manuscripta math. 44, 71 - 93 (1983)

manuscripta mathematica

HOMOLOGICAL AND TOPOLOGICAL PROPERTIES OF LOCALLY INDICABLE GROUPS

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The classes of locally indicable groups, conservative groups and D-groups have each been defined in a different context, and have been studied for various reasons. These three classes are shown to coincide. The corresponding mod p versions of the classes are also shown to coincide, for any prime p. Applications to topology are given. In particular, new light is shed on work of Adams on a problem of Whitehead concerning asphericity in 2-complexes.

1. Introduction

The object of this paper is to show that three group-theoretic notions, which have arisen independently in different contexts, are in fact equivalent. This equivalence sheds new light on the work of Adams [1] on Whitehead's problem about aspherical 2-complexes, and on other related topics.

The first notion under consideration is that of a (locally) indicable group, which was introduced by Higman in his work [8] on the zero-divisor and unit problems for group rings. Let R be $\mathbb{Z}$ or $\mathbb{F}_p$. We then call a group G R-indicable if R is a homomorphic image of G. A group is locally R-indicable if every nontrivial finitely generated subgroup is R-indicable. Let LI(R) denote the class of locally R-indicable groups. We also refer to (locally) $\mathbb{Z}$-indicable groups as (locally) indicable, and write LI for LI($\mathbb{Z}$).

The second notion is that of a group conservative over an abelian group A, defined by Adams [1] as follows. A G-covering is a regular covering of 2-complexes whose group of covering transformations is isomorphic to G. A group G is conservative over A if $H_2(\tilde{K},A) = 0$

whenever $\tilde{K} \to K$ is a G-covering such that $H_2(K,A) = 0$. A group is conservative if it is conservative over every abelian group A. Let $\underline{C}$ and $\underline{C}(A)$ denote the classes of conservative groups and groups conservative over A, respectively.

The third notion is that of a $\underline{D}(R)$-group, where R is a commutative ring with 1. In [17], Strebel defines a group G to be in the class $\underline{D}(R)$ if the functor $R \otimes_{RG}$ - detects injective homomorphisms between projective RG-modules. That is, whenever $\phi : M \to N$ is an RG-homomorphism between RG-projectives, such that $(1 \otimes \phi) : R \otimes_{RG} M \to R \otimes_{RG} N$ is injective, then ϕ itself is injective. As Strebel points out, it is sufficient that the above property hold when M and N are free of finite rank. If $G \in \underline{D}(R)$ for all R, then we say $G \in \underline{D}$ or G is a $\underline{D}$-group.

Locally indicable groups have recently been shown to be of interest in connection with equations over groups and 1-relator products [5, 7, 11, 12, 14]. Conservative groups have been studied in [6], under a slightly different, but equivalent definition. The properties of $\underline{D}$-groups have been applied to various problems in [4] and [17].

The equivalence between $\underline{D}(R)$ and $\underline{LI}(R)$ has been discovered independently by Gersten [7]. His methods rely on tower constructions [11] and cyclic covers.

Our results are as follows.

1.1. Comparison and reduction

THEOREM 1 *The classes* $\underline{LI}(R)$, $\underline{C}(R)$ *and* $\underline{D}(R)$ *coincide for* $R = \mathbb{Z}$ *or* $R = \mathbb{F}_p$.

With the help of this theorem we show how to relate the classes $\underline{C}(A)$ and $\underline{D}(R)$ for any abelian group A and commutative ring R with 1, to the classes $\underline{LI}$ and $\underline{LI}(\mathbb{F}_p)$.

THEOREM 2 (i) *If* $A = 0$, *then* $\underline{C}(A)$ *is the class of all groups.*

(ii) *If A is a torsion group, the orders of whose elements involve only finitely many primes* $p_1, \ldots, p_n$, *then* $\underline{C}(A) = \bigcap_{i=1}^{n} \underline{LI}(\mathbb{F}_{p_i})$.

(iii) *Otherwise* $\underline{C}(A) = \underline{LI}$

THEOREM 3 (i) *If R has characteristic* 0, *then* $\underline{D}(R) = \underline{LI}$.

(ii) If R has characteristic $n > 0$, then

$$\underline{D}(R) = \bigcap_{\substack{p \text{ prime} \\ p|n}} \underline{LI}(\mathbb{F}_p).$$

REMARKS 1 *If* R_1 *and* R_2 *are rings whose additive group structures are isomorphic, then* $\underline{D}(R_1) = \underline{D}(R_2)$.

2. *Clearly* $\underline{LI} \subset \underline{LI}(\mathbb{F}_p)$ *for all* p. *It follows that* $\underline{LI} = \underline{C} = \underline{D}$.

1.2. Applications

For any group G let r(G) denote the union of all the finitely generated, nonindicable subgroups of G. Then r(G) is a fully invariant subgroup of G, which we call the *locally indicable residual* of G. An equivalent definition of r(G) is that it is the smallest normal subgroup N of G such that G/N is locally indicable.

A 2-complex X is called *almost acyclic* if $H_2(X, \mathbb{F}_p) = 0$ for every prime p, or equivalently if $H_2(X) = 0$ and $H_1(X)$ is torsion-free [9]. The class of almost acyclic 2-complexes is denoted $\underline{P}$ in [6].

PROPOSITION 3.1. *Let* $\tilde{X} \to X$ *be a regular covering of 2-complexes such that* $\tilde{X}$ *is almost acyclic. Then* $\mathrm{cd}\left(\pi_1(X) / r(\pi_1(\tilde{X}))\right) \leq 2$.

COROLLARY 3.2. *Let* A *be any finitely generated central subgroup of* $\pi_1(X)$. *Then there exists a finitely generated perfect subgroup* P *of* $\pi_1(\tilde{X})$ *such that* $A/(A \cap P)$ *is free abelian of rank* $\rho \leq 2$. *The group* P *can be chosen so that* $\rho = 0$ *unless* $[\pi_1(X), \pi_1(X)] / r(\pi_1(\tilde{X}))$ *is free, and so that* $\rho \leq 1$ *unless* $\pi_1(X) / r(\pi_1(\tilde{X})) \cong \mathbb{Z} \times \mathbb{Z}$ *and* $\pi_1(X)/\pi_1(\tilde{X})$ *is infinite.*

These results apply in particular in the case where X is a subcomplex of an aspherical 2-complex Y. Then $\pi_1(\tilde{X})$ can be taken to be the kernel of $\pi_1 X \to \pi_1 Y$, and it is conjectured that $r(\pi_1 \tilde{X}) = 1$. These results should be compared with [4], Theorem 3.6. and Corollary 3.7.

An RG-module M is *perfect* if $R \otimes_{RG} M = 0$. In general it is not known whether nonzero finitely generated perfect projective $\mathbb{Z}G$-modules exist. Note that no finitely generated perfect projective $\mathbb{Z}G$-module is stably free.

In contrast, for any group G containing a nontrivial, finitely generated, perfect subgroup, methods of [21] can be used to construct a nonzero, countably generated, perfect projective $\mathbb{Z}G$-module.

COROLLARY 3.4. (a) *If every nontrivial, finitely generated subgroup of* G *has a proper subgroup of finite index, then there are no nonzero, finitely generated, perfect projective* $\mathbb{Z}G$-*modules*.

(b) *If* G *has a transfinite subnormal series*

$$G = G_0 \rhd G_1 \rhd \ldots \rhd G_\alpha \rhd G_{\alpha+1} \rhd \ldots \text{ such that } \bigcap G_\alpha = \{1\}$$

and each quotient $G_\alpha/G_{\alpha+1}$ *is locally* p_α-*indicable for some prime* p_α, *then there are no non-zero perfect* $\mathbb{Z}G$-*projectives*.

REMARK *The best-known example of a group which does not satisfy the hypotheses of Corollary* 3.4. (a) *is Higman's group*

$$G = \langle a, b, c, d \mid a^2 = a^b, b^2 = b^c, c^2 = c^d, d^2 = d^a \rangle .$$

However, results of Waldhausen [19] *show that finitely generated* $\mathbb{Z}G$-*projectives are stably free, so in particular there are no nonzero, finitely generated perfect* $\mathbb{Z}G$-*projectives*.

These results are related to [4] via the notion of a Cockroft-property for n-complexes. We say that a connected n-complex X is *Cockroft* if $\pi_i(X) = 0$ for $2 \leq i < n$ and the Hurewicz map $\pi_n(X) \to H_n(X)$ vanishes. For n = 2, this notion was introduced in [3].

THEOREM 3.5. *Let* X *be a Cockroft* n-*complex such that* $cd(\pi_1(X)) \leq n$. *Then* $\pi_n(X)$ *is a perfect projective* $\mathbb{Z}\pi_1(X)$-*module*. *If in addition* $\pi_1(X)$ *has a subnormal series as in Corollary* 3.4. (b) *above, then* X *is aspherical*.

We are grateful to K.A. Brown, P. Linnell and R. Strebel for various comments and suggestions which have enabled us to make several improvements to the paper, and to eliminate a number of errors.

2. Proofs

Subsection 2.1. is devoted to proving $\underline{D}(R) = \underline{LI}(R)$ and 2.2. to proving $\underline{D}(R) = \underline{C}(R)$, where R is either $\mathbb{Z}$ or $\mathbb{F}_p$ for some prime p. The proof of Theorem 1 is presented here as a chain of Lemmas. In 2.3. below we prove Theorem 2 and in 2.4. we prove Theorem 3.

2.1. Equivalence of $\underline{D}(R)$ and $\underline{LI}(R)$.

For the case $R = \mathbb{Z}$, Strebel proved in [17] that $\underline{D}(R) \subset \underline{LI}(R)$. His arguments remain valid also for the case $R = \mathbb{F}_p$ and we shall not repeat them here.

LEMMA 2.1. $\underline{LI}(R) \subset \underline{D}(R)$

Comment on the proof: our proof is indirect and uses induction on a certain measure of complexity. Suppose G is a locally R-indicable group outside $\underline{D}(R)$. The key idea is to reduce the problem to a situation where a finitely generated subgroup $H \subset G$ may be considered. If H is locally R-indicable, then either H = 1 or H has a normal subgroup K with H/K isomorphic to the additive group C of R. The fact [17] that $C \in \underline{D}(R)$ can now be used to pass from H to K with a resulting reduction in complexity.

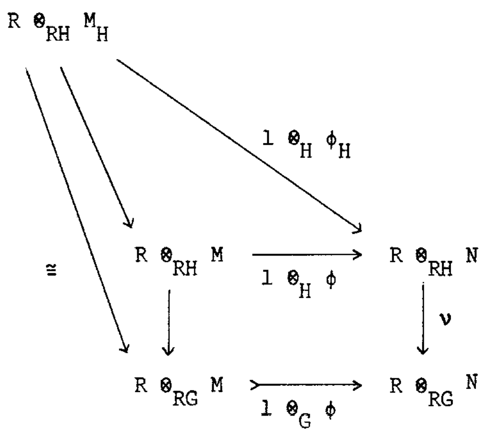
Proof of Lemma 2.1. Suppose G is an $\underline{LI}(R)$-group not belonging to $\underline{D}(R)$. Then there exist free RG-modules M and N and a non-injective RG-homomorphism $\phi : M \to N$ such that $1 \otimes \phi : R \otimes_{RG} M \to R \otimes_{RG} N$ is injective. Let m denote a nonzero element in the kernel of ϕ, and let X be an RG-basis for M. We now define the complexity of the data (m,X,G) to be the set S = S(m,X,G) of elements of G appearing in the unique R-linear expression

$$m = \sum_{ij} \lambda_{ij} g_{ij} x_i, \quad \lambda_{ij} \in R \setminus \{0\}, \quad g_{ij} \in G, \quad x_i \in X .$$

This set is always finite and it is nonempty unless m = 0. The size of the complexity is the number of its elements.

Let H be any subgroup of G. Then we denote by M_H the free RH-module with basis X and by ϕ_H the restriction of ϕ to M_H.

The commutative diagram



shows that $\nu \circ (1 \otimes_H \phi_H)$, and hence $1 \otimes_H \phi_H$ itself, is injective.

If in addition $S(m,X,G) \subset H$ holds, then m lies in the kernel of ϕ_H and $S(m,X,H) = S(m,X,G)$. Furthermore, $H \neq 1$, otherwise $1 \otimes_H \phi_H$ would equal ϕ_H and hence would be both injective and non-injective.

Now if H is finitely generated in this situation, then by assumption on G, there exists a normal subgroup $K \lhd H$ with $C = H/K$ isomorphic to the additive group of R.

We now observe that the functor $(R \otimes_{RH} -)$ factorises as

$$(R \otimes_{RH} -) = (R \otimes_{RC} -) \circ (R \otimes_{RK} -).$$

The modules $R \otimes_{RK} M_H$ and $R \otimes_{RK} N$ are RC-free, and so $1 \otimes_K \phi_H : R \otimes_{RK} M_H \to R \otimes_{RK} N$ is injective, since $C \in \underline{D}(R)$, [17].

We finally are ready for the inductive argument: so suppose that we have chosen the data (m,X,G) such that the size of $S(m,X,G)$ is as small as possible. Suppose also that $1 \in S(m,X,G)$. (If necessary, we can satisfy this assumption by replacing X by gX for some $g \in S$, for $S(m,gX,G) = g^{-1} S(m,X,G)$, and so in particular $|S(m,gX,G)| = |S(m,X,G)|$) .

Let $H \subset G$ be the subgroup generated by $S(m,X,G)$. Then $H \neq 1$ and H is finitely generated. If T is any transversal for the normal subgroup K in H, then $Y_T = \{tx \ ; \ t \in T, \ x \in X\}$ is an RK-basis for M_H, and $S = S(m,X,H)$ is a disjoint union of subsets $S \cap Kt$, $t \in T$. Since $1 \in S \cap K$, we have $S \cap K \neq \emptyset$, and since S generates $H \neq K$, we have $S \not\subset K$. So $S \cap Kt \neq \emptyset$ for at least one other coset Kt disjoint from K. We may thus choose our transversal T such that $S \cap T$ contains at least two elements, say 1 and u. We now consider the set $S(m,Y_T,K)$. It may be expressed as

$$S(m,Y_T,K) = \bigcup_{t \in T} (S \cap Kt)\,t^{-1}.$$

But $1 \in (S \cap K) \cap (S \cap Ku)\,u^{-1}$, so the above union is not disjoint.

Thus

$$\begin{aligned} |S(m,Y_T,K)| &< \sum_{t\in T} |(S \cap Kt)\,t^{-1}| \\ &= \sum_{t\in T} |(S \cap Kt)| \\ &= |S(m,X,H)| \\ &= |S(m,X,G)|. \end{aligned}$$

This contradicts our assumption of a choice of (m,X,G) realising minimum complexity.

REMARK *This proof also shows that $\underline{LI} \subset \underline{D}(S)$ for any ring S, using the fact* [17], *Prop. 1.3. that the infinite cyclic group belongs to $\underline{D}(S)$. It follows that $\underline{LI} \subset \underline{D}$.*

2.2. Equivalence of $\underline{D}(R)$ and $\underline{C}(R)$

In this subsection we will prove that $\underline{D}(R) = \underline{C}(R)$ when R is either $\mathbb{Z}$ or $\mathbb{F}_p$. The proof of the inclusion $\underline{D}(R) \subset \underline{C}(R)$ is elementary, and applies to an arbitrary ring R.

LEMMA 2.2. *Let R be a commutative ring with 1. Then $\underline{D}(R) \subset \underline{C}(R)$.*

Proof Suppose $G \in \underline{D}(R)$, and let $\tilde{K} \to K$ be a G-covering of 2-complexes, such that $H_2(K;R) = 0$. Then the cellular R-chain complex $C_*(\tilde{K})$ of $\tilde{K}$ consists of free RG-modules and RG-homomorphisms, while that of K is obtained by applying the functor $R \otimes_{RG} -$ to $C_*(\tilde{K})$:

$$C_*(K) \cong R \otimes_{RG} C_*(\tilde{K}).$$

Furthermore, since $C_3(K) = 0 = H_2(K;R)$, the boundary homomorphism $C_2(K) \to C_1(K)$ is injective. Since $G \in \underline{D}(R)$ it follows that $C_2(\tilde{K}) \to C_1(\tilde{K})$ is also injective, that is $H_2(\tilde{K};R) = 0$.

Hence $G \in \underline{C}(R)$, as required.

LEMMA 2.3. *Let R denote either* $\mathbb{Z}$ *or* $\mathbb{F}_p$ *for some prime p, and suppose* $\phi : M \to N$ *is an RG-homomorphism between free RG-modules of finite rank.*

(a) *There exist a free RG-module F, a G-covering* $K' \to K$ *of 2-complexes, and a commutative diagram*

$$\begin{array}{ccccc} C_2(K') & & \xrightarrow{\;d_2\;} & & C_1(K') \\ \cong\downarrow & & & & \downarrow\cong \\ M & \xrightarrow[\phi]{} & N & \hookrightarrow & N \oplus F \end{array}$$

of RG-modules.

(b) *Furthermore, in the case* $R = \mathbb{F}_p$, *we may choose the 2-complex K to be almost acyclic.*

COROLLARY 2.4. *Let R be either* $\mathbb{Z}$ *or* $\mathbb{F}_p$ *for some prime p. Then* $\underline{C}(R) \subset \underline{D}(R)$.

Proof of 2.4. If in Lemma 2.3., $1 \otimes \phi$ is injective, then so is $1 \otimes d_2 : R \otimes_{RG} C_2(K') \to R \otimes_{RG} C_1(K')$. But $R \otimes_{RG} C_*(K') \cong C_*(K)$, so

$$H_2(K,R) = \ker(1 \otimes d_2) = 0.$$

If, in addition, $G \in \underline{C}(R)$, then $\ker d_2 = H_2(K',R) = 0$, so d_2 is injective and hence so is ϕ. It follows that $G \in \underline{D}(R)$, as desired.

Proof of 2.3. Consider first the case $R = \mathbb{Z}$. Choose $\mathbb{Z}G$-bases X, Y for M, N respectively and a set Z of generators for G. Let F denote the free $\mathbb{Z}G$-module with basis Z, and write $\phi(x)$ in the form

$$\phi(x) = \sum_{i=1}^{n(x)} \lambda(x,i)\, g(x,i)\, y(x,i)$$

for each $x \in X$ with $\lambda(x,i) \in \mathbb{Z}$, $g(x,i) \in G$, $y(x,i) \in Y$.

For each pair (x,i) choose a word $h(x,i)$ over the generating set Z of G representing the element $g(x,i)$ of G and define $W(x)$ to be the word

$$W(x) = \prod_{i=1}^{n(x)} h(x,i)\; y(x,i)^{\lambda(x,i)}\, h(x,i)^{-1}$$

over the disjoint union $Y \cup Z$.

Let K be the geometric realisation of the presentation

$$\langle Y \cup Z \mid W(x),\ x \in X \rangle \qquad (1)$$

and let $\Gamma = \pi_1 K$ be the group presented by it.

The map $Z \to G$ extends to a map $\Theta : Y \cup Z \to G$ with $\Theta(Y) = \{1\}$ and so to an epimorphism $\Theta : \Gamma \twoheadrightarrow G$ (since each relator $W(x)$ is a product of conjugates of elements of Y.)

Then the covering $K' \to K$ corresponding to $\ker \Theta$ is a G-covering, and the second boundary homomorphism of the cellular $\mathbb{Z}$-chain complex of K' is given by the matrix of Fox derivatives of the presentation (1), reduced to $\mathbb{Z}G$ via the canonical map $\mathbb{Z}\Gamma \to \mathbb{Z}G$. It is easy to check that this matrix determines the composite $M \to N \hookrightarrow N \oplus F$ with respect to the bases X, Y, Z for M, N, F respectively.

Now consider the case $R = \mathbb{F}_p$. By Lemma 2.5. below we may choose a $\mathbb{Z}G$-homomorphism $\phi_0 : M_0 \to N_0$ between free $\mathbb{Z}G$-modules and $\mathbb{F}_pG$-isomorphisms

$$\mu : \mathbb{F}_p \otimes_{\mathbb{Z}} M_0 \to M, \quad \nu : \mathbb{F}_p \otimes_{\mathbb{Z}} N_0 \to N$$

such that $\nu^{-1} \circ \phi \circ \mu = 1 \otimes \phi_0 : \mathbb{F}_p \otimes_{\mathbb{Z}} M_0 \to \mathbb{F}_p \otimes_{\mathbb{Z}} N_0$, and such that $1 \otimes \phi_0 : \mathbb{Z} \otimes_{\mathbb{Z}G} M_0 \to \mathbb{Z} \otimes_{\mathbb{Z}G} N_0$ is split injective. Applying the above argument to ϕ_0 gives a G-covering $K' \to K$ such that the cellular $\mathbb{F}_p$-chain complex of K' has the desired form, which proves (a). It also follows from the fact that $(1 \otimes \phi_0) : \mathbb{Z} \otimes_{\mathbb{Z}G} M_0 \to \mathbb{Z} \otimes_{\mathbb{Z}G} N_0$ is split injective, that K is almost acyclic, which proves (b).

LEMMA 2.5. *Let* $\phi : M \to N$ *be an* $\mathbb{F}_pG$*-homomorphism between free* $\mathbb{F}_pG$*-modules of finite ranks* m *and* n *respectively, such that* $\overline{\phi} = 1 \otimes \phi : \mathbb{F}_p \otimes_{\mathbb{F}_pG} M \to \mathbb{F}_p \otimes_{\mathbb{F}_pG} N$ *is injective. Then there exists a* $\mathbb{Z}G$*-homomorphism* $\phi_0 : M_0 \to N_0$ *between free* $\mathbb{Z}G$*-modules, and isomorphisms* $\mu : \mathbb{F}_p \otimes_{\mathbb{Z}} M_0 \to M$, $\nu : \mathbb{F}_p \otimes_{\mathbb{Z}} N_0 \to N$, *such that the square*

$$\begin{array}{ccc} \mathbb{F}_p \otimes_{\mathbb{Z}} M_0 & \xrightarrow{1 \otimes \phi_0} & \mathbb{F}_p \otimes_{\mathbb{Z}} N_0 \\ \mu \downarrow & & \downarrow \nu \\ M & \xrightarrow[\phi]{} & N \end{array}$$

commutes, and such that $\overline{\Phi}_0 = 1 \otimes \phi_0 : \mathbb{Z} \otimes_{\mathbb{Z}G} M_0 \to \mathbb{Z} \otimes_{\mathbb{Z}G} N_0$ *is split injective.*

Proof Choose an $\mathbb{F}_pG$-basis X of M, and let $\overline{X} = \varepsilon(X)$ be the induced basis of $\overline{M} = \mathbb{F}_p \otimes_{\mathbb{F}_pG} M$, where $\varepsilon : \mathbb{F}_pG \to \mathbb{F}_p$ is the augmentation map. Since $\overline{\phi}$ is an injective map between $\mathbb{F}_p$-vector spaces, the set $\overline{\phi}(\overline{X})$ extends to a basis $\overline{Y}$ of $\overline{N} = \mathbb{F}_p \otimes_{\mathbb{F}_pG} N$; and the matrix of $\overline{\phi}$ with respect to the bases $\overline{X}$ and $\overline{Y}$ has the form

$$\begin{pmatrix} I_m \\ 0 \end{pmatrix} ,$$

where I_m denotes the identity $m \times m$ matrix.

The basis $\overline{Y}$ may be lifted to a basis Y, say, of N, such that $\varepsilon(Y) = \overline{Y}$. Let (f_{ij}) denote the matrix of ϕ with respect to the bases X and Y. Then

$$\varepsilon(f_{ij}) = \begin{pmatrix} I_m \\ \\ 0 \end{pmatrix} .$$

Since the canonical map $\pi : \mathbb{Z}G \to \mathbb{Z}G/p\mathbb{Z}G \cong \mathbb{F}_pG$ is surjective, we may choose $F_{ij} \in \mathbb{Z}G$ for all (ij), such that $\pi(F_{ij}) = f_{ij}$. Let $\phi_1 : (\mathbb{Z}G)^m \to (\mathbb{Z}G)^n$ denote the $\mathbb{Z}G$-homomorphism defined by the matrix (F_{ij}), and let $\mu : \mathbb{F}_p \otimes_{\mathbb{Z}} (\mathbb{Z}G)^m \cong (\mathbb{F}_pG)^m \to M$, $\nu : \mathbb{F}_p \otimes_{\mathbb{Z}} (\mathbb{Z}G)^n \cong (\mathbb{F}_pG)^n \to N$ be the isomorphisms obtained by sending the canonical bases onto X, Y respectively. Then clearly the square

$$\begin{array}{ccc} (\mathbb{F}_pG)^m & \xrightarrow{1 \otimes \phi_1} & (\mathbb{F}_pG)^n \\ \mu \big\downarrow & & \big\downarrow \nu \\ M & \xrightarrow[\phi]{} & N \end{array}$$

commutes.

The map $\overline{\phi}_1 = 1 \otimes \phi_1 : \mathbb{Z} \otimes_{\mathbb{Z}G} (\mathbb{Z}G)^m \to \mathbb{Z} \otimes_{\mathbb{Z}G} (\mathbb{Z}G)^n$ need not in general be split injective. Define an integer matrix (b_{ij}) by

$$(b_{ij}) = \begin{pmatrix} I_m \\ \\ 0 \end{pmatrix} - \varepsilon(F_{ij}).$$

Then each b_{ij} is a multiple of p, since

$$\varepsilon(F_{ij}) = \begin{pmatrix} I_m \\ \\ 0 \end{pmatrix} \text{mod. } p.$$

Now let $\phi_0 : (\mathbb{Z}G)^m \to (\mathbb{Z}G)^n$ be given by the matrix $(F_{ij} + b_{ij})$. Then $\overline{\phi}_0 : \mathbb{Z}^m \to \mathbb{Z}^n$ is given by the $\mathbb{Z}$-matrix

$$\begin{pmatrix} I_m \\ \\ 0 \end{pmatrix},$$

so is split injective, while $(1 \otimes \phi_0) = (1 \otimes \phi_1) : (\mathbb{F}_pG)^m \to (\mathbb{F}_pG)^n$. This completes the proof.

2.3. Proof of Theorem 2

Assertion (i) of the theorem is immediate, and requires no further comment. We will first prove the theorem in the case where A is finitely generated. This follows from some easy remarks.

(a) $\underline{C}(A_1 \oplus A_2) = \underline{C}(A_1) \cap \underline{C}(A_2)$,
because $H_2(X ; A_1 \oplus A_2) = H_2(X ; A_1) \oplus H_2(X ; A_2)$ for any 2-complex X.

(b) $\underline{C}(\mathbb{Z}/p^n\mathbb{Z}) = \underline{C}(\mathbb{F}_p)$ for any $n \geq 1$,
because $H_2(X ; \mathbb{Z}/p^n\mathbb{Z}) = 0$ if and only if $H_2(X ; \mathbb{F}_p) = 0$, for any 2-complex X. This is seen by an easy inductive argument, using the long exact sequences.

$$0 \to H_2(X ; \mathbb{Z}/p^{n-1}\mathbb{Z}) \to H_2(X ; \mathbb{Z}/p^n\mathbb{Z}) \to H_2(X ; \mathbb{F}_p) \to \dots$$

(c) $\underline{C}(\mathbb{Z}) \subset \underline{C}(\mathbb{F}_p)$ for every prime p;
by Theorem 1, because any indicable group is p-indicable for every prime p.

Now (a) and (b) together show that $\underline{C}(A) = C(\mathbb{F}_p)$ whenever A is a (non zero) finite abelian p-group. Then (a) and Theorem 1 show that the theorem holds for any finite A. If A is finitely generated but infinite, then

$$A \cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \oplus A_1 \dots \oplus A_n,$$

where each A_i is a finite abelian p_i-group for a (possible empty) set of primes $\{p_1, \dots, p_n\}$. Hence

$$\underline{C}(A) = \underline{C}(\mathbb{Z}) \cap \ldots \cap \underline{C}(\mathbb{Z}) \cap \underline{C}(\mathbb{F}_{p_1}) \cap \ldots \cap \underline{C}(\mathbb{F}_{p_n}) \text{ by (a)}$$
$$= \underline{C}(\mathbb{Z}) \text{ by (c)}$$

A direct limit argument extends the result to the case of infinitely generated A: we have $H_2(X ; A) = 0$ if and only if $H_2(X ; B) = 0$ for every finitely generated subgroup B of A, for any 2-complex X. Hence $\underline{C}(A)$ is the intersection of all the classes $\underline{C}(B)$, where B varies over all finitely generated subgroups of A. In particular, if A contains elements of prime order for an infinite set π of primes, then

$$\begin{aligned}
\underline{C}(A) &\subset \bigcap_{p \in \pi} \underline{C}(\mathbb{F}_p) \\
&= \bigcap_{p \in \pi} \underline{LI}(\mathbb{F}_p) \text{ by Theorem 1} \\
&= \underline{LI} \\
&= \underline{C}(\mathbb{Z}) \\
&\subset \bigcap \underline{C}(B) \quad \text{by (c)} \\
&= \underline{C}(A).
\end{aligned}$$

<u>REMARK</u> <u>It follows immediately from Theorem 2 that a group G is conservative if and only if $G \in \underline{C}(\mathbb{F}_p)$ for every prime p. By Lemma 2.3. (b) this holds if and only if every G-covering of an almost-acyclic 2-complex is almost-acyclic, so our definition of conservative agrees with that in [6].</u>

<u>2.4. Proof of Theorem 3.</u>

(i)
$$\begin{aligned}
\underline{D}(R) &\subset \underline{C}(R) && \text{by Lemma 2.2.} \\
&= \underline{C}(\mathbb{Z}) && \text{by Theorem 2} \\
&= \underline{LI} && \text{by Theorem 1} \\
&\subset \underline{D}(R) && \text{by the remark in 2.1.}
\end{aligned}$$

(ii) Let R_p denote the p-primary component of R. Then R is isomorphic to a finite product of rings R_p $(p|n)$, and it follows that $\underline{D}(R) = \bigcap_{p|n} \underline{D}(R_p)$. We are thus reduced to the case where $R = R_p$ for some prime p, so R has characteristic p^k for some $k \geqslant 1$. Since we already know, from Lemmas 2.2. and 2.3. and Theorem 2, that $\underline{D}(R) \subset \underline{C}(R) = \underline{C}(\mathbb{F}_p) = D(\mathbb{F}_p)$, we argue by induction on k to show that $\underline{D}(\mathbb{F}_p) \subset \underline{D}(R)$.

If $k = 1$, then every projective RG-module is a projective $\mathbb{F}_p G$-module and every RG-homomorphism is an $\mathbb{F}_p G$-homomorphism. Moreover, the natural $\mathbb{F}_p$-isomorphism $(\mathbb{F}_p \otimes_{\mathbb{F}_p G} -) \cong (R \otimes_{RG} -)$ implies that $\underline{D}(\mathbb{F}_p) \subseteq \underline{D}(R)$.

Now suppose $k \geq 2$ and $\underline{D}(\mathbb{F}_p) \subseteq \underline{D}(S)$ for any ring S of characteristic p^{k-1}. Let I denote the annihilator of p in R so that the ring $S = R/I$ is isomorphic (as an R-module) to the ideal pR of R. We assume that the group G belongs to $\underline{D}(\mathbb{F}_p)$ and we consider an RG-homomorphism $\phi : M \to N$ between free RG-modules such that $1 \otimes \phi : R \otimes_{RG} M \to R \otimes_{RG} N$ is injective.

The exact sequence $0 \to I \to R \to R/I \to 0$ together with the map $\phi : M \to N$ gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & IM & \to & M & \to & M/IM & \to & 0 \\ & & \downarrow{\scriptstyle \phi_I} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_S} & & \\ 0 & \to & IN & \to & N & \to & N/IN & \to & 0 . \end{array}$$

Its rows stay exact.

First note that ϕ_I is a map between free $\mathbb{F}_p G$-modules and $1 \otimes \phi_I : R \otimes_{RG} IM \to R \otimes_{RG} IN$ is injective being a restriction of $1 \otimes \phi$.

Using again the natural isomorphism $(\mathbb{F}_p \otimes_{\mathbb{F}_p G} -) \cong (R \otimes_{RG} -)$ it follows from $G \in \underline{D}(\mathbb{F}_p)$ that ϕ_I is injective.

Secondly, the map ϕ_S is an SG-module homomorphism between free SG-modules. Now there is a commutative square of additive group homomorphisms

$$\begin{array}{ccc} S \otimes_{SG} M/IM & \xrightarrow{\;1_S \otimes \phi_S\;} & S \otimes_{SG} N/IN \\ \cong \downarrow & & \downarrow \cong \\ pR \otimes_{RG} M & \xrightarrow{\;\overline{\phi}\;} & pR \otimes_{RG} N \end{array}$$

where $\overline{\phi}$ is a restriction of $1 \otimes \phi$. Hence $1_S \otimes \phi_S$ is injective. Also S has characteristic p^{k-1}, so $\underline{D}(\mathbb{F}_p) \subset \underline{D}(S)$ by inductive hypothesis. Thus ϕ_S is injective and so is ϕ by the Five - Lemma.

This shows that $\underline{D}(\mathbb{F}_p) \subseteq \underline{D}(R)$ and the proof is complete.

3. Applications

3.1. Whitehead's question

Let L be an aspherical 2-complex and K a subcomplex of L. An open question of J.H.C. Whitehead [20] asks whether K is necessarily itself aspherical. Let G denote the kernel of the inclusion-induced map $i : \pi_1 K \to \pi_1 L$. Then a theorem of Adams [1] says that, if K is not aspherical, then G has a non-trivial perfect subgroup. Indeed, Adams' construction determines a normal subgroup $P_A(G)$ for any group G, namely the smallest normal subgroup $N \lhd G$ such that G/N is conservative. The subgroup $P_A(G)$ is not in general perfect, but it is perfect in the case G = Ker i above.

Now Theorem 1 gives a purely group-theoretical interpretation of $P_A(G)$: it is precisely the locally indicable residual r(G). Note that r(G) is <u>not</u> in general equal to the maximal perfect subgroup $P_1(G)$, even in the situation of Whitehead's question. Indeed, Adams [1] gives an example of a pair $K \subseteq L$ with L (and also K) aspherical, and $P_1(G) = [G,G] = \{1\}$. But in this example G is a torsion-free 1-relator group, so by a theorem of Brodskiĭ [5] is locally indicable, in other words $r(G) = \{1\}$.

Now it is unknown whether G = Ker i is always locally indicable in the situation of Whitehead's question, so Adams' result may turn out to be stronger than has been generally realised.

If $K \subset L$ and G are as above, then the regular covering K_G of K corresponding to $G \lhd \pi_1 K$ is isomorphic to a subcomplex of the universal cover $\tilde{L}$ of L, which is contractible. In particular, K_G is almost-acyclic. Thus the following results generalise [4], Theorem 3.6. and Corollary 3.7.

<u>PROPOSITION 3.1.</u> <u>Let X be a 2-complex and G a normal subgroup of $\pi_1 X$ such that the corresponding regular covering $\tilde{X}$ of X is almost-acyclic. Then $(\pi_1 X) / r(G)$ has cohomological dimension at most 2.</u>

<u>Proof</u> Let X' denote the covering of $\tilde{X}$ corresponding to the subgroup r(G) of $G = \pi_1(\tilde{X})$. Then $H_2(X') = 0$ because $H_2(\tilde{X}) = 0$ and $G/r(G) \in \underline{C}$. Also $H_1(X') = H_1(r(G)) = 0$, since r(G) is perfect [10]. In other words, X' is acyclic, so the cellular chain complex of X' is a free $\mathbb{Z}(\pi_1 X / r(G))$-resolution of $\mathbb{Z}$, of length 2.

COROLLARY 3.2. *Let X and G be as in the Proposition, and let A be a finitely generated central subgroup of $\pi_1 X$. Then there exists a finitely generated perfect subgroup P of G such that $A/(A \cap P)$ is free abelian of rank $d \leqslant 2$. Furthermore, we may assume $d \leqslant 1$ unless $\pi_1(X) / r(G) \cong \mathbb{Z} \times \mathbb{Z}$ and G has infinite index; and we may assume $d = 0$ unless $[\pi_1 X, \pi_1 X] / r(G)$ is free.*

Proof By [2] and the Proposition, the centre of $\Gamma = \pi_1(X) / r(G)$ is either:

a) Γ if Γ is abelian,

b) trivial; or

c) $\mathbb{Z}$, only if $[\Gamma,\Gamma]$ is free.

Let B be the kernel of $A \to \pi_1 X \to \Gamma$. Then B is a finitely generated subgroup of r(G), so is contained in a finitely generated, non-indicable subgroup Q of G, which is in turn contained in a finitely generated perfect subgroup P of G [10]. Since $P \subset r(G)$, we have $A/(A \cap P) \cong A/B$, which is a finitely generated central subgroup of Γ. The conclusions of the corollary are immediate in cases b) and c) above, but some further comment is necessary in case a), when Γ is abelian. Since $cd(\Gamma) \leqslant 2$, either Γ has rank 1, so A/B has rank $\leqslant 1$, or $\Gamma \cong \mathbb{Z} \times \mathbb{Z}$. In the latter case we must also prove that G has infinite index in $\pi_1 X$. But if $|\pi_1 X : G|$ is finite, then $G/[G,G] = G/r(G) \cong \mathbb{Z} \times \mathbb{Z}$. Also $H_2(G) = 0$, so it follows [15, 16] that G has the same lower central factors as the free group of rank 2. But this contradicts $G/\left[G, [G,G]\right] = G/r(G) \cong \mathbb{Z} \times \mathbb{Z}$ (since r(G) is perfect [10, 17].

3.2. Perfect projective modules

If $G \in \underline{D}(R)$ and M is a perfect projective $\mathbb{Z}G$-module, then consideration of the map $R \otimes_{\mathbb{Z}} M \to 0$ shows that $R \otimes_{\mathbb{Z}} M = 0$, and so $M = 0$. In this section we use this remark and the results of section 2 to show that no nonzero perfect projective $\mathbb{Z}G$-module exists for a large class of groups G, and apply the result to a question in topology.

PROPOSITION 3.3. *Let P be a perfect projective $\mathbb{Z}G$-module. Then*

(i) *There exists a subgroup H of G minimal with respect to the property that P is perfect as a $\mathbb{Z}H$-module.*

(ii) *The group H is trivial if and only if the module P is zero.*

(iii) *No nontrivial homomorphic image of H is locally p-indicable for any prime p.*

(iv) *The group H is perfect.*

(v) *If P is finitely generated as a $\mathbb{Z}G$-module, then H is finitely generated as a group and has no proper subgroup of finite index.*

Proof (i) We consider an idempotent endomorphism $\phi : F \to F$ of some free $\mathbb{Z}G$-module F, factorising as $F \twoheadrightarrow P \rightarrowtail F$. Fix a $\mathbb{Z}G$-basis X of F. For any $x \in X$ define the set $\Lambda(x,X)$ to consist of all the elements $\lambda_i \neq 0$ in $\mathbb{Z}G$ which occur in the representation

$$\phi(x) = \sum_{i=1}^{m(x)} \lambda_i x_i .$$

For any subgroup $H \subset G$ let $IH \subset \mathbb{Z}H$ denote the augmentation ideal and let $JH = \mathbb{Z}G \otimes_{\mathbb{Z}H} IH$ denote the left ideal of $\mathbb{Z}G$ generated by IH. Then $\mathbb{Z}(G/H) = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z} = \mathbb{Z}G/JH$ and so P is perfect as a $\mathbb{Z}H$-module if and only if the set $\Lambda(X) = \bigcup_{x \in X} \Lambda(x,X)$ is contained in JH.

If $H_0 \geqslant H_1 \geqslant \dots$ is a descending chain of subgroups of G, with intersection $H_\infty = \bigcap_i H_i$, then $JH_0 \geqslant JH_1 \geqslant \dots$. is a descending chain of left ideals of $\mathbb{Z}G$, and it is easy to check that its intersection is JH_∞. Assertion (i) is now an immediate consequence of Zorn's lemma.

(ii) Clearly $J\{1\} = \{0\}$, so $H = \{1\}$ if and only if $\Lambda = \emptyset$, in other words, if and only if $P = \{0\}$.

(iii) This follows from the remark at the beginning of the section, for if $K \lhd H$ with H/K locally p-indicable, then $H/K \in \underline{D}(\mathbb{F}_p)$ by Theorem 1 and $\overline{P} = \mathbb{Z} \otimes_{\mathbb{Z}K} P = \mathbb{Z}(H/K) \otimes_{\mathbb{Z}H} P$ is a perfect projective $\mathbb{Z}(H/K)$-module. Hence $\overline{P} = 0$, so P is a perfect $\mathbb{Z}K$-module, and so $K = H$ by minimality of H.

(iv) If H is not perfect, then there exists a nonzero homomorphism $H \to C_{p^\infty}$ for some prime p, contradicting (iii).

(v) Since $\Lambda(X) \subset JH$, each $\phi(x)$ may be expressed as a finite sum

$$\phi(x) = \sum_{\alpha=1}^{r(x)} g_\alpha(x)\,(1 - h_\alpha(x))$$

($g_\alpha(x) \in G$, $h_\alpha(x) \in H$). Since P is finitely generated, the basis X may be chosen to be finite, so the set $\Lambda(X)$ is finite, and there are only finitely many elements $h_\alpha(x)$ of H involved in the above representations. Let $K \subset H$ be the subgroup generated by the elements $h_\alpha(x)$. Then $\Lambda(X) \subset JK$, so P is a perfect $\mathbb{Z}K$-module, and so $K = H$ by minimality.

Finally, suppose N is a normal subgroup of finite index in H. It follows from [18], Theorem 3 that there are no nonzero, finitely generated, perfect $\mathbb{Z}(H/N)$-projectives, and the argument used in (iii) above shows that $N = H$.

<u>COROLLARY 3.4.</u> (a) <u>If every nontrivial, finitely generated subgroup of G has a proper subgroup of finite index, then there are no nonzero, finitely generated, perfect projective $\mathbb{Z}G$-modules.</u>

(b) <u>If G has a transfinite subnormal series $G = G_0 \triangleright G_1 \triangleright \ldots$ such that $\bigcap_\alpha G_\alpha = \{1\}$ and each $G_\alpha/G_{\alpha+1}$ is locally p_α-indicable for some prime p_α, then there are no nonzero perfect projective $\mathbb{Z}G$-modules.</u>

Suppose X, Y are Cockroft 2-complexes with isomorphic fundamental groups, such that Y is aspherical. It is an open question [4], Question 2, whether X is necessarily also aspherical. The answer is known to be in the affirmative if both X and Y are finite [3], or if $G = \pi_1 X$ has no perfect subgroups [4]. We generalise the latter result as follows.

<u>THEOREM 3.5.</u> <u>Let X be a Cockroft n-complex ($n \geq 2$) such that $cd(\pi_1 X) \leq n$. Then $\pi_n(X)$ is a perfect projective $\mathbb{Z}(\pi_1 X)$-module.</u>

<u>Proof</u> Let $\tilde{X}$ denote the universal cover of X. Then $H_n\tilde{X} \cong \pi_n\tilde{X} \cong \pi_n X$ by the Hurewicz theorem, and $H_i\tilde{X} = 0$ for $1 \leq i \leq n-1$. Hence the cellular chain complex $C_*\tilde{X}$ yields an exact sequence

$$0 \to \pi_n X \to C_n\tilde{X} \to \ldots\ldots \to C_0\tilde{X} \to \mathbb{Z} \to 0.$$

Since $cd(\pi_1 X) \leq n$ and each $C_i\tilde{X}$ is a free $\mathbb{Z}(\pi_1 X)$-module, it follows that $\pi_n X$ is projective as a $\mathbb{Z}(\pi_1 X)$-module. (Actually, this part of the proof requires only $cd(\pi_1 X) \leq n + 1$).

Now add cells in dimensions $(n+1)$ and above to X to obtain an aspherical complex Y. From the commutative diagram

$$\begin{array}{ccccccc}
C_{n+2}\tilde{Y} & \xrightarrow{\tilde{d}} & C_{n+1}\tilde{Y} & \twoheadrightarrow & \pi_n X & \rightarrowtail & C_n\tilde{X} \\
\downarrow & & \downarrow\!\!\downarrow & & \downarrow 0 & & \downarrow \\
C_{n+2}Y & \xrightarrow[d]{} & C_{n+1}Y & \longrightarrow & H_n X & \longrightarrow & C_n X
\end{array}$$

we deduce that the map $C_{n+1}Y \to C_n Y = C_n X$ is the zero map and hence

$$\begin{aligned}
\mathbb{Z} \otimes_{\mathbb{Z}(\pi_1 X)} (\pi_n X) &= \mathbb{Z} \otimes_{\mathbb{Z}(\pi_1 X)} (\text{coker } \tilde{d}) \\
&= \text{coker } d \text{ (by right exactness)} \\
&= H_{n+1} Y \\
&= H_{n+1}(\pi_1 X) \\
&= 0 \text{ (since } cd(\pi_1 X) \leq n).
\end{aligned}$$

COROLLARY 3.6. *If, in addition to the hypotheses of the theorem, the group* $G = \pi_1 X$ *has a subnormal series as in Corollary* 3.4. (b), *then* X *is aspherical.*

4. Examples

4.1.

Every finite p-group is locally p-indicable [17], whereas the class LI contains only torsion-free groups, as does the class $LI(\mathbb{F}_p) \cap LI(\mathbb{F}_q)$ where p and q are distinct primes. It is not a priori clear that the class $LI(\mathbb{F}_p)$ contains torsion-free groups which are not in LI, or even that the class $LI(\mathbb{F}_p) \cap LI(\mathbb{F}_q)$ strictly contains LI, but we will give examples in this section to demonstrate that both inclusions are strict. More generally, for any nonempty set Π of primes, let $LI(\Pi)$ denote the intersection $\bigcap_{p \in \Pi} LI(\mathbb{F}_p)$, and TF the class of torsion-free groups. Then we will show that the classes $LI(\Pi) \cap TF$, for finite Π, are all distinct and properly contain LI. (For infinite Π it is clear that $LI(\Pi) \cap TF = LI(\Pi) = LI$).

Let $n \geq 1$ be an integer, and let G_n denote the fundamental group

$$\langle a, b, c \mid a^n = b^n = c^n = abc \rangle$$

of the Brieskorn 3-manifold [13]

$$M = M(n,n,n) = \left\{(Z_1,Z_2,Z_3) \in \mathbb{C}^3 \ ; \ Z_1^n + Z_2^n + Z_3^n = 0, |Z_1|^2+|Z_2|^2+|Z_3|^2 = 1\right\}.$$

<u>LEMMA 4.1. If</u> n <u>is divisible by the integer</u> $m \geq 3$, <u>then</u> G_n <u>has a locally indicable normal subgroup of index</u> m.

<u>Proof</u>. The 3-manifold M is a nilmanifold if $n = 3$, or hyperbolic if $n > 3$ [13]. In either case it is aspherical and hence so is any covering manifold. Now suppose K is an indicable subgroup of G_n. Then the corresponding covering M_K of M is aspherical and $H^1(M_K) = H^1(K) \neq 0$. It follows [12], Theorem 6.1. that $K = \pi_1 M_K$ is <u>locally</u> indicable. Thus it is sufficient to find an indicable normal subgroup of index m.

Let $\alpha = e^{2\pi i/m}$ be a primitive m'th root of unity, and define a (right) affine action of G_n on the complex plane $\mathbb{C}$ by

$$\begin{aligned} Z^a &= Z\alpha \\ Z^b &= Z\alpha + 1 \\ Z^c &= Z\alpha^{-2} + \alpha^{-2}. \end{aligned}$$

Provided $m \geq 3$, these three transformations satisfy the defining relations of G_n, and so do indeed define an action of G_n. The image of G_n in $\mathrm{Aff}(\mathbb{C})$ is a Bieberbach group whose translation subgroup T is nontrivial and of index m. Thus the inverse image of T in G_n is an indicable normal subgroup of index m.

<u>COROLLARY 4.2. The group</u> G_n <u>is locally</u> p-<u>indicable for any odd prime factor</u> p <u>of</u> n. <u>If</u> $4 \mid n$ <u>then</u> G_n <u>is locally</u> 2-<u>indicable</u>. <u>If</u> $n \neq 3$ <u>then</u> G_n <u>is not locally indicable</u>.

<u>Proof</u>. The first two assertions are immediate from Lemma 4.1., since the classes $\underline{LI}(\mathbb{F}_p)$ are extension-closed. For the third assertion, a direct computation shows that

$$G_n^{ab} \cong (\mathbb{Z}/n\mathbb{Z})^2 \times \mathbb{Z}/(n-3)\mathbb{Z},$$

which is finite of order $n^2|n - 3|$, provided $n \neq 3$.

LEMMA 4.3. *If $n > 3$ and $p \neq 3$ is a prime divisor of $n - 3$, then G_n is not locally p-indicable.*

Proof. Write $n - 3 = m.q$, where q is a power of 3, and $3 \nmid m$. Then let K be the kernel of the map $\phi : G_n \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ given by $\phi(a) = \phi(b) = \phi(c) = 1 + m\mathbb{Z}$. Note that K has a transversal in G consisting of the central elements a^{in} $(0 \leq i \leq m - 1)$, since m and n are coprime. It follows that $[K,K] = [K,G]$, so from the exact sequence [15, 16]

$$0 = H_2(\mathbb{Z}/m\mathbb{Z}) \to \frac{K}{[K,G]} \to \frac{G}{[G,G]} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

of abelian groups, we can deduce that
$|K^{ab}| = |K : [K,G]| = |G^{ab}| / m = n^2 q$, which is comprime to m. Hence G_n is not locally p-indicable for any prime factor p of m, that is for any prime factor $p \neq 3$ of $n - 3$.

COROLLARY 4.4. *Let Π be a nonempty finite set of primes. Then there exists a torsion-free group G such that Π is precisely the set of primes p for which G is locally p-indicable.*

Proof. Let n be the square of the product of all the primes in Π, and take $G = G_n$. Then $G_n \in \underline{LI}(\Pi)$ by Corollary 4.2., and is torsion-free since M is an aspherical 3-manifold. Conversely, suppose p is a prime such that $G_n \in \underline{LI}(\mathbb{F}_p)$. Then certainly $p \mid |G_n^{ab}| = n^2(n - 3)$. If $p \mid (n - 3)$ then we must have $p = 3$ by Lemma 4.3., so in any case $p \mid n$, that is $p \in \Pi$.

4.2.

Higman's group $H = \langle a,b,c,d \mid a^2 = a^b, b^2 = b^c, c^2 = c^d, d^2 = d^a \rangle$ is not locally indicable (indeed not locally p-indicable for any prime p) so H is not conservative, by Theorem 1. In particular, by the Remark in 2.3., there exists an H-covering $\tilde{K} \to K$ of 2-complexes with K almost-acyclic and $\tilde{K}$ not almost-acyclic. Indeed, the arguments of [17], Proposition 1.9. and of 2.2. enable us to construct an explicit example of such a covering.

Let K be a 2-complex with a single 0-cell; 8 1-cells s, t, u, v, w, x, y, z ; and 4 2-cells α, β, γ, δ, with attaching maps given by the words

$$\alpha \ : \ wsws^{-1}tw^{-1}t^{-1}z^{-1}v^2zv^{-2}$$

$$\beta \ : \ xtxt^{-1}ux^{-1}u^{-1}w^{-1}s^2ws^{-2}$$

$$\gamma \ : \ yuyu^{-1}vy^{-1}v^{-1}x^{-1}t^2xt^{-2}$$

$$\delta \ : \ zvzv^{-1}sz^{-1}s^{-1}y^{-1}u^2yu^{-2}$$

respectively.

Let $\tilde{K} \to K$ be the H-covering defined by the epimorphism $\pi_1 K \longrightarrow\!\!\!\!\!\!\!> H$ which maps s, t, u, v to a, b, c, d respectively, and each of w, x, y, z to 1.

Then $\tilde{K}$ has 0-cells p_h $(h \in H)$; 1-cells s_h, t_h, u_h, v_h, w_h, x_h, y_h, z_h $(h \in H)$; and 2-cells α_h, β_h, γ_h, δ_h $(h \in H)$. The 1-cells s_h, t_h, u_h, v_h join p_h to p_{ha}, p_{hb}, p_{hc}, p_{hd} respectively, while w_h, x_h, y_h, z_h are loops based at p_h. The 2-cells are attached along paths lifted from the attaching maps of α, β, γ, δ. For example

$$\alpha_h \ : \ w_h s_h w_{ha} s_h^{-1} t_h w_{hb}^{-1} t_h^{-1} z_h^{-1} v_h v_{hd} z_{hd^2} v_{hd}^{-1} v_h^{-1}$$

The action of H on $\tilde{K}$ is by left translation of the indices, thus: $g(p_h) = p_{gh}$ etc. The 2-complex K is clearly almost-acyclic, but $H_2(\tilde{K}) \cong \mathbb{Z}H$, generated by the 2-cycle $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 - \alpha_a - \beta_b - \gamma_c - \delta_d$.

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* Supported by an SERC Postdoctoral Fellowship.

(Received December 7, 1983)