \dot{0} \cap \dot{2}, \dot{1} \cap \dot{2}, \dot{0} \cap \dot{1} \cap \dot{2}$.

We put $q(0) = \{\dot{0}, \dot{1}, \dot{0} \cap \dot{1}\}$, $q(1) = \{\dot{0}, \dot{1}, \dot{0} \cap \dot{1}\}$, $q(2) = \{\dot{0}, \dot{2}, \dot{0} \cap \dot{2}\}$

Then (S,q) is a pseudo-topological space which is sober.

The map $c : (S,q) \to (S,q)$ with $c(0) = 0$, $c(1) = 1$, $c(2) = 0$ is continuous. The difference kernel of the pair

$$(S,q) \underset{c}{\overset{id_S}{\rightrightarrows}} (S,q)$$

is the subset $S^* = \{0,1\}$ of S supplied with the trace structure q^* which is the co-discrete (=indiscrete) convergence structure, hence - because of conv $\dot{0}$ = conv $\dot{1}$ - not sober. We observe that S^* is open in (S,q).

1.9 REMARK:

In a topological space (X,t) the filter F_A associated with an irreducible nonempty closed set A admits an ultrafilter refinement. Obviously, for every ultrafilter refinement U of F_A holds conv$U$ = A. In consequence, in characterizing "soberness" for topological spaces we can restrict ourselves to irreducible ultrafilters, thus arriving at a concept of "weakly sober" for convergence spaces (X,q):

(X,q) is "weakly sober", iff for every irreducible ultrafilter U on (X,q) there is a unique $x \in X$ with conv U = conv $\dot{x}$. The above results on products, closed subspaces, and the counterexample carries over to "weakly sober". A topological space is weakly sober, iff it is sober.

1.10 REMARK:

The underlying set functor U : Conv→Ens is a topological functor (see [18]). The non-cogenerators in Conv are those spaces (X,q) for which every continuous map $(S^*,q^*) \to (X,q)$ is constant - according to [17] 3.1: In consequence, (S,q) is not U-complete according to [18] 2.5 (1) in connection with [17] 3.8 (last statement). This is in contrast to the situation with topological spaces where "sober" means "complete" in the sense of [18] (in particular [18] 3.1)

In this connection, one is inclined to ask for an *internal* description of those spaces (M,t) which have a superspace (X,t') admitting a filter F with either

(a) M = conv F

or (b) $M \in F$, M = conv F

From the above we immediately deduce that (b) is equivalent to "(M,t) is irreducible and non-empty". Now, it is surprising to see that every topological space satisfies (a)

1.11 THEOREM:

Let (M,t) be a topological space (w.l.o.g. $M \cap \mathbb{N} = \emptyset$ for the set $\mathbb{N}$ of natural numbers). There exists a (distinguished) topology t' on $M \cup \mathbb{N}$ such that the following holds:

(i) the restriction of t' on M is t;

(ii) there is a filter F on $(M \cup \mathbb{N}, t')$ with M = conv$F$;

(iii) if t satisfies T_1, then so does t'.

Proof:

$K \subseteq M \cup \mathbb{N}$ is declared to be t'-open, iff either

(a) $K \cap M = \emptyset$

or (b) $K \cap M$ is t-open, and $\mathbb{N}$-K is finite.

Let F be the filter generated by those t'-open sets K with $\mathbb{N}$ - K finite. Since $M \cup (\mathbb{N}-\{n\}) \in F$ for every $n \in \mathbb{N}$, we have $\{n\} \notin U$ for every ultrafilter refinement U of F, hence conv U = conv F = M.

§ 2

Our description of sober (topological) spaces in terms of irreducible filters naturally leads to the question which topological spaces are characterized by the following requirement:

(I2) If $\underline{F}$ is an irreducible filter on the topological space X*) then there is a (unique) point x in X with

$$\underline{F} = \dot{x}$$

The answer is that X has to be both sober and T_D. A topological space X is called a T_D-space, iff every one point subset is the intersection of an open and a closed subset. T_D-spaces were introduced by C.E.Aull and W.J.Thron [4] (cf. [15,37]). T_D is strictly between T_o and T_1.

For the convenience of the reader we review from the literature the following characterizations of T_D. A point x of a topological space X is an accumulation point of a subset M of X, iff every neighborhood of x meets M in at least one point different from x. The b-topology associated with a topological space X is the topology on X generated by the open sets and the closed sets of X - [38] (see also [1o] p.288).

2.1 PROPOSITION [4,15,31] :

For a topological space X the following are equivalent:

(i) X is T_D.

(ii) For every subset M of X holds: h(M) (= the set of all accumulation points of M) is closed.

(iii) For every subset M of X and every point x of X holds: If x is an accumulation point of the closure of M, then x is an accumulation point of M. (Note that $h \circ cl = cl \circ h$.)

*) For notational convenience we designate both the topological space (X,t) and its underlying set X by the same symbol X.

(iv) The b-topology on X associated with the space X is discrete.

2.2 THEOREM:

Let X be a topological space, then the following are equivalent:

(a) For every irreducible filter $\underline{F}$ on X, there is a (unique) point x in X with $\underline{F} = \dot{x}$.

(b) X is both sober and T_D.

(c) X is sober and it is not (homeomorphic to) the universal sobrification of any subspace Y of X, unless Y is homeomorphic to X.

(d) Every subspace of X is sober.

(e) Every space finer than X is sober.

(f) Whenever Y is a T_o-space with $\underline{O}(Y) \cong \underline{O}(X)$, then Y is homeomorphic to X (cf Kowalsky [23] 5.1, Thron [39] §2).

Proof:

(a) $\Rightarrow$ (b): (i) Let $y \in X$. We consider the filter $\underline{F}_A$ on A:= cl{y} constructed in 1.1. Since $\underline{F}_A$ is irreducible, there is an element x in X with $\underline{F}_A = \dot{x}$ by virtue of our assumption, hence there is an open set O in X with $\{x\} = O \cap A \in \underline{F}_A$. Since A = cl{y}, we have $y \in O$, hence $\{y\} \subseteq O \cap A$. In consequence, y = x and $\{y\} = O \cap A$, i.e. (X,t) is T_D (in particular, it is T_o).

(ii) Let $\underline{F}$ be an irreducible filter on X, then $\underline{F} = \dot{x}$ for some $x \in X$, hence conv $\underline{F}$ = conv $\dot{x}$. Suppose that conv $\dot{x}$ = conv $\dot{y}$, then by axiom (C) $x,y \in$ conv $(\dot{x} \cap \dot{y})$ and $\{x,y\} \in \dot{x} \cap \dot{y}$, i.e. $\dot{x} \cap \dot{y}$ is irreducible. In consequence $\dot{x} \cap \dot{y} = \dot{z}$, hence x = y = z. Now X is sober by 1.5.

(b) $\Rightarrow$ (a) Let $\underline{G}$ be an irreducible filter on (X,t). Since X is sober, there is a point x in X with conv $\underline{G}$ = cl$\{x\} \in \underline{G}$, hence every open set O with $O \cap$ cl$\{x\} \neq \emptyset$, i.e. with $x \in O$ belongs to $\underline{G}$. Since X is T_D, there is an open neighborhood U of x with $\{x\} = U \cap$ cl$\{x\} \in \underline{G}$, hence $\underline{G} = \dot{x}$.

(b) $\Rightarrow$ (c): If X is (homeomorphic to) the universal sobrification of its subspace Y with $Y \not\approx X$, then Y is not sober. Since every b-closed subspace of a sober space is sober [16], the b-closure of Y in X is sober, hence different from Y. In particular, the b-topology on X is not discrete.

(c) $\Rightarrow$ (f) : Suppose $\underline{O}(X) \cong \underline{O}(Y)$, then ${}^{s}Y \approx {}^{s}X = X$. W.l.o.g. Y is a subspace of X, hence $Y \approx X$.

(f) $\Rightarrow$ (b): Suppose X is not sober, then $X \not\approx {}^{s}X$, but $\underline{O}(X) \cong \underline{O}({}^{s}X)$. If X is sober, but not T_D, then there is a proper b-dense subset Y of X not homeomorphic to X (since Y is not sober - [16]). By [18] 3.1.2 X is homeomorphic to ${}^{s}Y$, hence $\underline{O}(X) \cong \underline{O}(Y)$.

(b) $\Rightarrow$ (d): For a subspace of a sober space "b-closed" and "sober" are equivalent [16].

(b) $\Rightarrow$ (e): Let Y be finer than X, and let $M(M_Y)$ be an irreducible, nonempty subspace of Y which is closed in Y. The topology which X induces on $M(M_X)$ is coarser than M_Y, hence irreducible. In consequence, $cl_X\{x\} = cl_X(M)$ for some $x \in X$ (since $cl_X(M)$ is also irreducible in X). Since there is an open set U in X with $\{x\} = U \cap cl_X\{x\}$ (T_D), we have $x \in M$. In consequence $cl_Y\{x\} \subseteq cl_Y(M) = M$. Suppose $b \in M - cl_Y\{x\}$, then there exists a set O open in Y with $b \in O$, $x \notin O$. In consequence, $\emptyset = O \cap U \cap M \subseteq O \cap U \cap cl_X\{x\}$ and $b \in O \cap M$, $x \in U \cap M$, i.e. M_Y is not irreducible. Consequently, we have $M = cl_Y\{x\}$, and Y is sober.

(e) $\Rightarrow$ (b): Suppose $\{x\}$ is not the intersection of an open and a closed set $(x \in X)$, and let Y denote the finer space on the carrier of X generated by the open sets of X and by $\{x\}$. ("cl", "open" always refers to X). $S = cl_X\{x\} - \{x\}$ receives the same topology from Y as from X. S is closed in Y. Since $\{x\}$ is not closed in X, $S \neq \emptyset$. Suppose S is not irreducible: let O, U be open in X with $O \cap S \neq \emptyset$, $U \cap S \neq \emptyset$, $O \cap U \cap S = \emptyset$, hence $(O \cap U) \cap cl_X\{x\} = \{x\}$ in contrast to our assumption. Since Y is sober, there is a $y \in S$ with $cl_Y\{y\} = S$, i.e. every neighborhood of

$z \in cl_X\{x\}$ with $z \neq x$ meets y. Let U be an open neighborhood of x, then there is a $z \in U \cap cl_X\{x\}$ with $z \neq x$, hence U meets y. In consequence, $cl_X\{y\} = cl_X\{x\}$, hence $x = y(T_o)$ - contradiction.

Now we study (I3) for topological spaces (2.3(d)):

2.3 LEMMA:

For a topological space X the following conditions are equivalent:

(a) X is both sober and T_1

(b) Every irreducible non-empty subspace of X is of cardinality 1.

(c) For every irreducible space Y every continuous map $Y \to X$ is constant.

(d) Every irreducible filter on X has exactly one convergence point.

By 2.3(c) the category Sob-T_1 of sober T_1-spaces and continuous maps is a "disconnectedness" in the sense of G.Preuß [34,36], H.Herrlich [12] §14 and A.V.Arhangel'skiĭ - R.Wiegandt [1] which is induced by the class I of irreducible spaces. However, I is not a "connectedness", since I is not "second-additive" ([1] 3.1o(ii) - a counterexample with three points).
$T_o + L^{**}$ (W.Thron [39] p.675) = sober + T_1, furthermore cf G.Preuß [35] 5.3.

Recall [12] that a full isomorphism-closed subcategory X of the category Top is epi-reflective in Top, iff
(i) subspaces of members of X belong to X,
(ii) products of members of X belong to X;
X is (extremal epi)-reflective in Top, iff in addition to (i) and (ii), holds
(iii) every refinement of a member of X belongs to X.
Every disconnectedness is (extremal epi)-reflective in Top [1] 3.7, [12] 14.2.5.

A product of a family of non-empty topological spaces is T_D, iff every space of the family is T_D and all but finite members of the family are T_1-according to [15,37]. A space is T_1, iff all of its powers are T_D [37]. Thus, a productive (resp. reflective) full subcategory of Top consisting of sober T_D-spaces consists of T_1-spaces. Now we have:

2.4 THEOREM:

The category Sob-T_1 of sober T_1-spaces is the greatest epi-reflective subcategory of Top which consists only of sober spaces. Sob-T_1 is strictly smaller than $\underline{T}_1$, strictly greater than $\underline{T}_2$.

(For $\underline{T}_2 \neq$ Sob-T_1 $\neq \underline{T}_1$ see the proof of 2.1o below).

2.5 COROLLARY:

The intersection of an epi-reflective, resp. an (extremal epi)-reflective subcategory $\underline{Y}$ of Top consisting of T_1-spaces with Sob is an epi-reflective, resp. an (extremal epi)-reflective subcategory of Top.

2.6 REMARK:

The Sierpinski space D is both sober and T_D, but not T_1. We observe that Sob is neither stable under refinements nor under subspaces - see [15] 2 (example).

2.7 REMARK:

In [14] the following decreasing chain of classes of "weak Hausdorff spaces" between T_1 and T_2 is discussed: semi-T_2 [29,9] (unique sequential limits), t_2 [27] (subspaces which are continuous images of compact T_2-spaces are closed), T_2' [3,28,32,4o] , resp. LM-T_2 [24,25] (quasi-compact subspaces are closed, resp. T_2). Except for T_2' they form (extremal epi)-reflective subcategories of Top.

Adding the soberness requirement to semi-T_2,t_2,T_2' and LM-T_2, one obtains a chain of four full subcategories of Top strictly between Sob-T_1 and $\underline{T}_2$. Examining several examples given in the literature ([28],[24] 3.6,[26],[9] , especially [14]) one easily proves that these new properties are pairwise different and do not coincide with one of the before-mentioned "weak Hausdorff spaces". (2.8(a) below is a useful criterion.) In this connection the question arises whether there is an (extremal epi)-reflective subcategory $\underline{X}$ of $\underline{Top}$ with $\underline{X} \cap \underline{Sob}=\underline{T}_2$, $\underline{X} \neq \underline{T}_2$?

2.8 LEMMA:

(a) A T_1-space X containing a co-finite T_2-subspace M is sober.

(b) A T_o-space X is sober, iff X contains a co-finite open sober subspace M.

Proof:

Since every T_2-space is sober, (a) is a consequence of (b).

(b): Let A be an irreducible closed non-empty subspace of X. Suppose $A \cap M = \emptyset$, then A is a finite T_o-space, hence sober [16] 1.8, hence $A = cl\{x\}$ for a unique $x \in A$. Suppose now that $A \cap M \neq \emptyset$. Since $A \cap M$ is open in A, $A \cap M$ is irreducible. Since M is sober, there is a unique point $x \in M$ with $cl_M\{x\} = A \cap M$, i.e. $M \cap cl\{x\} = M \cap A$, hence $M \cap A$ and $A - cl\{x\}$ are disjoint open subsets of A. Since A is irreducible, $A - cl\{x\} = \emptyset$. Since $x \in A$, $cl\{x\} \subseteq A$, hence $cl\{x\} = A$. Since X is T_o, x is the unique generic point of A.

2.9 PROPOSITION:

A space X is sober, iff every point p of X has a sober neighborhood U_p.

Proof:

The interior M_p of U_p is open, hence b-closed in U_p, hence sober. Let A be an irreducible, closed subset of X with $p \in A$. Then $p \in A \cap M_p$ and the same considerations as in the proof of 2.8(b) apply.

Thron's example [39] p.675/676 suggests the following theorem 2.1o. Recall that a topological space X is called minimal with respect to a class of spaces $\underline{P}$, iff $X \in \underline{P}$ and there is no space in $\underline{P}$ which is coarser than X.

2.1o THEOREM:

In $\underline{Sob-T_1}$ minimal spaces are finite discrete spaces.

Proof:

If X is a T_1-space, $p \in X$, then those open subsets O with $p \notin O$ and all co-finite subsets of X determine a coarser T_1-space X'. {If X-{p} is not discrete, X' is not T_2 (X = [o,1] in [39] p.675/676).} Since X is T_1, X - {p} is open in X and it receives the same topology from X and X'. If X is both sober and T_1, then X - {p} is also sober, since it is open, hence b-closed. Thus X' is also sober, since X - {p} is co-finite in X' (see 2.8). Thus an open set O of a space X which is minimal in Sob-T_1 with $p \in O$ for some $p \in X$ must be co-finite. Since the co-finite topology on an infinite set is irreducible and T_1, hence not sober, such a minimal space must be finite.

2.11 ADDENDA:

I. There is an "ultrafilter version" 2.2(a') of 2.2(a) equivalent to 2.2(a):
X is T_o and for every irreducible ultrafilter $\underline{F}$ on X there is a (unique) $x \in X$ with $\underline{F} = \dot{x}$.*)
Similarly in 2.3(d) filters can be replaced by ultrafilters. Note that these statements are on the level of t o p o l o g i c a l spaces.

II. "WEAKLY IRREDUCIBLE FILTERS AND STRONGLY SOBER SPACES":
A filter $\underline{F}$ on a convergence space (X,q) is "weakly irreducible", iff for every $M \in \underline{F}$ holds $M \cap \text{conv } \underline{F} \neq \emptyset$ (hence conv $\underline{F} \neq \emptyset$) - this is in a sense "dual" to the definition of open. An ultrafilter is weakly irreducible iff it is irreducible. A filter $\underline{F}$ on a topological space is weakly irreducible, iff it admits an irreducible ultrafilter refinement $\underline{U}$ with conv $\underline{F}$ = conv $\underline{U}$ (hence conv $\underline{F}$ is irreducible).
A convergence space (X,q) is "strongly sober", iff for every weakly irreducible filter $\underline{F}$ on (X,q) there is a unique point $x \in X$ with conv $\underline{F}$ = conv $\dot{x}$. 1.6, 1.7 carry over; (S,q)(1.8) is strongly sober ($\dot{0} \cap \dot{2}$ is weakly irreducible with "generic" point 2, but not irreducible).
Replacing "irreducible" in (I1), (I2), (I3) by "weakly irreducible", we get (J1), (J2), (J3). For topological spaces we have: (J1) = "sober", (J2) = "discrete", (J3) = "sober+T_1" (note that every neighborhood filter of a point is weakly irreducible).

*) "T_o" is not superfluous as every finite space shows.

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