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THE GREEN FUNCTION FOR UNIFORMLY ELLIPTIC EQUATIONS

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The authors discuss a generalization of the usual Green function to equations with only measurable and bounded coefficients. The existence and uniqueness as well as several other important properties are shown. Such a Green function proves useful in connection with quasilinear elliptic systems of "diagonal type".

0. Introduction

In this paper we are concerned with the Green function for the following kind of uniformly elliptic operator

$$(*) \qquad Lu = - \sum_{i,j=1}^{n} D_j(a^{ij} D_i u)$$

in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$. Here the a^{ij} are supposed to be bounded measurable functions such that the matrix (a^{ij}) is uniformly positive definite in Ω. This concept of a Green function is a straightforward generalization of the familiar one in potential theory.

For two reasons we only consider the case $n \geq 3$. The first one is that the different behaviour of the Green function in dimension two (there does not exist a Green function for $\Omega = \mathbb{R}^2$) sometimes requires a different method. The second one is the fact that the applications mentioned below only treat the case of dimension at least three.

In section one we give a proof of the fundamental existence and uniqueness theorem. Furthermore, we derive various interesting properties of the Green function as well as a series of useful inequalities.

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Section two contains an investigation of the regular points for an elliptic operator as in (*), and it is shown that the concept of "regular point" is the same for all such operators.

In the final chapter we strengthen the hypotheses on the regularity of the coefficients a^{ij}, and we are able to prove additional estimates for the Green function and its derivatives in this case.

In recent years the Green function has been used with considerable success in the theory of quasilinear elliptic systems of "diagonal type", c.f. [7], [8], [9], [11], [12], [3]. Variational inequalities can also be attacked by this means as is shown in [10], [14]. Furthermore, these methods proved useful in the theory of harmonic mappings between Riemannian manifolds [5], [6].

The basic facts about the Green function for symmetric operators were already proved in [15] by Littman, Stampacchia and Weinberger. The case of non-symmetric coefficients, however, could not be attacked by their methods, while we are able to handle both cases in a unified manner. In addition we provide complete proofs of various facts about the Green function, which have frequently been used in the papers cited above. In this way our paper may be considered as a reference for the Green function and its properties, so that it may become a useful tool for other authors.

We would like to point out that most of the material presented here is already contained in [18], [19], [20], [4].

The first named author wants to express his gratitude to Linköping University for the kindness and hospitality during his stay there in the spring of 1979.

1. Existence, uniqueness and basic properties of the Green function

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, we shall consider elliptic operators of the following type.

(1.1) $$Lu = -\sum_{i,j=1}^{n} D_j(a^{ij}D_iu).$$

The coefficients shall satisfy $a_{ij} \in L_\infty(\Omega)$ and there exist $0 < \lambda \le \mu$ such that

(1.2) $$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \text{ and } a^{ij}\xi_i\eta_j \le \mu|\xi||\eta|$$

holds for all $\xi, \eta \in \mathbb{R}^n$ and almost all $x \in \Omega$. The following theorem will be proved.

(1.1) Theorem. *There exists a unique function* $G: \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$, $G \ge 0$, *such that for each* $y \in \Omega$ *and any* $r > 0$

(1.3) $$G(\cdot,y) \in H_2^1(\Omega \setminus B_r(y)) \cap \overset{\circ}{H}{}_1^1(\Omega)$$

and for all $\phi \in C_c^\infty(\Omega)$

(1.4) $$a(G(\cdot,y),\phi) = \phi(y).$$

Here and in the sequel we use the abbreviation

$$a(u,v) := \int_\Omega a^{ij} D_i u\, D_j v$$

whenever the right hand side is defined. This so called Green function enjoys the following properties:
For each $y \in \Omega$ $(G(x) := G(x,y))$

(1.5) $$G \in L^*_{\frac{n}{n-2}}(\Omega) \textit{ with } \|G\|_{L^*_{\frac{n}{n-2}}} \le K(n)\lambda^{-1},$$

(1.6) $$\nabla G \in L^*_{\frac{n}{n-1}}(\Omega) \textit{ with } \|\nabla G\|_{L^*_{\frac{n}{n-1}}} \le K(n,\mu,\lambda),$$

(1.7) $$G \in \overset{\circ}{H}{}_s^1(\Omega) \textit{ for each } s \in [1, \tfrac{n}{n-1}[.$$

For all $x,y \in \Omega$ *we have*

(1.8) $$G(x,y) \le K_1(n,\mu/\lambda)\lambda^{-1}|x-y|^{2-n};$$

and for all $x,y \in \Omega$ *satisfying* $|x-y| \le \frac{1}{2}$ *dist*$(y,\partial\Omega)$

(1.9) $$G(x,y) \ge K_2(n,\mu/\lambda)\mu^{-1}|x-y|^{2-n}$$

Remark. A careful inspection of the proof shows that e.g. in the case of a symmetric operator one can take

(1.10) $$K_1(n,\mu/\lambda) = K(n)(\mu/\lambda)^{\frac{n-2}{2}}(1+\log(\mu/\lambda)).$$

The constant in the estimate from below is worse because we use the Harnack inequality in which the best constant is

$c(n)^{1+(\mu/\lambda)^{1/2}}$.

Let us at first introduce some notation and state a well known lemma in the form in which we are going to apply it.

Ω is always a bounded domain in $\mathbb{R}^n$, $n \geq 3$. If A is a measurable subset of $\mathbb{R}^n$ and $u \in L_1(A)$,

$$\fint_A u := \mathbb{L}^n(A)^{-1} \int_A u(x)\, d\,\mathbb{L}^n(x);$$

here $\mathbb{L}^n$ denotes the n-dimensional Lebesgue measure. Let $B_\rho(x)$ be the ball of radius ρ centered at x.

For $p > 1$ we define the Banach space $L_p^*(\Omega)$ by

$$L_p^*(\Omega) := \{f : \Omega \to \mathbb{R} \cup \{\infty\},\ f \text{ measurable and } \|f\|_{L_p^*(\Omega)} < \infty\}$$

where $\|f\|_{L_p^*(\Omega)} := \sup\limits_{t>0} t[\mathbb{L}^n\{x \in \Omega : |f(x)| > t\}]^{1/p}$.

We note

$$(1.11) \qquad \|f\|_{L_p^*(\Omega)} \leq \|f\|_{L_p(\Omega)} \quad \text{and}$$

$$(1.12) \qquad \|f\|_{L_{p-\varepsilon}(\Omega)} \leq \left(\frac{p}{\varepsilon}\right)^{\frac{1}{p-\varepsilon}} [\mathbb{L}^n(\Omega)]^{\frac{\varepsilon}{p(p-\varepsilon)}} \|f\|_{L_p^*(\Omega)}$$

for $0 < \varepsilon \leq p-1$.

The following lemma due to Moser [16] will be used frequently.

<u>(1.2) Lemma</u>. *Let* $u \in H_2^1(\Omega)$ *be a non-negative subsolution of* $Lu \leq 0$. *There exists a constant* $K(n)$, *such that for* $\alpha > 1$ *and* $B_\rho(x) \subset\subset \Omega$

$$(1.13) \qquad \sup_{B_{\rho/2}(x)} u^\alpha \leq \left(\frac{\alpha}{\alpha-1}\right)^n \left(\frac{\mu}{\lambda}\right)^n K(n) \fint_{B_\rho(x)} u^\alpha .$$

In the case of symmetric coefficients one can replace $\left(\frac{\mu}{\lambda}\right)^n$ *by* $\left(\frac{\mu}{\lambda}\right)^{n/2}$.

We are now ready to give the

Proof of Theorem (1.1)

Let $y \in \Omega$ be fixed. Due to the ellipticity condition (1.2) we may consider $a(\cdot,\cdot)$ as a continuous, positive definite bilinear form on $\overset{\circ}{H}{}^1_2(\Omega) \times \overset{\circ}{H}{}^1_2(\Omega)$. For fixed $\rho > 0$ $(B_\rho := B_\rho(y))$

$$\overset{\circ}{H}{}^1_2(\Omega) \ni \phi \longmapsto \fint_{B_\rho} \phi$$

is a bounded linear functional on $\overset{\circ}{H}{}^1_2(\Omega)$. By the Lax-Milgram-Theorem there exists a unique function $G^\rho \in \overset{\circ}{H}{}^1_2(\Omega)$, such that for all $\phi \in \overset{\circ}{H}{}^1_2(\Omega)$

$$(1.14) \qquad a(G^\rho,\phi) = \fint_{B_\rho} \phi .$$

Inserting $|G^\rho| \in \overset{\circ}{H}{}^1_2(\Omega)$ as a test function in (1.14) we find

$$(1.15) \qquad a(G^\rho,G^\rho) = \fint_{B_\rho} G^\rho \le \fint_{B_\rho} |G^\rho| = a(G^\rho,|G^\rho|).$$

Accordingly we have for a $K \ge 1$

$$(1.16) \qquad a(G^\rho,G^\rho) = a(G^\rho,|G^\rho|/K) = a(|G^\rho|/K,G^\rho).$$

We get

$$(1.17) \qquad a(|G^\rho|/K,|G^\rho|/K) = K^{-2}a(G^\rho,G^\rho) \le a(G^\rho,|G^\rho|/K),$$

$$(1.18) \qquad a(|G^\rho|/K - G^\rho,|G^\rho|/K - G^\rho) \le 0,$$

which implies $|G^\rho| = K\, G^\rho$, so that

$$(1.19) \qquad G^\rho \ge 0.$$

We shall now give an estimate for $\|G^\rho\|_{L^*_{\frac{n}{n-2}}(\Omega)}$.

For that purpose we use for $t > 0$ as a test function in (1.14) $\phi(x) := [1/t - 1/G^\rho(x)]^+ := \max\{0, 1/t - 1/G^\rho(x)\}$. With the notation $\Omega_t := \{x \in \Omega : G^\rho(x) > t\}$ we get

$$(1.20) \qquad \int_{\Omega_t} a^{ij} D_i G^\rho D_j G^\rho (G^\rho)^{-2} = \fint_{B_\rho} \phi \le \frac{1}{t} .$$

Therefore

$$(1.21)\qquad \int_{\Omega_t} |\nabla G^\rho|^2/(G^\rho)^2 \le t^{-1}\lambda^{-1} .$$

Next we consider $v(x) := [\log(G^\rho(x)) - \log t]^+, v \in \overset{\circ}{H}{}^1_2(\Omega)$ and in view of Sobolev's inequality and (1.21) we estimate

$$(1.22)\qquad \left[\int_{\Omega_t} (\log(G^\rho/t))^{\frac{2n}{n-2}}\right]^{\frac{n-2}{n}} \le K(n)\lambda^{-1}t^{-1}.$$

But this implies

$$(1.23)\qquad (\log 2)^2 \,[\mathbb{L}^n(\Omega_{2t})]^{\frac{n-2}{n}} \le K\lambda^{-1}t^{-1}.$$

Setting $s = 2t$ we get from (1.23) for all $s > 0$

$$(1.24)\qquad s\,[\mathbb{L}^n(\Omega_s)]^{\frac{n-2}{n}} \le K\,\lambda^{-1}.$$

Thus we have shown

$$(1.25)\qquad \|G^\rho\|_{L^*_{\frac{n}{n-2}}(\Omega)} \le K(n)\lambda^{-1}.$$

To estimate the Dirichlet integral of G^ρ we use G^ρ as a test function and Sobolev's inequality

$$\lambda\int_\Omega |\nabla G^\rho|^2 \le \int_\Omega a^{ij}D_iG^\rho D_jG^\rho = \fint_{B_\rho} G^\rho \le$$

$$(1.26)\qquad \le K(n)\rho^{-n}\left(\int_{B_\rho} (G^\rho)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \rho^{n(1-\frac{n-2}{2n})} \le$$

$$\le K(n)\,\rho^{\frac{2-n}{2}}\left(\int_\Omega |\nabla G^\rho|^2\right)^{1/2} .$$

That is

$$(1.27)\qquad \int_\Omega |\nabla G^\rho|^2 \le K(n)\lambda^{-2}\rho^{2-n}.$$

We now give a pointwise estimate for G^ρ

$$(1.28)\qquad G^\rho(x) \le K(n,\mu/\lambda)\lambda^{-1}\,|x-y|^{2-n} \text{ if } |x-y| \ge 2\rho.$$

Let $R := |x-y| \ge 2\rho$. First we consider the case $B_{R/2}(x) \subset \Omega$.

As G^ρ is a solution of $Lu=0$ in $\Omega\setminus B_\rho$, we may use Lemma (1.2) to get

$$(1.29)\qquad (G^\rho(x))^\alpha \le K(\alpha,n)\,K(\mu/\lambda) \fint_{B_{R/4}(x)} (G^\rho)^\alpha$$

where we restrict α to be less than $n/(n-2)$, so that the L_α-norm can be estimated by the $L^*_{\frac{n}{n-2}}$ - norm.

$$(1.30)\qquad \fint_{B_{R/4}(x)} (G^\rho)^\alpha \le K(\alpha,n)R^{\alpha(2-n)}\|G^\rho\|^\alpha_{L^*_{\frac{n}{n-2}}} .$$

Taking into account (1.25) we have proved (1.28). If $B_{R/2}(x)\not\subset\Omega$ we consider a region $\tilde\Omega$ so large that $B_{R/2}(x)\subset\tilde\Omega$. As we may extend the operator L to $\tilde\Omega$, we get a function $\tilde G^\rho$. Restricting $\tilde G^\rho$ to Ω we see that $L(G^\rho-\tilde G^\rho) = 0$ in Ω. But on $\partial\Omega$ $0=G^\rho\le\tilde G^\rho$ and the maximum principle implies $G^\rho \le \tilde G^\rho$ throughout Ω. As (1.28) is true for $\tilde G^\rho$ with the same constants we have proved it also for G^ρ.

We shall now show

$$(1.31)\qquad \|\nabla G^\rho\|_{L^*_{\frac{n}{n-1}}(\Omega)} \le K(n,\mu,\lambda).$$

We choose a function η satisfying $\eta\equiv 1$ outside of B_R, $\eta\equiv 0$ in $B_{R/2}$ and $|\nabla\eta|\le\frac{K}{R}$, and insert $G^\rho\eta^2$ into (1.14). If $R\ge 4\rho$ we get

$$(1.32)\qquad \int_{\Omega\setminus B_R} |\nabla G^\rho|^2 \le K(\tfrac{\mu}{\lambda})^2\, R^{-2} \int_{B_R\setminus B_{R/2}} (G^\rho)^2 \le K(n,\mu,\lambda)R^{2-n},$$

where we have used (1.28). If $R<4\rho$ we use (1.27) to get the same estimate.

For $t>0$ let $\Omega_t := \{x\in\Omega : |\nabla G^\rho|>t\}$ and $R=t^{-\frac{1}{n-1}}$. We have $t^2\mathbb{L}^n(\Omega_t\cap\Omega\setminus B_R)\le K(n,\mu,\lambda)t^{\frac{n-2}{n-1}}$ which is the same as

$$(1.33)\qquad \mathbb{L}^n(\Omega_t\cap\Omega\setminus B_R) \le [K(n,\mu,\lambda)t^{-1}]^{\frac{n}{n-1}} .$$

Trivially $\mathbb{L}^n(\Omega_t \cap B_R) \leq K(n)R^n = [K(n)t^{-1}]^{\frac{n}{n-1}}$. Therefore we have proved (1.31).

Now (1.31) implies in virtue of (1.12) that we have for each $s \in [1, \frac{n}{n-1}[$ a uniform bound on $\|G^\rho\|_{\overset{\circ}{H}{}^1_s(\Omega)}$ with respect to ρ. Considering sequences $\rho_\nu \to 0$ and $s_\nu \to \frac{n}{n-1}$ we find by a diagonal process a subsequence $\{G^{\rho_\mu}\}$ of $\{G^{\rho_\nu}\}$ and a function $G \in \overset{\circ}{H}{}^1_s(\Omega)$ for all $s < \frac{n}{n-1}$, such that

$$(1.34) \qquad G^{\rho_\mu} \rightharpoonup G \text{ in } \overset{\circ}{H}{}^1_s(\Omega),\ s \in [1, \tfrac{n}{n-1}[\ .$$

We have proved (1.7). To show (1.4) let $\phi \in C_c^\infty(\Omega)$. Then $a(\cdot,\phi)$ is a continuous linear functional on $\overset{\circ}{H}{}^1_s(\Omega)$ for $s < \frac{n}{n-1}$. In particular (1.34) shows $a(G^{\rho_\mu},\phi) \to a(G,\phi)$. As $a(G^{\rho_\mu},\phi) = \displaystyle\fint_{B_{\rho_\mu}} \phi \to \phi(y)$ we have

$$(1.35) \qquad \int_\Omega a^{ij} D_i G(x,y) D_j \phi(x)\, d\mathbb{L}^n(x) = \phi(y).$$

As a consequence of (1.25) and (1.31) we now derive (1.5) and (1.6).

The L_p-norms are weakly lower semicontinuous, so that we get ($p := \frac{n}{n-2}$, $0 < \varepsilon < p-1$, $\Omega_t := \{x \in \Omega : G(x) > t\}$)

$$\|G\|^{p-\varepsilon}_{L_{p-\varepsilon}(\Omega_t)} \leq \liminf_{\mu\to\infty} \|G^{\rho_\mu}\|^{p-\varepsilon}_{L_{p-\varepsilon}(\Omega_t)} \leq$$

$$(1.36) \qquad \leq \liminf_{\mu\to\infty} \left(\tfrac{p}{\varepsilon}\right)(\mathbb{L}^n(\Omega_t))^{\frac{\varepsilon}{p}} \|G^{\rho_\mu}\|^{p-\varepsilon}_{L^*_p(\Omega)}$$

$$\leq \left(\tfrac{p}{\varepsilon}\right)(\mathbb{L}^n(\Omega_t))^{\frac{\varepsilon}{p}} (K(n)\lambda^{-1})^{p-\varepsilon}.$$

This implies

$$t(\mathbb{L}_n(\Omega_t))^{\frac{1}{p}} \le \left(\frac{p}{\varepsilon}\right)^{\frac{1}{p-\varepsilon}} K(n)\lambda^{-1}. \tag{1.37}$$

Letting $\varepsilon \to p-1$ we get (1.5). The same argument gives (1.6). As consequences of (1.28) and (1.32) we get (1.8) and

$$\int_{\Omega \setminus B_R(y)} |\nabla G|^2 \le K(n,\mu,\lambda) R^{2-n} \quad \text{if} \quad B_R(y) \subset\subset \Omega. \tag{1.38}$$

In fact in view of (1.32) we may also assume $G^{\rho_\mu} \rightharpoonup G$ in $H^1_2(\Omega \setminus B_R(y))$, so that (1.38) follows immediately from the weak lower semicontinuity of the Dirichlet integral. With the help of Rellich's theorem we may even assume $G^{\rho_\mu}(x) \to G(x)$ for almost all $x \in \Omega$. Now (1.8) is an easy consequence of (1.28), because $G(\cdot,y)$ is Hölder continuous in $\Omega \setminus \{y\}$. This follows from the famous de Giorgi-Nash regularity theorem, because $L\,G(\cdot,y) = 0$ in $\Omega \setminus B_R(y)$.

Apart from the uniqueness and property (1.9) Theorem (1.1) is proved.

We are now going to give the

Proof of (1.9)

For that purpose we consider any function $G \ge 0$ satisfying (1.3) and (1.4), because we have not yet proved the uniqueness.

So let $x,y \in \Omega$, $|x-y| < \frac{1}{2}\,\mathrm{dist}(y,\partial\Omega)$ and $r = |x-y|$. Consider a cut-off function $\eta \in C_c^\infty(\Omega)$ which is one on $B_r(y) \setminus B_{\frac{r}{2}}(y)$ and zero outside of $B_{\frac{3r}{2}}(y) \setminus B_{\frac{r}{4}}(y)$ and having the properties $0 \le \eta \le 1$ and $|\nabla\eta| \le \frac{K}{r}$. Inserting $G(\cdot,y)\eta$ as a testfunction into (1.4) we get at once

$$\int_{\frac{r}{2} \le |z-y| \le r} |\nabla G(z,y)|^2 \, d\mathbb{L}^n(z) \le K\left(\frac{\mu}{\lambda}\right)^2 \frac{1}{r^2} \int_{\frac{r}{4} \le |z-y| \le \frac{3r}{2}} G(z,y)^2 d\mathbb{L}^n(z)$$

$$\le K\left(\frac{\mu}{\lambda}\right)^2 r^{n-2} \Big(\sup_{\frac{r}{4} \le |z-y| \le \frac{3r}{2}} G(z,y)^2 \Big). \tag{1.39}$$

Choosing a similar cut-off function φ which is one on

$B_{\frac{r}{2}}(y)$ and zero outside of $B_r(y)$ we conclude

$$1 = \int_{\frac{r}{2}\le|z-y|\le r} a^{ij}D_iG(\cdot,y)D_j\varphi \le \mu\frac{K}{r} \int_{\frac{r}{2}\le|z-y|\le r} |\nabla G(\cdot,y)| \le$$

$$(1.40) \qquad \le \mu\ K\ r^{\frac{n}{2}-1}(K(\tfrac{\mu}{\lambda})^2\ r^{n-2}(\sup_{\frac{r}{4}\le|z-y|\le\frac{3r}{2}} G(z,y)^2))^{\frac{1}{2}} \le$$

$$\le K(n,\mu/\lambda)\mu\ r^{n-2} \inf_{\frac{r}{4}\le|z-y|\le\frac{3r}{2}} G(z,y) \le$$

$$\le K(n,\mu/\lambda)\mu\ |x-y|^{n-2}\ G(x,y).$$

Here we have used the Harnack inequality. This completes the proof of (1.9).

Now it only remains to show the uniqueness of the Green function.

<u>Proof</u> of Uniqueness

We already know that there exists one function $G \ge 0$ having the properties (1.3), (1.4). Let $\tilde{G}$ be another such function. Let $y \in \Omega$ be fixed. We write $G(x) = G(x,y)$ as well as $\tilde{G}(x) = \tilde{G}(x,y)$ and define

$$(1.41) \qquad m(\rho) := \inf_{\partial B_\rho(y)} \tilde{G}.$$

We treat two cases seperately.

I. There exist $K, \rho_o > 0$, such that $\rho \le \rho_o$ implies $m(\rho) \ge K$. We define a new function v^ρ by

$$(1.42) \qquad v^\rho(x) := \begin{cases} \inf(\tilde{G}(x), m(\rho)), & \text{if } |x-y| \ge \rho \\ m(\rho) & , \text{ if } |x-y| < \rho. \end{cases}$$

As the capacity of the ball of radius ρ is $c_n\rho^{n-2}$, we get

$$(1.43) \qquad \int_\Omega a^{ij}D_iv^\rho D_jv^\rho \ge \lambda \int_\Omega |\nabla v^\rho|^2 \ge \lambda c_n \rho^{n-2}(m(\rho))^2.$$

Using (1.4) we see

(1.44) $m(\rho) = \int_\Omega a^{ij}D_i\tilde{G}D_jv^\rho = \int_\Omega a^{ij}D_iv^\rho D_jv^\rho \geq \lambda c_n \rho^{n-2}(m(\rho))^2,$

because $\nabla\tilde{G} = \nabla v^\rho$ on the set where $\nabla v^\rho \neq 0$.
The Harnack inequality gives for $|x-y| = \rho$

(1.45) $\tilde{G}(x) \leq K(n,\mu/\lambda)\lambda^{-1}|x-y|^{2-n}.$

If therefore $c > 0$ is small enough we get from (1.9) and the maximum principle

(1.46) $G - c\,\tilde{G} \geq 0$ in Ω.

Let $c_o := \sup\{c : G - c\,\tilde{G} \geq 0 \text{ in } \Omega\}$. We have $u := G - c_o\tilde{G} \geq 0$ in Ω; let us show $c_o = 1$.

Looking again at $m^*(\rho) := \inf_{\partial B_\rho(y)} u$ we have to distinguish two cases.

(i) There are constants $K^*, \rho_1 > 0$, such that $\rho \leq \rho_1$ implies $m^*(\rho) \geq K^*$. The arguments used above for $\tilde{G}$ apply again and we would have $(1-c_o) > 0$ and therefore $u(x) \geq K\;(n,\mu/\lambda)\mu^{-1}|x-y|^{2-n}$ for $|x-y| \leq \frac{1}{2}\operatorname{dist}(y,\partial\Omega)$. But this implies using (1.45) $G - (c_o+\delta)\tilde{G} \geq 0$ in Ω for some positive δ. This contradicts the maximality of c_o, so we must have

(ii) There exists a sequence $\rho_\nu \to 0$ such that $\lim_{\nu\to\infty} m^*(\rho_\nu) = 0$. As u is a solution of $Lu = 0$ in $\Omega\setminus\{y\}$ we get using the Harnack inequality again

(1.47) $\int_{B_{2\rho_\nu}\setminus B_{\rho_\nu}} |\nabla u| \leq K(n,\mu/\lambda)\rho_\nu^{n-1}\; m^*(\rho_\nu).$

We now choose a cut-off function η equal to zero on $B_{\rho_\nu}(y)$ and equal to one on $\Omega\setminus B_{2\rho_\nu}(y)$. Inserting $u\eta$ as a test function we get

(1.48) $\int_{\Omega\setminus B_{\rho_\nu}} a^{ij}D_iuD_ju\eta = -\int_{B_{2\rho_\nu}\setminus B_{\rho_\nu}} a^{ij}D_iuD_j\eta u,$

which implies, again by the Harnack inequality and (1.47),

$$(1.49)\qquad \int_{\Omega\setminus B_{2\rho_\nu}} |\nabla u|^2 \le K(n,\mu/\lambda)\rho_\nu^{n-2}\ (m^*(\rho_\nu))^2,$$

from which we conclude $\nabla u \equiv 0$ in Ω. Therefore $G = c_o\tilde{G}$ from which $c_o=1$ by (1.4) and $G=\tilde{G}$. The second case

II. $\lim_{\nu\to\infty} m(\rho_\nu) = 0$ for a sequence $\rho_\nu \searrow 0$

leads immediately to a contradiction in the same way as above and the uniqueness of the Green function has been proved.

As a consequence of the next theorem we get for symmetric coefficients ($a^{ij} = a^{ji}$) the symmetry of the Green function $G(x,y) = G(y,x)$.

<u>(1.3) Theorem</u>. *Let* $L_t := -\sum_{i,j=1}^{n} D_j(a^{ji}D_i)$ *be the adjoint operator to* L *and consider the Green functions* G *and* G_t *corresponding to* L *and* L_t.
Then for all points $x,y\in\Omega$ *we have*

$$(1.50)\qquad G(x,y) = G_t(y,x).$$

<u>Proof</u> of Theorem (1.3)

Let $x,y\in\Omega$, $x\neq y$. We have sequences $\{\rho_\nu\}$, $\{\sigma_\mu\}$ tending to zero ($\rho_\nu,\sigma_\mu < \frac{1}{3}|x-y|$) and corresponding functions $G^{\rho_\nu}(\cdot,y)$, $G_t^{\sigma_\mu}(\cdot,x)$ which converge a.e. to $G(\cdot,y)$ and $G_t(\cdot,x)$ respectively.
Inserting them as test functions we get

$$(1.51)\qquad \fint_{B_{\rho_\nu}(y)} G_t^{\sigma_\mu}(\cdot,x) = \fint_{B_{\sigma_\mu}(x)} G^{\rho_\nu}(\cdot,y) =: a_{\nu\mu}.$$

Letting σ_μ tend to zero we have ($G^{\rho_\nu}(\cdot,y)$ is continuous on $B_{\sigma_\mu}(x)$, $G_t^{\sigma_\mu}(\cdot,x)\longrightarrow G_t(\cdot,x)$)

$$(1.52)\qquad \fint_{B_{\rho_\nu}(y)} G_t(\cdot,x) = G^{\rho_\nu}(x,y).$$

As $G_t(\cdot,x)$ is continuous on $B_{\rho_\nu}(y)$ we conclude

$$(1.53) \qquad G_t(y,x) = \lim_{\nu\to\infty} G^{\rho_\nu}(x,y).$$

That is $\lim_{\nu\to\infty}\lim_{\mu\to\infty} a_{\nu\mu} = G_t(y,x)$. In the same manner we get $\lim_{\mu\to\infty}\lim_{\nu\to\infty} a_{\nu\mu} = G(x,y)$. To prove (1.50) one only has to observe that the double sequence $\{a_{\nu\mu}\}$ converges uniformly in μ with respect to ν. This is true, because we can bound the Hölder-norm of $G^{\rho_\nu}(\cdot,y)$ on $B_{\sigma_\mu}(x)$ independent of ν. This completes the proof. Using (1.52) for $\rho < \text{dist}(y,\partial\Omega)$ and (1.50) we get the following representation formula.

<u>(1.4) Proposition</u>. *For any* $x,y \in \Omega, \rho > 0$ *such that* $\rho < \text{dist}(y,\partial\Omega)$

$$(1.54) \qquad G^\rho(x,y) = \fint_{B_\rho(y)} G(x,z)\,d\mathbb{L}^n(z).$$

This implies that there exist $K = K(n,\mu/\lambda) > 0$ *such that*

$$(1.55) \qquad G^\rho(x,y) \le K\lambda^{-1}\rho^{2-n},$$

$$(1.56) \qquad G^\rho(x,y) \le K\lambda^{-1}|x-y|^{2-n}.$$

<u>Proof</u> of Proposition (1.4)

Using (1.54) and (1.8) we conclude

$$\rho^n G^\rho(x,y) \le K\lambda^{-1}\int_{B_\rho(y)} |x-z|^{2-n}d\mathbb{L}^n(z) \le$$
$$\le K\lambda^{-1}\int_{B_\rho(x)} |x-z|^{2-n}d\mathbb{L}^n(z) =$$
$$= K\lambda^{-1}\rho^2.$$

To prove (1.56) we only have to consider the case $|x-y| \ge 2\rho$.

$$G^\rho(x,y) \le K\lambda^{-1}\fint_{B_\rho(y)} |x-z|^{2-n}d\mathbb{L}^n(z) =$$

$$= K\lambda^{-1}|x-y|^{2-n} \int_{B_\rho(y)} \left(\frac{|x-y|}{|x-z|}\right)^{n-2} d\mathbb{L}^n(z) \le$$

$$\le K\lambda^{-1}|x-y|^{2-n} \int_{B_\rho(y)} \left(1+\frac{|y-z|}{|x-z|}\right)^{n-2} d\mathbb{L}^n(z) \le$$

$$\le K\lambda^{-1}|x-y|^{2-n} .$$

In view of (1.54), the estimates for G and the maximum principle we have the

<u>(1.5) Corollary</u>

There exist constants $K_1(n,\mu/\lambda), K_2(n,\mu/\lambda) > 0$ *such that*

(1.57) $G^\rho(x,y) \le K_1\ G(x,y)$ *if* $|x-y| \le \frac{1}{2}\operatorname{dist}(y,\partial\Omega)$,

(1.58) $G(x,y) \le K_2\ G^\rho(x,y)$ *if* $|x-y| \ge \rho$ *and* $\rho < \frac{1}{5}\operatorname{dist}(y,\partial\Omega)$.

For the rest of this section we shall assume a certain regularity of the boundary $\partial\Omega$.

<u>(1.6) Assumption</u> (Exterior cone condition)

There are numbers $h > 0$, $0 < \vartheta < \frac{\pi}{2}$ such that the following is true:

For each $z \in \partial\Omega$ there exists a cone $C(z,h,\vartheta)$ with the property $\Omega \cap C = \emptyset$, $\overline{\Omega} \cap \overline{C} = \{z\}$.

Here $C(z,h,\vartheta)$ denotes the open cone with cusp z, radius h and opening angle ϑ.

In the proof of the next theorem we shall use

<u>(1.7) Lemma</u>. *Let* $R>0$ *and* $D_R := B_R(0) \setminus C(0,R/2,\vartheta)$. *Consider the weak solution* $u_R \in H^1_2(D_R)$ *of* $L_t u = 0$ *in* D_R *satisfying the boundary conditions* $u_R \equiv 1$ *on* $\partial B_R(0)$, $u_R \equiv 0$ *on* $\partial C(0,R/2,\vartheta)$.

Then there are constants $K(n,\mu,\lambda,\vartheta)>0, \alpha(n,\mu,\lambda,\vartheta) \in \,]0,1[$ *such that for* $x \in D_R$

(1.59) $$u_R(x) \le K|x|^\alpha R^{-\alpha}.$$

Remark. This lemma is of course very well known, but it can easily be derived from Lemma (2.3) in section 2 of our paper. The exterior cone condition guarantees that all points of $\partial\Omega$ are regular.
We are now prepared to prove the following theorem about the boundary regularity of G.

(1.8) Theorem. *There are constants* $K(n,\mu,\lambda,\text{diam}\,\Omega,\ \partial\Omega) > 0$ *and* $\alpha(n,\mu,\lambda,\vartheta)$, $0 < \alpha < 1$, *such that for all* $x,y \in \Omega$ $(\delta(y) := \text{dist}(y,\partial\Omega))$

(1.60) $$G(x,y) \le K\ \delta^{\alpha}(y)\ |x-y|^{2-n-\alpha}.$$

Proof of Theorem (1.8)
Let $h = h(\partial\Omega)$ be the height of the cone in Assumption (1.6). For fixed $x \in \Omega$ we consider the function $G(\cdot) := G(x,\cdot)$ which is the Green function corresponding to L_t.
We have to distinguish between four cases.

(i) If $\delta(y) \ge h$ we use the boundedness of Ω and the upper estimate (1.8) for G to draw the conclusion.

(ii) If $\delta(y) \ge |x-y|/4$ we get by (1.8) the desired inequality.

(iii) If $\delta(y) < h$ and $\delta(y) < |x-y|/4 < h$ we fix $y^* \in \partial\Omega$ with $|y-y^*| = \delta(y)$ and set $R := |x-y|/4$. As $\partial\Omega$ satisfies the cone condition, we may work with the cone $C_R := C(y^*,R/2,\vartheta)$. We apply the preceding Lemma (1.7) in the region $B_R(y^*)\setminus C_R$. In $\Omega \cap (B_R(y^*)\setminus C_R)$ we have $L_tG = 0$ and for $z \in \partial B_R(y^*) \cap \bar{\Omega}$ $G(z) \le K|x-y|^{2-n}$, while $G_{|\partial\Omega} \equiv 0$. As $G(\cdot)$ and $K|x-y|^{2-n}u_R(\cdot)$ are solutions of $L_tu = 0$ in $\Omega \cap (B_R(y^*)\setminus C_R)$ and $G \le K|x-y|^{2-n}u_R$ on the boundary, we get by the maximum principle and (1.59) (remembering the choice of y^* and R)
$$G(x,y) \le K\ |x-y|^{2-n}u_R(y) \le K\ \delta^{\alpha}(y)\,|x-y|^{2-n-\alpha}.$$

(iv) If $\delta(y) < h < 4h \le |x-y|$ we fix $y^* \in \partial\Omega$ as before and set $R = h$. We again apply the preceding lemma and the same arguments as in (iii) imply

$$G(x,y) \le K|x-y|^{2-n} u_R(y) \le K|x-y|^{2-n} \delta^\alpha(y) h^{-\alpha}$$

$$= K\delta^\alpha(y)|x-y|^{2-n-\alpha}\left(\frac{\text{diam } \Omega}{h}\right)^\alpha .$$

This finishes the proof of the theorem. In the next theorem we give a Hölder estimate, which is valid outside of the singularity.

(1.9) Theorem. *There are constants* $K(n,\mu,\lambda,\text{diam } \Omega,\partial\Omega) > 0$, $\alpha(n,\mu,\lambda,\vartheta)$, $0 < \alpha < 1$, *such that for all* $x,y,z \in \Omega$

$$(1.61) \qquad |G(x,y)-G(z,y)| \le K|x-z|^\alpha (|x-y|^{2-n-\alpha} + |z-y|^{2-n-\alpha}).$$

Proof of Theorem (1.9)
The proof is rather lengthy because we have to treat several cases seperately. Because of the symmetry of (1.61) with respect to x and z we may suppose $G(x,y) \ge G(z,y)$.

Case 1. $|x-z| \ge |x-y|/2$

We easily get by (1.8) for each $0 < \alpha < 1$:

$$|G(x,y)-G(z,y)| = G(x,y)-G(z,y) \le K|x-y|^{2-n} \le$$

$$\le K|x-z|^\alpha |x-y|^{2-n-\alpha} \le$$

$$\le K|x-z|^\alpha (|x-y|^{2-n-\alpha} + |z-y|^{2-n-\alpha}).$$

Case 2. $|x-z| < |x-y|/2$ and $\delta(x) \le |x-z|$

Using Theorem (1.8) we have

$$|G(x,y)-G(z,y)| \le G(x,y) \le K \delta^\alpha(x)|x-y|^{2-n-\alpha} \le$$

$$\le K|x-z|^\alpha (|x-y|^{2-n-\alpha} + |z-y|^{2-n-\alpha}).$$

Case 3. $|x-z| < |x-y|/2$ and $\delta(x) > |x-z|$

We set $R := \min\{\delta(x), |x-y|/2\}$. As $L\,G(\cdot,y)=0$ in $B_R(x)$ we derive in a standard fashion, using the Harnack inequality ($\gamma = \gamma(n,\mu/\lambda)$) for $0 < \rho \le r \le R$

$$(1.62) \qquad \max_{B_\rho(x)} G(\cdot,y) - \min_{B_\rho(x)} G(\cdot,y) \le$$

$$\le \gamma \left(\frac{\rho}{r}\right)^\alpha \{\max_{B_r(x)} G(\cdot,y) - \min_{B_r(x)} G(\cdot,y)\},$$

which implies for $\rho := |x-z|$ and $r := R$

$$(1.63) \qquad |G(x,y)-G(z,y)| \le \gamma |x-z|^{\alpha} R^{-\alpha} \max_{B_R(x)} G(\cdot,y).$$

If $R = |x-y|/2$ we get from (1.63) and (1.7)

$$|G(x,y) - G(z,y)| \le K|x-z|^{\alpha} \; |x-y|^{2-n-\alpha} \le$$
$$\le K|x-z|^{\alpha}(|x-y|^{2-n-\alpha}+|z-y|^{2-n-\alpha}).$$

If $R = \delta(x)$ we use the boundary estimate (1.60) to get for $z' \in B_R(x)$

$$G(z',y) \le K\; \delta^{\alpha}(z')\,|z'-y|^{2-n-\alpha} \le$$
$$(1.64) \qquad \le K(|z'-x|+\delta(x))^{\alpha} \; (|x-y| - |x-z'|)^{2-n-\alpha} \le$$
$$\le K\; \delta(x)^{\alpha}|x-y|^{2-n-\alpha} \; .$$

Now (1.63) together with (1.64) implies the statement of Theorem (1.9).

We now use Theorem (1.8) to show how the integrals of G respectively ∇G behave, if we approach the boundary.

<u>(1.10) Theorem</u>

(i) *Let* $1 \le p < \frac{n}{n-2}$. *Then we have: For each* $\varepsilon > 0$ *there exists* $\varepsilon' > 0$ *such that for all* $y \in \Omega$ *with* $\delta(y) < \varepsilon'$

$$\int_{\Omega} G(x,y)^{p} \, d\mathbb{L}^{n}(x) < \varepsilon.$$

(ii) *Let* $1 \le p < \frac{n}{n-1}$. *Then we have:*

For each $\varepsilon>0$ *there exists* $\varepsilon'>0$ *such that for all* $y \in \Omega$ with $\delta(y) < \varepsilon'$

$$\int_{\Omega} |\nabla G(x,y)|^{p} \, d\mathbb{L}^{n}(x) < \varepsilon.$$

<u>Proof</u> of Theorem (1.10)

(i) We get by (1.60) and (1.8) ($d(\Omega) := \operatorname{diam} \Omega$)

$$\int_{\Omega} G(x,y)^{p} = \int_{\Omega \setminus B_{\delta}(y)} G(x,y)^{p} + \int_{B_{\delta}(y)} G(x,y)^{p} \le$$

$$\le K\,\delta^{\alpha p}\int_{\Omega\setminus B_\delta(y)}|x-y|^{(2-n-\alpha)p}+K\int_{B_\delta(y)}|x-y|^{(2-n)p}\le$$

$$\le K\,\{d(\Omega)^{(2-n-\alpha)p+n}\,\delta^{\alpha p}-\delta^{(2-n)p+n}\}+K\delta^{(2-n)p+n}\ .$$

As $(2-n)p+n>0$ we get the desired conclusion.

(ii) By (1.6) $|\nabla G|\in L^*_{\frac{n}{n-1}}(\Omega)$, so that using (1.12) we get for any measurable subset $\Omega'\subset\Omega$

$$(1.65)\qquad \int_{\Omega'}|\nabla G(\cdot,y)|^p\le$$

$$\le K(p,n)\ (\mathbb{L}^n(\Omega'))^{(\frac{n}{n-1}-p)/p}\,\|\nabla G\|^p_{L^*_{\frac{n}{n-1}}(\Omega)}\ .$$

Letting $\Omega'=\Omega\cap B_r(y)$ we may in view of (1.66) choose $r(\varepsilon)>0$ so small that

$$(1.66)\qquad \int_{\Omega\cap B_r(y)}|\nabla G(\cdot,y)|^p<\varepsilon/2\ .$$

The remaining integral is estimated in a standard way.

$$\int_{\Omega\setminus B_r(y)}|\nabla G(\cdot,y)|^p\le$$

$$\le(\mathbb{L}^n(\Omega))^{1-p/2}\Big[\int_{\Omega\setminus B_r(y)}|\nabla G(\cdot,y)|^2\Big]^{\frac{p}{2}}\le$$

$$(1.67)\qquad \le K(\Omega)\Big[Kr^{-2}\int_{\Omega\setminus B_{r/2}(y)}G(\cdot,y)^2\Big]^{\frac{p}{2}}\le$$

$$\le Kr^{-p}\Big[\int_{\Omega\setminus B_{r/2}(y)}G(x,y)\,|x-y|^{2-n}\Big]^{\frac{p}{2}}\le$$

$$\leq K\, r^{-\frac{np}{2}} \left[\int_\Omega G(\cdot,y) \right]^{\frac{p}{2}} \leq \frac{\varepsilon}{2} \quad ,$$

if $\delta(y)$ is small enough. Here we have used (1.8) and (i). (1.66) and (1.67) prove the theorem.

2. Regular points

In this part of our paper we make the same assumptions about the operator L as before, that is (1.1) and (1.2). First we want to show how one can compare the Green functions for different elliptic operators. This is done in Theorem (2.1). After that we shall investigate the regular points for the elliptic operator L. It turns out that they are the same as for the Laplacian Δ. This problem was already investigated in Littman, Stampacchia and Weinberger [15]; but as before their method only works for symmetric operators, while we present a proof covering both cases.

Taking the estimates (1.8) and (1.9) into account we get the following result.

(2.1) Theorem *Let* G *and* G' *be the Green functions corresponding to the operators* L *and* L' *which are both supposed to satisfy* (1.1) *and* (1.2) *with constants* λ,μ *and* λ',μ' *respectively. Denote by* K_1, K_2, K_1', K_2' *the corresponding constants in* (1.8) *and* (1.9).
Then we have for any $x,y \in \Omega$ *satisfying* $|x-y| < \frac{1}{2}\operatorname{dist}(y,\partial\Omega)$

$$(2.1) \qquad (K_2/K_1')(\lambda'/\mu)G'(x,y) \leq G(x,y) \leq (K_1/K_2')(\mu'/\lambda)G'(x,y).$$

Now let us turn to the definition of capacity for the operator L. The usual definition of variational capacity does not apply because L is not supposed to be symmetric. Our definition is inspired by the theory of variational inequalities.
Let $E \subset \Omega$ be compact and denote by $\mathbb{K}_E$ the following closed convex subset of $\overset{\circ}{H}{}_2^1(\Omega)$

(2.2) $\mathbb{K}_E := \{v \in \overset{\circ}{H}{}^1_2(\Omega) : v \geq 1 \text{ on } E \text{ in the sense of } H^1_2\}$

We are looking for a solution of the variational inequality

(2.3) $u \in \mathbb{K}_E, \quad a(u,v-u) \geq 0 \quad \text{for all} \quad v \in \mathbb{K}_E$.

Standard arguments in the theory of variational inequalities show the following (c.f. Kinderlehrer and Stampacchia [13]). There exists a unique solution $u \in \mathbb{K}_E$ of the variational inequality (2.3). This function u is called the equilibrium potential of E and satisfies $u \equiv 1$ on E in the sense of H^1_2. In $\Omega \setminus E$ it equals the unique solution of the Dirichlet problem $Lu = 0$ with boundary values zero on $\partial\Omega$ and one on ∂E.
From (2.3) we get for all $\phi \in C^\infty_c(\Omega)$ with $\phi \geq 0$ on E

(2.4) $$a(u,\phi) \geq 0 .$$

By the Riesz representation theorem there exists a nonnegative measure μ with supp $\mu \subset E$ such that

(2.5) $$a(u,\phi) = \int_E \phi \, d\mu$$

for any $\phi \in C^\infty_c(\Omega)$.

(2.2) Definition *The capacity of* E *with respect to the operator* L *is defined as*

(2.6) $$\Gamma_L(E) := \mu(E) = a(u,u).$$

By continuity (2.5) is true for $\phi \in \overset{\circ}{H}{}^1_2(\Omega) \cap C^0(E)$ and we get for $\phi = G^\rho(x,\cdot)$

(2.7) $$\fint_{B_\rho(x)} u(y)\,dy = \int_E G^\rho(x,y)\,d\mu(y).$$

Taking the lim inf on both sides of (2.7) we have

(2.8) $$u(x) = \int_E G(x,y)\,d\mu(y).$$

The measure μ is called the equilibrium measure of E and because of (2.8) the name "equilibrium potential" for u is justified.

From (2.5) one has for any $\phi \in \overset{\circ}{H}{}^1_2(\Omega)$ with $\phi \geq 1$ on E in the sense of H^1_2

(2.9) $$a(u,\phi) \geq \Gamma(E) ,$$

and the reverse inequality if $\phi \leq 1$ on E in the sense of H^1_2. This implies that if we consider the equilibrium potential u_t of E with respect to the adjoint operator L_t we get

(2.10) $$\Gamma_{L_t}(E) = a(u,u_t) = \Gamma_L(E) ,$$

which any reasonable definition of capacity should of course satisfy.

Let us now show that the above defined capacity Γ is a Choquet-capacity, i.e. it satisfies the following three conditions (c.f. [2]):

(i) $\Gamma(\emptyset) = 0$ and $E_1 \subset E_2$ implies $\Gamma(E_1) \leq \Gamma(E_2)$.

(ii) For every antitone sequence $\{E_j\}$ of compact subsets of Ω we have
$$\Gamma(\bigcap_{j=1}^{\infty} E_j) = \lim_{j\to\infty} \Gamma(E_j).$$

(iii) $\Gamma(E_1 \cup E_2)+\Gamma(E_1 \cap E_2) \leq \Gamma(E_1)+ \Gamma(E_2)$.

Property (i) is obvious. To prove (ii) we note that $(E = \bigcap_{j=1}^{\infty} E_j)$
$$\Gamma(E) \leq \lim_{j\to\infty} \Gamma(E_j).$$

Thus it remains to show the reverse inequality. For that purpose denote by u the potential of E with respect to L_t and by u_j the potential of E_j with respect to L. Choose $\varepsilon,\delta>0$ and $\phi \in C_c^{\infty}(\Omega)$, $\phi \geq 1$ on E, s.t. $\|\phi-u\|_{H^1_2} \leq \varepsilon/\{(\mu/\lambda)\Gamma(E_1)^{1/2}\}$ (if $\Gamma(E_1)=0$ there is nothing to prove). Setting $\psi = (1+\delta)\phi$, (2.9) and (2.10) imply for $j \geq j_o(\varepsilon,\delta)$

$$\begin{aligned} \Gamma(E_j) \leq a(u_j,\psi) &= (1+\delta)a(u_j,\phi) = \\ &= (1+\delta)\{a(u_j,\phi-u)+a(u_j,u)\} \\ &\leq (1+\delta)\{\varepsilon + \Gamma(E)\}. \end{aligned}$$

Letting $j\to\infty$, $\delta\searrow 0$, $\varepsilon\searrow 0$, we get (ii). For the proof of (iii) consider the potential u of $E_1 \cup E_2$ with respect to L and the potentials u_1, u_2 of E_1, E_2 with respect to L_t. From (2.9) and (2.10) one concludes

$$\begin{aligned}&\Gamma(E_1 \cup E_2) + \Gamma(E_1 \cap E_2) \leq \\ &\leq a(u,\max(u_1,u_2)) + a(u,\min(u_1,u_2)) = \\ &= a(u,\max(u_1,u_2) + \min(u_1,u_2)) = \\ &= a(u,u_1+u_2) = a(u,u_1) + a(u,u_2) = \\ &= \Gamma(E_1) + \Gamma(E_2) ,\end{aligned}$$

where the inequality is due to the fact that

$$\Gamma(E_1 \cap E_2) = \Gamma_t(E_1 \cap E_2) = a(u,v) \leq a(u,\min(u_1,u_2)) .$$

Here v denotes the potential of $E_1 \cap E_2$ with respect to L_t, and we have applied the maximum principle. Therefore Γ is a Choquet-capacity.

Finally one sees from (2.5) that one can compare the capacities with respect to different elliptic operators

$$\lambda\Gamma_\Delta(E) \leq \Gamma_L(E) \leq \mu\, c(L)\, \Gamma_\Delta(E) , \tag{2.11}$$

where $c(L) = 1$ if L is symmetric and $c(L) = \mu/\lambda$ if L is not symmetric. Here Γ_Δ is the ordinary capacity.

Remark. It is easy to see that this capacity is identical to the "traditional" capacity

$$\Gamma(E) = \sup\{\mu(E) : \operatorname{supp}\mu \subset E,\ \int G(x,y)d\mu(y) \leq 1 \text{ in } \Omega\} .$$

In fact if ν is any such measure with potential u^ν and u is the equilibrium potential of E with respect to L_t and μ its associated measure, then

$$\begin{aligned}\mu(E) &\geq \int_E u^\nu(x)d\mu(x) = \int_E\int_E G(x,y)d\mu(x)d\nu(y) = \\ &= \int_E u(y)d\nu(y) = \nu(E) .\end{aligned}$$

The justification for the equalities is an easy exercise.

From now on we assume that the coefficients of L have been extended to a large ball B which contains $\overline{\Omega}$ well in its interior and consider the Green function G of L on B. Fix $x_o \in \partial\Omega$ (we may assume $x_o=0$) and set for $r>0$ $C_r := (\mathbb{R}^n \setminus \Omega) \cap B_R$. The following lemma is the basic step in the proof of Theorem (2.4).

<u>(2.3) Lemma</u>. *Consider the solution* $u \in H_2^1(\Omega)$ *of* $Lu = 0$ *in* Ω, $0 \le u \le 1$ *on* $\partial\Omega$ *and* $u=0$ *on* $\partial\Omega \cap B_\rho$ $(B_\rho := B_\rho(0))$. *There exist constants* $0 < \alpha_o(n,\mu,\lambda) < \frac{1}{2}$ *and* $K(n,\mu,\lambda) > 0$ *such that for all* $\alpha \le \alpha_o$ *and* $r < \alpha\rho$ *we have*

$$\sup_{\Omega\cap B_r} u \le \exp\left[\frac{K}{\log \alpha} \int_r^{\alpha\rho} \frac{\Gamma(C_t)}{t^{n-1}}\, dt\right] . \tag{2.12}$$

<u>Proof</u> of Lemma (2.3)

Let $\alpha < \frac{1}{2}$, $\rho' \le \rho$ and v the equilibrium potential of $C_{\alpha\rho'}$. By (2.8) and (1.8) we get for $|x| = \rho'$

$$v(x) \le K(n,\mu,\lambda)\rho'^{2-n}\, \Gamma(C_{\alpha\rho'}) . \tag{2.13}$$

As $v \le 1$ by the maximum principle, (2.13) gives for $x \in \Omega \cap \partial B_{\rho'}$

$$[1-v(x)]\sup_{\Omega \cap B_{\rho'}} u \ge [1-K\rho'^{2-n}\, \Gamma(C_{\alpha\rho'})]\; u(x) , \tag{2.14}$$

and (2.14) is also true on $\partial\Omega \cap B_{\rho'}$, because u=0 there. From the maximum principle we get (2.14) on $\Omega \cap B_{\rho'}$. By (1.9) we have for $|x| = 2\alpha\rho'$

$$v(x) \ge K'(\alpha\rho')^{2-n}\Gamma(C_{\alpha\rho'}) . \tag{2.15}$$

For $x \in \Omega \cap \partial B_{2\alpha\rho'}$, (2.14) and (2.15) imply

$$\begin{aligned} u(x) &\le \sup_{\Omega\cap B_{\rho'}} u \left[\frac{1-K'(\alpha\rho')^{2-n}\Gamma(C_{\alpha\rho'})}{1-K\,\rho'^{2-n}\,\Gamma(C_{\alpha\rho'})}\right] \le \\ &\le \sup_{\Omega\cap B_{\rho'}} u \left[1-K^*(\alpha\rho')^{2-n}\,\Gamma(C_{\alpha\rho'})\right] , \end{aligned} \tag{2.16}$$

where the last inequality is true for $\alpha \le \alpha_o(n,\mu,\lambda)$. Using the maximum principle again we get (2.16) on $\Omega \cap B_{2\alpha\rho}$,. Using the notation $M(r) := \sup_{\Omega\cap B_r} u$ we have proved for $t \le \rho$, $\alpha \le \alpha_o$

$$(2.17) \qquad M(\alpha t) \le M(t)\ [1-K^*(\alpha t)^{2-n}\ \Gamma(C_{\alpha t})]\ .$$

This gives

$$(2.18) \qquad \frac{\log M(t)}{t} - \frac{\log M(\alpha t)}{t} \ge \frac{K^*}{\alpha^{n-2}} \frac{\Gamma(C_{\alpha t})}{t^{n-1}}\ .$$

Integration from r/α to ρ yields

$$(2.19) \qquad M(r) \le M(\rho)\ \exp\left[\frac{K^*}{\log\alpha} \int_r^{\alpha\rho} \frac{\Gamma(C_t)}{t^{n-1}}\,dt\right]\ ,$$

which implies (2.12).

<u>Remark</u>. We want to indicate the proof of Lemma (1.7). Choose $\Omega = D_R$, $\rho = \frac{2}{3}R$ and (2.12) gives for $r<\alpha\frac{2}{3}R$

$$(2.20) \qquad \sup_{\Omega\cap B_r} u_R \le \exp\left[-K\int_r^{\alpha\rho} \frac{\Gamma(C_t)}{t^{n-1}}\,dt\right]\ .$$

As C_t is a cone of height t and fixed opening angle ϑ, we have $\Gamma(C_t) \ge K(n,\mu,\lambda,\vartheta)t^{n-2}$. Now (2.20) implies

$$(2.21) \qquad \sup_{\Omega\cap B_r} u_R \le \exp\left[-\beta\int_r^{\alpha\rho}\frac{dt}{t}\right] = \left(\frac{r}{\alpha\rho}\right)^{\beta} = Kr^{\beta}\ R^{-\beta}\ .$$

If $|x| \ge \alpha\frac{2}{3}R$ we use $u_R \le 1$ to derive (1.60).

Let us now state the main theorem of this section.

<u>(2.4) Theorem</u>. *A point of* $\partial\Omega$ *is a regular point of the Dirichlet problem for* $Lu=0$ *if and only if it is regular for* $\Delta u = 0$.

The next theorem will immediately imply Theorem (2.4).

(2.5) Theorem. *The point* $x_o \in \partial\Omega$ *(we may assume* $x_o = 0$*) is a regular point for* $Lu = 0$ *if and only if* $(C_r := (\mathbb{R}^n \setminus \Omega) \cap B_r)$

$$(2.22) \qquad \int_0 \frac{\Gamma(C_r)}{r^{n-1}}\, dr = \infty .$$

Proof of Theorem (2.5)

Necessity of (2.22)

Suppose $\int_0 \frac{\Gamma(C_r)}{r^{n-1}}\, dr$ is finite and let μ_r be the equilibrium measure of C_r, v_r the corresponding potential. We set

$$(2.23) \qquad \hat{v}_r(0) := \lim_{\rho \to 0} \int_{C_r \setminus B_\rho} G(0,\cdot)\, d\mu_r .$$

Using (1.8) we find

$$(2.24) \qquad \hat{v}_r(0) \le K \left\{ \frac{\Gamma(C_r)}{r^{n-2}} + \int_0^r \frac{\mu_r(C_t)}{t^{n-1}}\, dt \right\}$$

and observing $\mu_r(C_t) \le \mu_t(C_t) = \Gamma(C_t)$ we get for small r

$$(2.25) \qquad \hat{v}_r(0) < 1.$$

Here we used the finiteness of $\int_0 \frac{\Gamma(C_t)}{t^{n-1}}\, dt$.

As $\lim_{\rho \to 0} G^\rho(0,x) = G(0,x)$ we get by Lebesgue's theorem that $\lim_{\rho \to 0} \int_{C_r} G^\rho(0,\cdot)\, d\mu_r = \int_{C_r} G(0,\cdot)\, d\mu_r$. This implies

$$(2.26) \qquad v_r(0) = \hat{v}_r(0) < 1$$

for some small r. Using $v_r(0) = \lim_{\rho \to 0} \inf \fint_{B_\rho} v_r$ we see

$$(2.27) \qquad \liminf_{\Omega \ni y \to 0} v_r(y) \le v_r(0) < 1.$$

Now v_r is a solution of $Lu = 0$ in Ω with $v_r \ge 0$ on $\partial\Omega$ and $v_r = 1$ on $\partial\Omega \cap B_r$ in the sense of H_2^1.
Consider the solution u of $Lu = 0$ with continuous boundary

values g defined by $g(x) = \max\left\{0, 1 - \frac{|x|}{r}\right\}$ for $x \in \partial\Omega$. From the maximum principle we infer $u \le v_r$ in Ω. But the boundary value one at O is not attained by (2.27) and therefore the origin is an irregular point of $\partial\Omega$.

Sufficiency of (2.22)

It is well known that it is sufficient to show the existence of a barrier u satisfying:

(2.28) $u \in H_2^1(\Omega)$, $Lu = 0$ in Ω;

(2.29) for any $r > 0$ there exists $\tau > 0$, such that $u \ge \tau$ on $\partial\Omega \setminus B_r$ in the sense of H_2^1;

(2.30) $\lim\limits_{y\to 0} u(y) = 0.$

Let $\rho \to 0$ and consider the equilibrium potential v of $C_{\alpha\rho}$ ($\alpha \le \alpha_o$). From Lemma (2.3) applied to 1-v we infer for $x \in \Omega \cap B_{\alpha^2\rho}$:

(2.31) $$1 - v(x) \le \exp\left[-K \int_{|x|}^{\alpha^2\rho} \frac{\Gamma(C_t)}{t^{n-1}}\, dt\right].$$

For $|x| = \rho$ we get for α small enough observing $\Gamma(C_{\alpha\rho}) \le K(\alpha\rho)^{n-2}$

(2.32) $$v(x) \le K\, \alpha^{n-2} < 1/2.$$

Now let u be the solution of $Lu = 0$ with boundary values $|x|$. Because of (2.32) we have

(2.33) $$\begin{aligned} u(x) &\le 2 \operatorname{diam} \Omega\, (1 - v(x)) && \text{on } \Omega \cap \partial B_\rho \\ u(x) &\le \rho && \text{on } \partial\Omega \cap B_\rho \end{aligned}$$

in the sense of H_2^1. The maximum principle and (2.31) yield

(2.34) $$u(x) \le \rho + 2 \operatorname{diam}\Omega \exp\left[-K \int_{|x|}^{\alpha^2\rho} \frac{\Gamma(C_t)}{t^{n-1}}\, dt\right]$$

for $x \in \Omega \cap B_{\alpha^2\rho}$. Now condition (2.22) shows

(2.35) $$\limsup_{\Omega \ni x \to 0} u(x) \le \rho.$$

As (2.28), (2.29) are naturally true for u and (2.30) follows from (2.35) and the arbitrariness of ρ, we get Theorem (2.5).
We have the following

(2.5) Corollary. *Condition* (2.22) *of Theorem* (2.5) *is equivalent to*

(2.36) $$\sum_{\nu=1}^{\infty} 2^{\nu(n-2)} \Gamma(\overline{C_{2^{-\nu}} \setminus C_{2^{-\nu-1}}}) = \infty$$

Proof of Corollary (2.5)
Using the fact that $\Gamma(C_t) \le Kt^{n-2}$ and a subdivision of the intervall $[0,1]$ we see that (2.22) implies

(2.37) $$\sum_{\nu=1}^{\infty} 2^{\nu(n-2)} \{\Gamma(C_{2^{-\nu}}) - \Gamma(C_{2^{-\nu-1}})\} = \infty$$

Using the subadditivity of Γ (2.37) gives (2.36). To show that (2.36) implies (2.22) we estimate

$$\int_0^{2^{-N}} \frac{\Gamma(C_t)}{t^{n-1}}\, dt \ge \frac{1}{2} \sum_{\nu=N}^{\infty} 2^{\nu(n-1)-\nu} \Gamma(C_{2^{-\nu-1}}) \ge$$

$$\ge K \sum_{\nu=N+2}^{\infty} 2^{\nu(n-2)} \Gamma(\overline{C_{2^{-\nu}}}) \ge$$

$$\ge K \sum_{\nu=N+2}^{\infty} 2^{\nu(n-2)} \Gamma(\overline{C_{2^{-\nu}} \setminus C_{2^{-\nu-1}}}) = \infty$$

Remark. In the case of the Laplace operator condition (2.36) is the famous Wiener criterion.

Proof of Theorem (2.4)
By (2.11) we have $\lambda \le \Gamma_L/\Gamma_\Lambda \le \mu(\mu/\lambda)$. Theorem (2.5) now proves Theorem (2.4).

3. The case of regular coefficients

In this section the assumptions about the operator L will be the same as in section 1. Moreover we shall assume that the coefficients are Dini-continuous which will enable us to derive some more pointwise estimates for the Green function and its derivative.
So we shall consider coefficients which satisfy

$$(3.1) \qquad |a^{ij}(x)-a^{ij}(y)| \le \omega(|x-y|) \text{ for any } x,y \in \Omega.$$

Here $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is supposed to be non-decreasing and to satisfy

$$(3.2) \qquad \omega(2t) \le K\,\omega(t) \text{ for some } K>0 \text{ and all } t>0,$$

$$(3.3) \qquad \int_0 \frac{\omega(t)}{t}\,dt < \infty .$$

Remark. In particular Hölder coefficients, where $\omega(t)=Kt^{\alpha}$, $0<\alpha<1$, are allowed.
We shall first prove two lemmas.

(3.1) Lemma. *Suppose* u *is a bounded solution of* $Lu=0$ *in* Ω. *There exists* $K(n,\mu,\lambda,\omega,\Omega)>0$ *such that for any* $x \in \Omega$ $(\delta(x):=\operatorname{dist}(x,\partial\Omega))$

$$(3.4) \qquad |\nabla u(x)| \le K\,\delta^{-1}(x)\sup_{\Omega}|u|.$$

Proof of Lemma (3.1)
We note that under the regularity assumptions $u \in C^1_{loc}(\Omega)$.
By considering a slightly smaller domain we may assume

$$\sup_{x\in\Omega} \delta(x)|\nabla u(x)| =: M_1 < \infty . \text{ Let } M_0:=\sup_{\Omega}|u| \text{ and}$$

choose $x_0 \in \Omega$ such that $\delta(x_0)|\nabla u(x_0)| > \frac{1}{2} M_1$. For $d \le \frac{1}{2}\delta(x_0)$ we define a cut-off function $\eta \ge 0$ on $B_d(x_0)$ which is one on $B_{d/2}(x_0)$ such that $|\nabla u| \le K d^{-1}$ and $|\nabla^2 u| \le K d^{-2}$.
Let F be the Green function of the operator with constant

coefficients $a_o^{ij} := a^{ij}(x_o)$ in $B_d(x_o)$. For $y \in B_d := B_d(x_o)$ we use $\eta F(\cdot,y)$ as a test function to get

$$0 = \int_{B_d} a^{ij} D_i u D_j(\eta F) = \int_{B_d} a_o^{ij} D_i u D_j F \eta + \int_{B_d} a_o^{ij} D_i u D_j \eta\, F +$$

$$+ \int_{B_d} (a^{ij} - a_o^{ij}) D_i u (D_j F \eta + D_j \eta\, F) =$$

$$= u(y)\eta(y) + \int_{B_d} a_o^{ij} D_i u D_j \eta F - \int_{B_d} a_o^{ij} D_i \eta D_j F u +$$

$$(3.5) \qquad + \int_{B_d} (a^{ij} - a_o^{ij}) D_i u (D_j F \eta + D_j \eta F).$$

Differentiating (3.5) with respect to y and setting $y = x_o$ we see

$$\nabla u(x_o) = \int_{B_d} a_o^{ij} D_i \eta\, \nabla_y D_j F(\cdot,x_o) u - \int_{B_d} a_o^{ij} D_i u D_j \eta \nabla_y F(\cdot,x_o)$$

$$(3.6) \qquad - \int_{B_d} (a^{ij} - a_o^{ij}) D_i u (\nabla_y D_j F(\cdot,x_o)\eta + D_j \eta \nabla_y F(\cdot,x_o)).$$

An integration by parts shows

$$- \int_{B_d} a_o^{ij} D_i u D_j \eta \nabla_y F(\cdot,x_o) =$$

$$(3.7) \qquad = \int_{B_d} a_o^{ij} u (D_i D_j \eta\, \nabla_y F(\cdot,x_o) + D_j \eta D_i \nabla_y F(\cdot,x_o)).$$

Using known inequalities for F (Widman [13]) we can conclude from (3.6) and (3.7)

$$(3.8) \qquad |\nabla u(x_o)| \le K\, M_o d^{-1} + K M_1 \delta(x_o)^{-1} \int_0^d \frac{\omega(t)}{t}\, dt.$$

In view of the choice of x_o (3.8) in turn gives

(3.9) $$M_1 \le K\, M_o \frac{\delta(x_o)}{d} + K\, M_1 \int_0^d \frac{\omega(t)}{t}\, dt.$$

Setting $d_o := \sup \{d \mid K \int_0^d \frac{\omega(t)}{t}\, dt \le 1/4\} = d_o(n,\mu,\lambda,\omega)$ which is possible by (3.3) and choosing $d := \inf\{\frac{1}{2}\delta(x_o), d_o\}$, (3.9) implies

(3.10) $$M_1 \le K\, M_o \frac{\delta(x_o)}{d} \le K\, M_o \sup\{2, \operatorname{diam} \Omega / d_o\}.$$

This proves Lemma (3.1).

(3.2) Lemma. *Let* u *be the solution of the Dirichlet problem* $Lu = 0$ *in* $D_r := B_{2r} \setminus \overline{B}_r$ $(r \le 1)$ *with* $u = 0$ *on* ∂B_r *and* $u = 1$ *on* ∂B_{2r}. *There exists* $K(n,\mu,\lambda,\omega) > 0$ *such that for any* $x \in D_r$

(3.11) $$|\nabla u(x)| \le \frac{K}{r}.$$

Proof of Lemma (3.2)
We shall assume $u \in C^1(\overline{D}_r)$, which can be proved directly. Furthermore, we only have to consider the case $r = 1$, because the general case is reduced to this by a homothety (here one uses the monotonicity of ω).

Let $M := \sup_D |\nabla u|$ and choose $x_o \in D$ such that $|\nabla u(x_o)| \ge \frac{1}{2} M$. Choose a cut-off function $\eta \in C_c^\infty(B_d(x_o))$ with $\eta \equiv 1$ on $B_{d/2}(x_o)$, $|\nabla\eta| \le Kd^{-1}$ and $|\nabla^2\eta| \le Kd^{-2}$. The number $d \le \frac{1}{4}$ will be determined later.

We may assume $u\eta = 0$ on ∂D. In fact this is obvious if $|x_o| \le \frac{3}{2}$. If $|x_o| > \frac{3}{2}$ we consider $(1-u)$ instead of u. This is a solution of the same equation but with boundary values one on ∂B_1 and zero on ∂B_2. We again have $\sup_D |\nabla(1-u)| = M$ and $|\nabla(1-u)(x_o)| \ge \frac{1}{2} M$.

Let F be the Green function in D corresponding to the operator with coefficients $a_o^{ij} := a^{ij}(x_o)$. For $y \in D$ we insert $\eta F(\cdot,y)$ as a testfunction and get as in the proof of Lemma (3.1)

$$u(y)\eta(y) = \int_D a_o^{ij}\, u(D_iD_j\eta F(\cdot,y)+D_j\eta D_iF(\cdot,y))$$

$$(3.12) \qquad + \int_D a_o^{ij} u D_jF(\cdot,y)D_i\eta -$$

$$- \int_D (a^{ij}-a_o^{ij})D_iu(D_jF(\cdot,y)\eta + D_j\eta F(\cdot,y)).$$

If we differentiate (3.12) with respect to y, set $y=x_o$ and estimate the remaining terms in the same way as in the proof of Lemma (3.1) we end up with

$$(3.13) \qquad \tfrac{1}{2}M \le |\nabla u(x_o)| \le Kd^{-1}+KM\int_0^d \frac{\omega(t)}{t}\,dt.$$

Now choose $d=d_o\ (n,\mu,\lambda,\omega)$ such that $K\int_0^d \frac{\omega(t)}{t}\,dt \le \frac{1}{4}$.

Now (3.13) implies the statement of the lemma.
For the next theorem we assume that the domain Ω satisfies an exterior sphere condition uniformly.

<u>(3.3) Theorem</u>

Let Ω *be as above and suppose that* L *satisfies* (1.1), (1.2), (3.1)-(3.3). *Then for the corresponding Green function* G *the following six inequalities are true for any* $x,y \in \Omega$ $(\delta(y):=\text{dist}(y,\partial\Omega))$.

(i) $G(x,y) \le K|x-y|^{2-n}$, $K=K(n,\mu,\lambda)$
(ii) $G(x,y) \le K\ \delta(x)\ |x-y|^{1-n}$, $K = K(n,\mu,\lambda,\omega,\Omega)$
(iii) $G(x,y) \le K\ \delta(x)\delta(y)|x-y|^{-n}$
(iv) $|\nabla_x G(x,y)| \le K|x-y|^{1-n}$
(v) $|\nabla_x G(x,y)| \le K\delta(y)|x-y|^{-n}$
(vi) $|\nabla_x\nabla_y G(x,y)| \le K|x-y|^{-n}$.

Proof of Theorem (3.3)

(i) was proved in section 1.

(ii) is proved in the same way as Theorem (1.8). One only has to use the annuloid domains and Lemma (3.2) instead of Lemma (1.7). We remark that (3.2) is needed to extend L to a neighbourhood of Ω.

(iii) is deduced from (ii) exactly as (ii) from (i).

(iv) If $\delta(x) \le |x-y|$ apply Lemma (3.1) to $G(\cdot,y)$ in the ball $B_{\frac{1}{2}\delta(x)}(x)$ and use (ii). If $|x-y| < \delta(x)$ apply Lemma (3.1) in $B_{\frac{1}{2}|x-y|}(x)$ and use (i).

(v) The proof is exactly the same as for (iv) the only difference being that one uses (iii) instead of (ii) and (ii) instead of (i).

(vi) We first note that for fixed $x \in \Omega$ $\nabla_x G(x,\cdot)$ is a solution of $Lu = 0$ in $\Omega \setminus \{x\}$. Now (vi) follows from (iv) and (v) as (iv) was implied by (i) and (ii). The theorem is proved.

We shall now consider the Green function in more special situations which are useful for applications.

(3.4) Theorem

Let Ω be convex and suppose that L *satisfies* (1.1), (1.2). *Denote by* G *the corresponding Green function.*

(i) *There is a constant* $K(n,\mu,\lambda) > 0$ *and a number* $0 < \alpha(n,\mu,\lambda) < 1$ *such that for any* $x_1, x_2, y \in \Omega$

$$(3.14) \qquad |G(x_1,y)-G(x_2,y)| \le K|x_1-x_2|^{\alpha} \sum_{i=1}^{2} |x_i-y|^{2-n-\alpha}.$$

(ii) *If* L *additionally satisfies* (3.1)-(3.3) *the constants in* (ii), (iii) *of Theorem* (3.3) *depend only on* n,μ,λ,ω, *while the constants in* (iv)-(vi) *of Theorem* (3.3) *also depend on* diam Ω.

Proof of Theorem (3.4)

If Ω is convex one may take a cone (a ball) of arbitrary height (diameter) and arbitrary opening angle. A careful inspection of the proofs of Theorem (1.8), (1.9) and (3.3) then implies the statements of Theorem (3.4).

In the following theorem we prove Hölder-continuity of the gradient of the Green function using a method of Campanato [1].

(3.5) Theorem

(i) *Let* $R^* > 0$ *and* $D_{R^*}(0) := \{x \in \mathbb{R}^n : |x| < R^*, x^n > 0\}$ *and suppose* L *satisfies* (1.1), (1.2). *Assume that the coefficients of* L *are Hölder continuous with exponent* $\alpha \in (0,1)$ *and denote by* $[a^{ij}]_{0,\alpha,\overline{D_{R^*}}}$ *the Hölder-seminorm.*

If $x_1, x_2 \in D_{R^*} \cap B_{R^*-\tau}$, $0 < \tau \le R^*/2$ *then for all* $y \in D_{R^*}$

(3.15) $$|\nabla_x G(x_1,y) - \nabla_x G(x_2,y)| \le K^* |x_1 - x_2|^\alpha \sum_{i=1}^{2} |x_i - y|^{1-n-\alpha},$$

where $K^* = K^*(n,\mu,\lambda,\alpha,[a^{ij}]_{0,\alpha,\overline{D}_{R^*}},[a^{ij}]_{0,\alpha,\overline{D}_{R^*}} R^{*\alpha}, R^*/\tau)$.

(ii) *If* Ω *has a* $C^{1,\alpha}$*-boundary and* L *is as in* (i) *we have for any* $x_1, x_2, y \in \Omega$ *the estimate* (3.15) *with* $K^* = K^*(n,\mu,\lambda,\alpha,[a^{ij}]_{0,\alpha,\overline{\Omega}}, \operatorname{diam}\Omega, \partial\Omega)$.

Proof of Theorem (3.5)

We shall only give the proof of (i), because it contains the essential arguments needed for the proof of (ii), c.f. [1].

Assume w.l.o.g. $|y-x_1| \le |y-x_2|$ and set

$$5R := \min\{|y-x_2|, \tau\}, \quad u(z) := G(z,y).$$

Let us first consider the case $|x_1-x_2| \ge R$. Then (iv) of Theorem (3.3) (denote the constant by K_1) yields

$$|\nabla u(x_1) - \nabla u(x_2)| \le K_1(|x_1-y|^{1-n} + |x_2-y|^{1-n}) \le$$

$$\le K_1 |x_2-y|^\alpha \sum_{i=1}^{2} |x_i-y|^{1-n-\alpha} \le$$

$$\le K_1 5^\alpha |x_1-x_2|^\alpha \sum_{i=1}^{2} |x_i-y|^{1-n-\alpha}$$

if $|y-x_2| < \tau$. If $\tau \le |y-x_2|$ we get the estimate

$$|\nabla u(x_1)-\nabla u(x_2)| \le K_1 (2R^*-\tau)^\alpha \sum_{i=1}^{2} |x_i-y|^{1-n-\alpha} \le$$
$$\le K_1 5^\alpha (\frac{2R^*}{\tau} - 1)^\alpha |x_1-x_2|^\alpha \sum_{i=1}^{2} |x_i-y|^{1-n-\alpha}.$$

Let now be $|x_1-x_2| < R$. We are going to prove Hölder-continuity of ∇u on $D_{R^*} \cap B_R(x_2)$.
Consider $x_o \in D_{R^*} \cap B_R(x_2)$ and $\sigma \le \frac{R}{2}$.
If $B_{R/2}(x_o) \subset D_{R^*}$ we proceed as follows.
By (iv) of Theorem (3.3) we get

$$\text{(3.16)} \qquad \|\nabla u\|_{L^2(B_\sigma(x_o))} \le c K_1 \sigma^{n/2} \sup_{z \in B_\sigma(x_o)} |z-y|^{1-n} \le$$
$$\le c K_1 \sigma^{n/2} R^{1-n},$$

because $z \in B_\sigma(x_o)$ implies $|z-y| \ge 5R-R-\sigma \ge \frac{7}{2} R$.
Here and in the sequel c denotes a constant depending only on n,μ,λ and sometimes on α. Let v be the solution of the Dirichlet problem

$$-D_j[a^{ij}(x_o)D_i v] = 0 \text{ on } B_\sigma(x_o),$$
$$v-u \in \overset{\circ}{H}{}^1_2 (B_\sigma(x_o)).$$

Then we have, c.f. [1], for 0<t<1

$$\int_{B_{t\sigma}(x_o)} |\nabla v|^2 \le c\, t^n \int_{B_\sigma(x_o)} |\nabla v|^2$$

and

$$\text{(3.17)} \qquad \|\nabla v-(\nabla v)_{t\sigma}\|_{L^2(B_{t\sigma}(x_o))} \le c\, t^{\frac{n}{2}+1} \|v-(\nabla v)_\sigma\|_{L^2(B_\sigma(x_o))},$$

where $(\ )_\rho$ denotes the mean value over $B_\rho(x_o) \cap D_{R^*}$.
Using the Hölder-continuity of the coefficients
$(\Lambda := \sup_{i,j}[a^{ij}]_{o,\alpha,\overline{D}_{R^*}})$ we get for $w := u-v \in \overset{\circ}{H}{}^1_2(B_\sigma(x_o))$

$$\text{(3.18)} \qquad \|\nabla w\|_{L^2(B_\sigma(x_o))} \le c\,\Lambda\,\sigma^\alpha \|\nabla u\|_{L^2(B_\sigma(x_o))} \le$$
$$\le c\,\Lambda\, K_1\, \sigma^{\frac{n}{2}+\alpha} R^{1-n}.$$

(3.17) and (3.18) imply for $0<t<1$

$$\|\nabla u-(\nabla u)_{t\sigma}\|_{L^2(B_{t\sigma}(x_o))} \le C\, t^{\frac{n}{2}+1} \|\nabla u-(\nabla u)_{\sigma}\|_{L^2(B_{\sigma}(x_o))} + B\,\sigma^{\frac{n}{2}+\alpha},$$

where we have set $B := C K_1 \Lambda R^{1-n}$.

Defining $\beta = \frac{n}{2}+\alpha$, $\gamma = \frac{n}{2}+1$ and $\varphi(\rho) := \rho^{-\beta}\|\nabla u-(\nabla u)_{\rho}\|_{L^2(B_{\rho}(x_o))}$ we have shown for $\sigma \le \frac{R}{2}$, $0<t<1$

$$t^{\beta}\,\varphi(t\sigma) \le C\, t^{\gamma}\,\varphi(\sigma) + B .$$

An elementary but important lemma, c.f. [1], then implies

$$\varphi(t\sigma) \le C\,\varphi(\sigma) + \tilde{C}(C,\alpha) B .$$

Setting $\sigma = R/2$, $\rho := t\sigma$ we get using (3.16)

$$\|\nabla u-(\nabla u)_{\rho}\|_{L^2(B_{\rho}(x_o))} \le C\,\rho^{\frac{n}{2}+\alpha}\{R^{-(\frac{n}{2}+\alpha)}\|\nabla u-(\nabla u)_{R/2}\|_{L^2(B_{R/2}(x_o))} + K_1\,\Lambda\, R^{1-n}\}$$

$$\le C\rho^{\frac{n}{2}+\alpha}\{R^{-(\frac{n}{2}+\alpha)} C K_1 R^{1-\frac{n}{2}} + K_1 \Lambda R^{1-n}\} \tag{3.19}$$

$$\le C K_1 \{1+\Lambda R^{\alpha}\} R^{1-n-\alpha} \rho^{\frac{n}{2}+\alpha}.$$

If on the other hand $\rho \ge R/2$ we get again by (3.16)

$$\|\nabla u-(\nabla u)_{\rho}\|_{L^2(B_{\rho}(x_o)\cap B_R(x_2)\cap D_R*)} \le \rho^{\frac{n}{2}+\alpha}\|\nabla u\|_{L^2(B_R(x_2))}\,\rho^{-(\frac{n}{2}+\alpha)}$$

$$\le \rho^{\frac{n}{2}+\alpha} C K_1 R^{1-\frac{n}{2}} R^{-(\frac{n}{2}+\alpha)} = C K_1 R^{1-n-\alpha} \rho^{\frac{n}{2}+\alpha}. \tag{3.20}$$

Let us now treat the case $B_{R/2}(x_o) \not\subset D_R*$ and introduce $\bar{x}_o := (x_o^1,\dots,x_o^{n-1},0)$. If $\sigma \le \sigma_o = x_o^n < R/2$ we proceed as above and get for $\rho < \sigma_o$

$$\|\nabla u-(\nabla u)_{\rho}\|_{L^2(B_{\rho}(x_o))} \le C\,\rho^{\frac{n}{2}+\alpha}\{\sigma_o^{-(\frac{n}{2}+\alpha)}\|\nabla u-(\nabla u)_{\sigma_o}\|_{L^2(B_{\sigma_o}(x_o))} + K_1\,\Lambda\, R^{1-n}\}. \tag{3.21}$$

If $\sigma_o \le \sigma \le R/2$ we have $B_\sigma(x_o) \cap D_{R^*} \subset D_{2\sigma}(\overline{x}_o) \subset D_{R^*}$.
As in (3.16) we get for any $\sigma \le R/2$

$$\|\nabla u\|_{L^2(D_{2\sigma}(\overline{x}_o))} \le C K_1 (2\sigma)^{\frac{n}{2}} R^{1-n} , \tag{3.22}$$

because $z \in D_{2\sigma}(\overline{x}_o)$ implies $|z-y| \ge 5R/2$.

Let v be the solution of $-D_j[a^{ij}(\overline{x}_o) D_i v] = 0$ on $D_{2\sigma}(\overline{x}_o)$, $w := u-v \in \overset{\circ}{H}{}^1_2(D_{2\sigma}(\overline{x}_o))$. For v we have the estimates, c.f. [1], for $0<t<1$,

$$\sum_{\ell=1}^{n-1} \int_{D_{t(2\sigma)}(\overline{x}_o)} |D_\ell v|^2 \le C\, t^{n+2} \sum_{\ell=1}^{n-1} \int_{D_{2\sigma}(\overline{x}_o)} |D_\ell v|^2 , \tag{3.23}$$

$$\int_{D_{t(2\sigma)}(\overline{x}_o)} |D_n v|^2 \le C\, t^n \int_{D_{2\sigma}(\overline{x}_o)} |D_n v|^2 , \tag{3.24}$$

$$\int_{D_{t(2\sigma)}(\overline{x}_o)} |D_n v-(D_n v)_{t(2\sigma)}|^2 \le C\, t^{n+2} \int_{D_{2\sigma}(\overline{x}_o)} |D_n v-(D_n v)_{2\sigma}|^2 . \tag{3.25}$$

For w we get as in (3.18)

$$\int_{D_{2\sigma}(\overline{x}_o)} |\nabla w|^2 \le C\, \Lambda^2 K_1^2 (2\sigma)^{n+2\alpha} R^{2(1-n)} . \tag{3.26}$$

Now (3.23) and (3.26) imply for $0<t<1$

$$\sum_{\ell=1}^{n-1} \int_{D_{t(2\sigma)}(\overline{x}_o)} |D_\ell v|^2 \le C\, t^{n+2} \sum_{\ell=1}^{n-1} \int_{D_{2\sigma}(\overline{x}_o)} |D_\ell u|^2 + B(2\sigma)^{n+2\alpha} , \tag{3.27}$$

where $B = C\Lambda^2 K_1^2 R^{2(1-n)}$.

Because of (3.25) and (3.26) we get for $0<t<1$

$$\int_{D_{t(2\sigma)}(\overline{x}_o)} |D_n u-(D_n u)_{t(2\sigma)}|^2 \le C t^{n+2} \int_{D_{2\sigma}(\overline{x}_o)} |D_n u-(D_n u)_{2\sigma}|^2 + B(2\sigma)^{n+2\alpha}, \tag{3.28}$$

where we may take the same B as in (3.27).

If $\beta := n+2\alpha$, $\gamma := n+2$, $\varphi(\rho) := \rho^{-\beta} \sum_{\ell=1}^{n-1} \int_{D_\rho(\overline{x}_o)} |D_\ell u|^2$ and

$\psi(\rho) := \rho^{-\beta} \int_{D_\rho(\overline{x}_o)} |D_n u - (D_n u)_\rho|^2$ we have shown, $0<t<1$, $\sigma \le R/2$,

$$t^\beta \varphi(t(2\sigma)) \le C t^\gamma \varphi(2\sigma) + B$$

and
$$t^\beta \psi(t(2\sigma)) \le C t^\gamma \psi(2\sigma) + B .$$

From this we conclude as above

$$\varphi(t(2\sigma)) \le C \varphi(2\sigma) + \tilde{C} B$$

and an analogous inequality for ψ.
Setting $\sigma = R/2$, $2\rho := t(2\sigma)$ we get from this

$$\int_{D_{2\rho}(\overline{x}_o)} |\nabla u - (\nabla u)_{2\rho}|^2 \le \sum_{\ell=1}^{n-1} \int_{D_{2\rho}(\overline{x}_o)} |D_\ell u|^2 +$$

$$\int_{D_{2\rho}(\overline{x}_o)} |D_n u - (D_n u)_{2\rho}|^2 \le$$

$$\text{(3.29)} \qquad \le C(2\rho)^{n+2\alpha} \{ R^{-(n+2\alpha)} \int_{D_R(\overline{x}_o)} |\nabla u|^2 + B \} \le$$

$$\le CK_1^2 \rho^{n+2\alpha} \{ R^{-(n+2\alpha)} R^n R^{2(1-n)} + \Lambda^2 R^{2(1-n)} \} =$$

$$= CK_1^2 \{ 1 + \Lambda^2 R^{2\alpha} \} R^{2(1-n-\alpha)} \rho^{n+2\alpha} .$$

If $\rho \ge \sigma_o$ we have $B_\rho(x_o) \cap D_{R*} \subset D_{2\rho}(\overline{x}_o)$ and (3.29) yields

$$\text{(3.30)} \qquad \|\nabla u - (\nabla u)_\rho\|_{L^2(B_\rho(x_o)\cap D_{R*})} \le CK_1 \{1+\Lambda R^\alpha\} R^{1-n-\alpha} \rho^{n/2+\alpha} .$$

Together with (3.21) this implies (3.30) for any $0<\rho\le R/2$.
Now (3.19), (3.20) and (3.30) imply, c.f. [1],

$$\text{(3.31)} \qquad [\nabla u]_{o,\alpha,\overline{B_R(x_2)\cap D_{R*}}} \le CK_1 \{ 1 + \Lambda R^\alpha \} R^{1-n-\alpha} .$$

Thus we may conclude

$$(3.32)\qquad |\nabla u(x_1) - \nabla u(x_2)| \le CK_1 \{ 1 + \Lambda R^\alpha \} R^{1-n-\alpha} |x_1 - x_2|^\alpha .$$

If $5R = |x_2 - y|$ we get $R^{1-n-\alpha} \le C \sum_{i=1}^{2} |x_i - y|^{1-n-\alpha}$, $R^\alpha \le C(2R^* - \tau)^\alpha$;

if $5R = \tau$ we get $R^\alpha = C\tau^\alpha$, $R^{1-n-\alpha} = C\tau^{1-n-\alpha}$ and

$\sum_{i=1}^{2} |x_i - y|^{1-n-\alpha} \ge (2R^* - \tau)^{1-n-\alpha}$.

Altogether we get (3.15) with

$$K^* = CK_1 \{ 1 + \Lambda R^{*\alpha} \} (2R^*/\tau - 1)^{n-1+\alpha} ,$$

where K_1 may be estimated by $K_1 \le C(n,\mu,\lambda,\alpha,\Lambda,\Lambda R^{*\alpha})$ and $C = C(n,\mu,\lambda,\alpha)$.

This finishes the proof.

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