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MINIMAL BOUNDARIES ENCLOSING A GIVEN VOLUME

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We prove some facts concerning surfaces of minimal area bounding regions of prescribed volume in $\mathbb{R}^n$. The main result we prove is that the mean curvature of such a surface is constant, if possibly a discontinuous function of the enclosed volume. The boundary behaviour of the solutions is also discussed.

Introduction

In this paper we study sets minimizing surface area with prescribed volume. This kind of problem naturally appears when considering (parametric) capillary surfaces in the absence of gravity (see [2], [3], [5], [6] for the non-parametric case). Soap bubbles constitute a typical example.

In Section 1 we recall some definitions and results concerning the existence and regularity of solutions. The constancy of the mean curvature of the free boundary of any such solution constitutes the main result of Section 2. We prove it by showing that otherwise it would be possible to decrease surface area by removing material from

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more curved regions to less ones. In Section 3 we discuss the boundary behaviour of the solutions; in particular, we show that any solution will meet tangentially the boundary of the given domain if it does not assume the prescribed boundary values. Finally, in Section 4 we give an example of non-uniqueness of the solutions to this problem. The same example shows that the mean curvature of the solution may be a discontinuous function of the enclosed volume.

1. Let Ω be an open bounded subset of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega$ and let Γ be a (Borel) subset of $\partial\Omega$. Our problem will then be:

(P)

$$\text{to minimize} \quad \mathcal{F}(E) = P_\Omega(E) + \int_{\partial\Omega} |\phi_E - \phi_\Gamma| \, dH_{n-1} ,$$

among the Borel subsets E of Ω satisfying

$$|E| = v, \quad v < |\Omega| .$$

Here ϕ_E denotes the characteristic function of the set E, $P_\Omega(E)$ is the perimeter of E in Ω , i.e.

$$P_\Omega(E) = \sup \{ \int_E \operatorname{div} g(x) dx, \quad g \in [C_0^1(\Omega)]^n, \ |g| \leq 1\}$$

As it is well-known, every set E with $P_\Omega(E) < +\infty$ has a trace integrable on $\partial\Omega$, which we still denote by ϕ_E (see [1], [4],).

Roughly speaking, we are looking for a solution set E taking on the prescribed boundary value Γ , such that the area of its free surface $\partial E \cap \Omega$ yields a mini -

mum when compared to the area of the free boundary of any admissible set, having the same volume in Ω and the same trace on $\partial\Omega$. In our formulation, the "Dirichlet condition" Γ is actually retained in the functional itself, as a "penalty term".

Alternatively, we might prescribe as "boundary condition" a set M of finite perimeter, thus looking for a solution set E, satisfying the volume constraint $|E| = v$ and minimizing the perimeter in the <u>closure</u> of Ω, among the sets which, outside Ω itself, do coincide with M (i.e. $E - \Omega = M - \Omega$).

Our starting point will be the following (known) result:

<u>THEOREM 1. For every $v < |\Omega|$ and $\Gamma \subset \partial\Omega$ there exists a solution E to problem (P); moreover, the boundary ∂E is analytic in Ω, except perhaps for a closed singular set whose Hausdorff dimension does not exceed $n - 8$.</u> (See [11], [12], [13], [15]).

2. Now assume that $x_o \in \partial E \cap \Omega$ is a <u>regular point</u> of the free boundary of a solution E to problem (P); that is, there exists a cylinder C centered at x_o, $C = B_\rho \times I$, with B_ρ an $n-1$ dimensional ball and I an open interval in $\mathbb{R}$, such that $E \cap C$ = epigraph of an analytic function $u : B_\rho \to I$. It is well known (see [7], [10]) that in this case there exists a <u>local</u> Lagrange multiplier λ, by means of which our <u>constrained</u> problem (P) can be <u>locally</u> converted into a free one; that is, u is a minimum of the functional

$$I(v) = \int_{B_\rho} \sqrt{1+|Dv|^2} + \lambda \int_{B_\rho} v$$

with $v \in BV(B_\rho)$ (the space of functions of bounded variation on B_ρ), $v = u$ on ∂B_ρ, graph $v \subset C$ (i.e. $|v-u|$ small).

It follows

$$\text{div } Tu = \lambda \quad \text{on } B_\rho \text{, where } Tu = \frac{Du}{\sqrt{1+|Du|^2}} \text{ ;}$$

that is, λ coincides with $n-1$ times the mean curvature of ∂E in C (measured with respect to the inner normal, so that convex sets have non negative boundary mean curvature).

The local character of λ can be easily understood with the aim of the following simple example: consider problem (P) with $\Omega = B_R$ and $\Gamma = \emptyset$; then, for each v, its solution will clearly be any ball $B_\rho \subset B_R$, with $\rho = (\omega_n^{-1} v)^{1/n}$.

On the other hand, the minimum of the functional

$$\mathcal{F}(E) + \lambda |E| ,$$

with $E \subset B_R$ uncostrained, is $E = \emptyset$ (corresponding to $\lambda \geqslant -\frac{n}{R}$) or $E = B_R$ (corresponding to $\lambda \leqslant -\frac{n}{R}$). There follows the non-existence of a "global multiplier" converting the constrained problem (P) into its usual free-associated.

Nevertheless, the local multiplier λ derived above does not actually depend on $\partial E \cap C$: the mean curvature of $\partial E \cap \Omega$ is the same at every regular point. This remarkable fact (which is rather obvious for connected regular parts of $\partial E \cap \Omega$) can be suggested by the following simple example: let Ω be the upper half-space in $\mathbb{R}^3$ and let Γ be the union of two disjoint disks contained in $\partial\Omega$. Then the solution E is the union of two pieces of a suitable ball in $\mathbb{R}^3$. In general we have the following

<u>THEOREM 2.</u> *Let E be a solution to problem (P), x_1 and x_2 regular points of $\partial E \cap \Omega$. Then*

$$\operatorname{div} Tu_1(x_1) = \operatorname{div} Tu_2(x_2),$$

where u_1, u_2 are functions describing ∂E near x_1 and x_2 , as seen in the preceding discussion.

The following results will be used in the proof of theorem 2; let

$$I_\lambda(v) = \int_{B_R} \sqrt{1 + |Dv|^2} + \lambda \int_{B_R} v + \int_{\partial B_R} |v - \phi|$$

with $B_R \subset \mathbb{R}^{n-1}$, $\phi \in C^\circ(\partial B_R)$, $\lambda \in \mathbb{R}$, $v \in BV(B_R)$. Then:

(α) if u minimizes I_λ , then $|\lambda| \leqslant \frac{n-1}{R}$;

(β) if $|\lambda| \leqslant \frac{n-2}{R}$, then there exists a unique function u minimizing I_λ , which moreover satisfies $u = \phi$ on ∂B_R.

Part (α) follows integrating by parts in the equation div Tu = λ; while (β) can be proved using e.g. the method in [9] .

<u>Proof of Theorem 2</u>

As x_1 and x_2 are regular points of ∂E in Ω , there will be two balls $B_1, B_2 \subset \mathbb{R}^{n-1}$ together with two open intervals $I_1, I_2 \subset \mathbb{R}$ such that

i) x_i is the center of the cylinder $C_i = B_i \times I_i$ and $\bar{C}_1 \cap \bar{C}_2 = \emptyset$.

ii) $E \cap C_i$ = epigraph of a suitable function $u_i \in C^2(\bar{B}_i, I_i)$

iii) u_i minimizes

$$I^{(i)}(v) = \int_{B_i} \sqrt{1 + |Dv|^2} + \lambda_i \int_{B_i} v + \int_{\partial B_i} |v - u_i| d H_{n-2}$$

with respect to any perturbation $v \in BV(B_i)$ whose graph is contained in C_i; where λ_i is a suitable Lagrange multiplier, as seen above. We have to show that $\lambda_1 = \lambda_2$. We can assume that the radius R_i of B_i is so small that

iv) $\left[(\min_{\bar{B}_i} u_i) - R_i \, , \, (\max_{\bar{B}_i} u_i) + R_i\right] \subset I_i$;

now suppose

v) $0 \leqslant \lambda_1 < \lambda_2$

and fix $\bar{R}_i$ in such a way that (see (α) above):

$$0 \leqslant \lambda_i \leqslant \frac{n-1}{R_i} < \frac{n-2}{\bar{R}_i} .$$

Now, if λ is chosen in the interval $\left(\lambda_1, \frac{n-2}{\bar{R}_1} \wedge \lambda_2 \right)$, then, using property (β), we can find an unique minimum $u_\rho^{(i)}$ of

$$(*^i) \qquad \int_{B_\rho^{(i)}} \sqrt{1+|Dv|^2} + \lambda \int_{B_\rho^{(i)}} v + \int_{\partial B_\rho^{(i)}} |v - u_i|$$

on $B_\rho^{(i)} \subset B_i$, for <u>any</u> $\rho \leqslant \bar{R}_i$; moreover, $u_\rho^{(i)} = u_i$ on $\partial B_\rho^{(i)}$.

Call

$$\tilde{u}_\rho^{(i)} = \begin{cases} u_\rho^{(i)} & \text{on } B_\rho^{(i)} \\ u_i & \text{on } B_i - B_\rho^{(i)} \end{cases}$$

Then, for every $\rho \leqslant \bar{R}_1$, we obtain from ([8], section 3) $\tilde{u}_\rho^{(1)} \leqslant u_1$ on B_1, while $\tilde{u}_\rho^{(2)} \geqslant u_2$ on B_2 if $\rho \leqslant \bar{R}_2$. Moreover, from iv) and property (α), it follows easily that graph $u_\rho^{(i)}$ is always contained in C_i.

Defining

$$v_\rho^{(i)} = \int_{B_\rho^{(i)}} |u_i - u_\rho^{(i)}|$$

we claim, that for suitable radii r_1, r_2 there holds

vi) $$v^{(1)}_{r_1} = v^{(2)}_{r_2} .$$

To see this, minimize $(*^1)$ with $\rho = \bar{R}_1$, thus getting $u^{(1)}_{\bar{R}_1}$ and the corresponding value $v^{(1)}_{\bar{R}_1} > 0$; then starting with $\rho = \bar{R}_2$, minimize $(*^2)$ with $\rho \leqslant \bar{R}_2$ decreasing to zero, thus getting $u^{(2)}_\rho$ and the corresponding values $v^{(2)}_\rho$; as $\rho \to 0$ implies $v^{(2)}_\rho \to 0$, there will be a value r_2, $0 < r_2 \leqslant \bar{R}_2$ such that $0 < v^{(2)}_{r_2} < v^{(1)}_{\bar{R}_1}$. Let's then come back to the solutions $u^{(1)}_\rho$ of $(*^1)$, with $\rho \leqslant \bar{R}_1$ decreasing to zero: $\rho \to 0$ implies again $v^{(1)}_\rho \to 0$ so that, in view of the continuity of $v^{(1)}_\rho$ with respect to ρ (which can be easily checked) there will be a value $r_1 < \bar{R}_1$ such that vi) holds true.
If we now set

$$E^* = \left(E \cup \left[\text{epi } u^{(1)}_{r_1} \cap C_1 \right]\right) \cap \left[\text{epi } u^{(2)}_{r_2} \cap C_2\right],$$

then vi) implies precisely $|E^*| = |E|$; clearly, $E \Delta E^* \subset\subset \Omega$, while

$$P_\Omega(E) - P_\Omega(E^*) = \int_{B^{(1)}_{r_1}} \sqrt{1 + |Du_1|^2} + \int_{B^{(2)}_{r_2}} \sqrt{1 + |Du_2|^2} +$$

$$- \left(\int_{B^{(1)}_{r_1}} \sqrt{1 + |Du^{(1)}_{r_1}|^2} + \int_{B^{(2)}_{r_2}} \sqrt{1 + |Du^{(2)}_{r_2}|^2} \right) +$$

$$+ \left(\lambda \int_{B^{(1)}_{r_1}} (u_1 - u^{(1)}_{r_1}) + \lambda \int_{B^{(2)}_{r_2}} (u_2 - u^{(2)}_{r_2}) \right)$$

(note that the last contribution vanishes, in view of vi)). In view of the strict minimality of $u^{(1)}_{r_1}$, $u^{(2)}_{r_2}$ (see $(*^{i})$) we get $P_{\Omega}(E) > P_{\Omega}(E^{*})$, which yields the expected contradiction to our assertion v).

A similar argument applies in the cases $\lambda_1 < \lambda_2 \leqslant 0$, $\lambda_1 < 0 < \lambda_2$.

3. Using a general regularity result for minimal boundaries in presence of smooth obstacles, we can show that if a solution E to problem (P) does not assume the prescribed boundary value Γ , then it must be tangent to $\partial\Omega$. Precisely:

<u>THEOREM 3.</u> *Let* $0 \in \partial\Omega$ *be a regular boundary point* (i.e. $\partial\Omega \cap B_r(0)$ is a C^1-manifold for a suitable $r > 0$). *If* $0 \notin \bar{\Gamma}$ *, then there exists* $0 < \rho < r$ *such that* $\partial E \cap B_\rho(0)$ *is also of class* C^1 *(possibly empty)*. *(Similarly, if* $0 \in \mathring{\Gamma}$ *, then* $\partial(\Omega - E) \cap B_\rho(0)$ *is of class* C^1).

Proof. We observe that $\tilde{E} = \mathbb{R}^n - E$ minimizes perimeter in $B_r(0)$, with a volume constraint and with respect to the obstacle $L = \mathbb{R}^n - \Omega$; that is to say, for every set F s.t. $F \Delta \tilde{E} \subset\subset B_r$, $F \supset L$ and $|F \cap B_r| = |\tilde{E} \cap B_r|$, there holds: $P_{B_r}(\tilde{E}) \leqslant P_{B_r}(F)$. The result is then a consequence of Theorem 2 of [16], which works (essentially with the same proof) in the presence of a volume constraint as well.

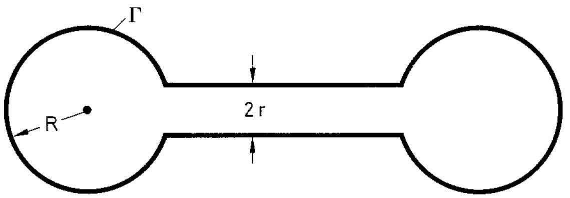
REMARK. Consider problem (P) with $\Omega = \{(y,t) : y \in A,\ t > 0\}$ and $\Gamma = \{(y,t) :$ either $y \in \partial A$ and $0 \leqslant t \leqslant \phi(y)$ or $y \in A$ and $t = 0\}$ (non-parametric case), where A is a domain in $\mathbb{R}^{n-1}$ with smooth boundary and ϕ is a continuous positive function on ∂A. If the mean curvature of ∂A is strictly positive, then the (unique) solution $\tilde{E}$ of :

$$P_\Omega(E) + \int_{\partial\Omega} |\phi_E - \phi_\Gamma| \, d\, H_{n-1} \to \min$$

(the "uncostrained solution") is known to satisfy the boundary condition, i.e. $\phi_{\tilde{E}} = \phi_\Gamma$ on $\partial\Omega$. Moreover, using the non-parametric theory (see e.g. [9]) one can derive the existence of $\varepsilon > 0$ (depending on the data ∂A and ϕ) s.t. if E_v denotes the solution to (P) corresponding to the constraint $|E| = v$, then $\left| v - |\tilde{E}| \right| < \varepsilon$ implies $\phi_{E_v} = \phi_\Gamma$ on $\partial\Omega$.

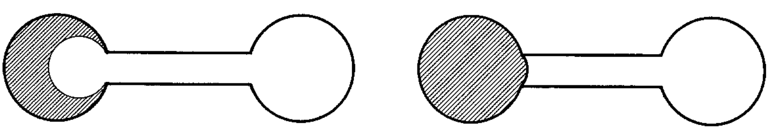
4. We can expect neither uniqueness of solutions to problem (P) nor continuous dependence of the mean curvature of $\partial E \cap \Omega$ as a function of the volume v. Consider the following

Example. Let the domain $\Omega \subset \mathbb{R}^n$ be constituted by two balls of the same radius R connected by a narrow cylindrical pipe, r being the radius of the circular section of the pipe and $2R$ its length. Let Γ be the surface of the left ball (figure A).



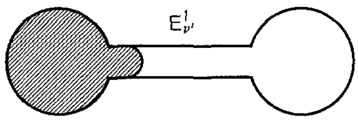
a

As the volume v increases starting from $v=0$, the solution E_v (whose free-boundary is of course a spherical cap) fills the left ball, partially at first and then completely; then it begins to enter the pipe (figures B and C).

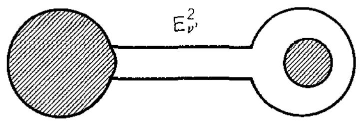


b c

However, it r is small enough, then the solution E_v cannot continue to expand in the pipe while remaining connected. Precisely, there will be a first value $v = v'$ admitting two different solutions with different mean curvature (figures D and E).

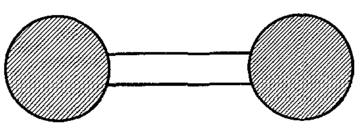


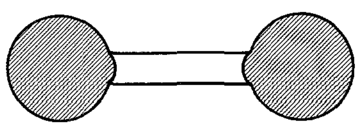
d



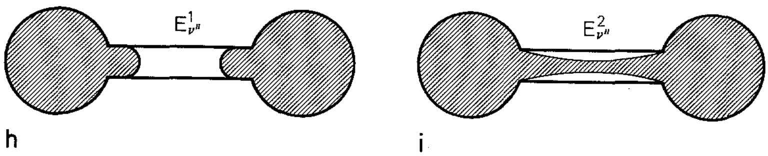
e

A further increasing of the volume will give rise to a contraction toward the left ball together with an expansion in the right ball of the disconnected solution $E^2_{v'}$, until it coincides exactly with the union of the two balls (figures F).


f

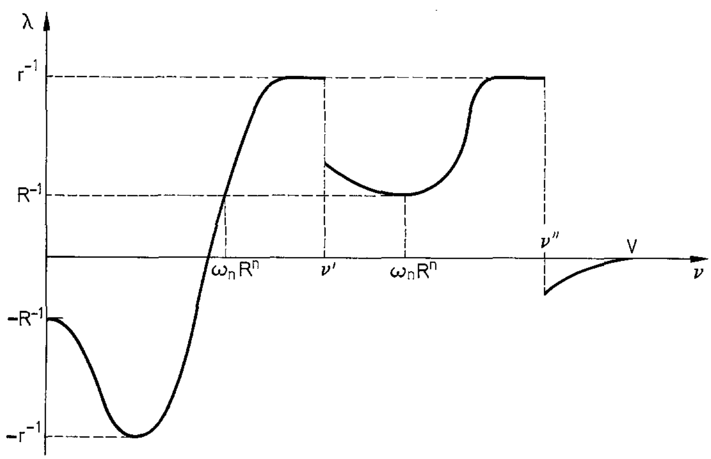

g

Then the solution enters again the pipe symmetrically from the opposite sides (figure G), until a new discontinuity appears, corresponding to a second value $v = v''$ (figures H and I).



The connected solution $E^2_{v''}$ will then expand and eventually fill Ω .

When $n = 2$, the "mean curvature versus volume" diagram results as follows:



REFERENCES

[1] ANZELLOTTI, G., GIAQUINTA, M., MASSARI, U., MODICA, G., PEPE, L.: Note sul problema di Plateau. Pisa: Editrice Tecnico Scientifica 1974

[2] CHEN, J.T.: On the existence of capillary free surfaces in the absence of gravity. Doctoral dissertation. Stanford University (1979)

[3] CONCUS, P., FINN, R.: On capillary free surfaces in the absence of gravity. Acta Math. 132, 177-198 (1974)

[4] DE GIORGI, E., COLOMBINI, F., PICCININI, L.: Frontiere orientate di misura minima e questione collegate. Pisa: Editrice Tecnico Scientifica 1972

[5] FINN, R.: Existence and non existence of capillary surfaces. Manuscripta Math. 28, 1-11 (1979)

[6] FINN, R., GIUSTI, E.: Non existence and existence of capillary surfaces. Manuscripta Math. 28, 13-20 (1979)

[7] GERHARDT, C.: On the capillarity problem with constant volume. Ann. Scuola Norm. Sup. Pisa (4) 2, 303-320 (1975)

[8] GIAQUINTA, M.: Regolarità delle superfici BV(Ω) con curvatura media assegnata. Boll. U.M.I. 8, 567-578 (1973)

[9] GIAQUINTA, M.: On the Dirichlet problem for surfaces of prescribed mean curvature. Manuscripta Math. 12, 73-86 (1974)

[10] GIUSTI, E.: Generalized solutions for the mean curvature equation. To appear

[11] GIUSTI, E.: The equilibrium configuration of liquid drops. To appear in J. für Reine und Angew. Math.

[12] GONZALEZ, E., MASSARI, U., TAMANINI, I.: On the regularity of boundaries of sets minimizing perimeter with a volume constraint. To appear

[13] MASSARI, U.: Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in $\mathbb{R}^n$. Arch. Rat. Mech. Anal. 55, 357-382 (1974)

[14] MASSARI, U., PEPE, L.: Successioni convergenti di ipersuperfici di curvatura media assegnata. Rend. Sem. Mat. Univ. Padova 53, 53-68 (1975)

[15] MIRANDA, M.: Existence and regularity of hypersurfaces of $\mathbb{R}^n$ with prescribed mean curvature. Proceedings of Symposia in Pure Mathematics, 23 (1973)

[16] MIRANDA, M.: Frontiere minimali con ostacoli. Ann. Univ. Ferrara 16, 29-37 (1971)

[17] SERRIN, J.: The problem of Dirichlet for quasilinear elliptic equations with many independent variables. Phil. Trans. Royal Soc. London 264, 413-496 (1969)

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