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Local Time-Decay of High Energy Scattering States for the Schrödinger Equation

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1. Introduction and Results

Consider the Schrödinger operator $H:=H_0+V_S+V_L$ in the Hilbert space $\mathscr{H}:=L^2(\mathbb{R}^n)$, $n\in\mathbb{N}$ where $H_0:=\overline{(-\Delta)\upharpoonright C_0^\infty(\mathbb{R}^n)}$ and V_S and V_L are real-valued functions, satisfying the following assumptions:

Let $N\in\mathbb{N}$, then

$$\text{(S)}\quad \begin{cases}\langle x\rangle^N V_S(H_0+1)^{-\delta} \text{ is bounded for some } \delta\in[0,\tfrac{1}{2})\\ \langle x\rangle^{2N} V_S(x)=o(|x|), \quad |x|\to\infty\end{cases}$$

and

$$\text{(L)}\quad \begin{cases} V_L\in C^{\left(N+2+\left[\frac{n}{2}\right]\right)}(\mathbb{R}^n) \text{ such that}\\ |D^\alpha V(x)|\leqq c_\alpha \langle x\rangle^{-|\alpha|-\varepsilon} \quad \text{for } \varepsilon>0,\\ \text{suitable } c_\alpha \text{ and any multiindex } \alpha \text{ with } |\alpha|\leqq N+1.\end{cases}$$

Note that above we used the notation $\langle x\rangle$ for the Sobolev weight $(1+|x|^2)^{\frac{1}{2}}$ and for the associated self-adjoint multiplication operator. (We will use it in this way throughout the paper).

Note also, that H is a self-adjoint operator in $\mathscr{H}$ with domain the Sobolev space $D(H)=\mathscr{H}_2(\mathbb{R}^n)$.

Let $\chi\in C^\infty(\mathbb{R})$ with $\chi(\lambda)=0$ if $\lambda\leqq\lambda_0$ and $\chi(\lambda)=1$ if $\lambda\geqq 2\lambda_0$ for $\lambda_0>0$ suitably large. We are interested in the time evolution

$$\phi(t):=e^{-itH}\chi(H)\phi \quad \text{for } t\in\mathbb{R},\ \phi\in\mathscr{H};$$

which is the (Hilbert space)-solution of the Schrödinger equation

$$-i\frac{d}{dt}\phi(t)=H\phi(t), \quad t\in\mathbb{R}, \quad \phi(0)=\chi(H)\phi\in D(H).$$

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By the Kato-Agmon-Simon Theorem (cf. [19], Theorem XIII.58), H has no positive eigenvalues, and it is also known (cf. Cycon [5], Mourre [14] or Perry-Sigal-Simon [17]) that H has no singular continuous spectrum. Hence, for any $\lambda_0>0$ and any $\phi\in L^2(\mathbb{R}^n)$ $\chi(H)\phi$ lies in the absolutely continuous spectral subspace for H.

Hence $\chi(H)\phi$ should be a scattering state, and one expects that it leaves every bounded region of $\mathbb{R}^n$ eventually for $t\to\pm\infty$ (see [2]).

This can be quantitatevely expressed as the time-decay of the "localized" time evolution i.e. for $s>0$ there exists a decay rate $\alpha>0$ such that

$$\|\langle x\rangle^{-S}e^{-itH}\chi(H)\phi\|\leqq C(1+|t|)^{-\alpha}\|\langle x\rangle^{s}\phi\|,\qquad t\in\mathbb{R}. \tag{1.1}$$

Intuitively, it is clear that the decay-rate α depends on the "smoothness" of the potential (besides depending on the "localization weight" $\langle x\rangle^{-s}$).

The aim of this paper is to prove a more precise quantitative statement of this kind, i.e. we will show

Theorem 1. *Let $N\in\mathbb{N}$, $N\geqq 3$ and $H:=H_0+V_S+V_L$, where V_S and V_L satisfy the assumption (S) and (L) respectively. Let $\chi\in C^\infty(\mathbb{R})$ as above for suitably large $\lambda_0>0$. Then*

$$\|\langle x\rangle^{-s}e^{-itH}\chi(H)\langle x\rangle^{-s}\|\leqq C(1+|t|)^{-s+\frac{s}{N}} \tag{1.2}$$

for any $s\in[0,N]$, $t\in\mathbb{R}$ and a suitable constant $C>0$.

Remarks. 1. While the operator $\langle x\rangle^{-s}$ on the L.H.S. of e^{-itH} in (1.2) can be understood as a "localization weight", the operator $\langle x\rangle^{-s}$ on the other side has for its range those states which are concentrated near the center at time zero.

2. We make the assumption that λ_0 is large for technical convenience. Actually by combining Theorem 1 with results of Jensen et al. [9] we can treat the case of $\chi\in C^\infty(\mathbb{R})$, $\operatorname{supp}\chi\leqq(\lambda_0,\infty)$ for any $\lambda_0>0$.

3. Theorem 1 implies that if one increases the smoothness of the potential (i.e. if one increases N), then the time-decay can be more and more increased (by increasing s) and one approaches more and more the best possible result of this type, i.e.

$$\|\langle x\rangle^{-s}e^{-itH}\langle x\rangle^{-s}\|\leqq C(1+|t|)^{-s}$$

which is known for the free Hamiltonian H_0.

Local time-decay of scattering states for Schrödinger operators with short-range potentials has been discussed by many authors, see for example [18, 7, 8, 15]. The long-range case seems to have not been considered until recently (see however [4]). There is also a paper by Kitada [11] which contains a result very similar to our Theorem 1 above. In fact our paper was stimulated by [11] but has a more general result than [11]. However, while Kitada employs quite an involved Fourier integral operator machinery, we use a completely different method.

The basic idea in our paper consists in proving some estimates for certain powers of the resolvent of H near the real axis and then transforming them into estimates for e^{-itH} by Cauchy's integral formula.

We obtain these estimates by following a strategy which goes back to Mourre [13, 14], i.e., one rotates the essential spectrum of H (and thus also the essential spectrum of the resolvent) away from the real axis by a "complex dilation" in order to find a differential inequality for certain powers of the "dilated" resolvent in suitably weighted spaces. This differential inequality can be solved by an iteration procedure which leads to the desired estimates.

We note that, besides using a finite order approximation of the usual dilation family $\theta \to e^{-\theta A} H e^{\theta A}$, we use the modified generator

$$A := \tfrac{1}{2}(x \cdot \tilde{G}(p) + \tilde{G}(p) \cdot x)$$

instead of the usual dilation generator $\frac{1}{2}(x \cdot p + p \cdot x)$, where $\tilde{G}(p)$ is like p near zero and $|\tilde{G}(p)| = 1$ for $|p| \geqq \tilde{\lambda}_0$ for suitable $\tilde{\lambda}_0 > 0$. This modified A has the advantage that it is bounded in the p-variable. This will be used explicitly in the derivation of the differential inequality. The "approximate dilated" Hamiltonian we use in this paper (compare [9] for a similar technique) is defined as

$$H(\theta) := H_L + \sum_{k=1}^{N} \frac{(-i\theta)^k}{k!} [H_L, iA]^{(k)}$$

where θ is real, $H_L := H_0 + V_L$ and $[H_L, iA]^{(k)}$ is the closure of the (k)-th commutator $\underbrace{[\ldots[[H, iA], iA], \ldots iA]}_{(k\text{-times})}$ (initially defined as forms on $S(\mathbb{R}^n) \times S(\mathbb{R}^n)$, the C^∞-functions of rapid decrease).

We should mention that $H(\theta)$ has as essential spectrum a line which starts at 0 with an angle -2θ (like the usual dilated Hamiltonian [19]) and ends up to be the lower part of a paraboly in the complex plane (see Fig. 1)

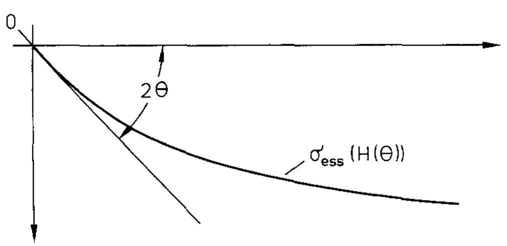

Fig. 1

This shape of $\sigma_{\text{ess}}(H(\theta))$ should be kept in mind if one looks at the (crucial) a priori estimate in Lemma 2.3.

The paper is organized as follows. In Sect. 2 we derive the basic estimates which lead to the main estimates for certain powers of the resolvent for the long-range case. Sect. 3 contains a short-range perturbation of the main estimates and in Sect. 4 we prove Theorem 1. The appendix in Sect. 5 is a collection of results for pseudo-differential operators which we add for the convenience of the reader.

One of us (H.C.) would like to thank R. Wüst and R. Seiler for many enlightening discussions.

We should note that after finishing the manuscript we learned that I.M. Sigal has used a very similar modified dilatation generator for to discuss resonances [21].

2. Some Estimates for the Long-Range Hamiltonian

We first define the "modified" dilation generator mentioned in the introduction. Let $\tilde{\lambda}_0 > 0$ and define

$$g(t) := c \int_0^t \tilde{g}(s)\,ds, \qquad t \geqq 0 \tag{2.1}$$

where

$$\tilde{g}(s) := \begin{cases} \exp\{-(s-\tilde{\lambda}_0)^{-2} - s^{-2}\} & \text{for } s \leqq \tilde{\lambda}_0 \\ 0 & \text{for } s > \tilde{\lambda}_0 \end{cases}$$

and

$$c := \left[\int_0^{\tilde{\lambda}_0} \tilde{g}(s)\,ds\right]^{-1}.$$

$g(t)$ is a C^∞-function with a zero of infinite order at $t=0$, $g(t)=1$ for $t \geqq \tilde{\lambda}_0$ and $g'(t)$ has a zero of infinite order at $t=\tilde{\lambda}_0$. If we now set

$$\tilde{G}(\xi) := g(|\xi|)(\xi|\xi|^{-1}), \qquad \xi \in \mathbb{R}^n$$

we can define a symmetric operator A on $S(\mathbb{R}^n)$ by

$$A := \tfrac{1}{2}(x \cdot \tilde{G}(P) + \tilde{G}(p) \cdot x), \qquad p := -i\nabla_x. \tag{2.2}$$

The construction of $\tilde{G}$ guarantees that A is a pseudodifferential operator with $A \in OPS(0, \{1-i\}_{i \in \mathbb{N}})$.

Moreover by the commutator Theorem of [20, Theorem X.36] with $N = x^2 + 1$, A is essentially self-adjoint on $S(\mathbb{R}^n)$ and has a unique self-adjoint realization on $L^2(\mathbb{R}^n)$ which we also denote by A.

Now let V_L satisfy the hypotheses of Theorem 1 and let $H_L := H_0 + V_L$.

For $k \in \mathbb{N}$, denote by B_1 the commutator

$$[H_L, iA] \quad \text{on } S(\mathbb{R}^n) \times S(\mathbb{R}^n),$$

and by B_k the commutator $[B_{k-1}, iA]$. The B_k extend to bounded operators from $D(H_0)$ to $L^2(\mathbb{R}^n)$ for $1 \leqq k \leqq N+1$ by the hypotheses on V and the explicit calculations we make below. We also use the notation $[H_0, iA]^{(k)}$ and $[V_L, iA]^{(k)}$ to denote the k-fold commutator of iA with H_0 and V_L respectively. For $N \in \mathbb{N}$, $N \geqq 2$, we define the "dilated" Hamiltonian and its resolvent by

$$\begin{aligned} H(\theta) &:= H_L + \sum_{k=1}^{N} \frac{(-i\theta)^k}{k!} B_k, \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ G(\theta, z) &:= (H(\theta) - z)^{-1}, \end{aligned} \tag{2.3}$$

(the estimates we make below show that $G(\theta, z)$ exists and is analytic in θ for the z we want to discuss).

Remark 2.1. An easy calculation shows that

$$[H_0, iA]^{(1)} = 2H_0^{\frac{1}{2}} g(H_0^{\frac{1}{2}}) =: f(H_0) \tag{2.4}$$

and all higher commutators are bounded functions of H_0. Using Corollary A_2 and A_4 in the appendix one also sees that

$$[V_L, iA]^{(k)} \in OPS(-k, \{-\varepsilon - i\}_{i=1}^k)$$

and are therefore compact operators. Thus, $H(\theta)$ is an analytic family of type (A) in the sense of Kato [10].

Now we prove a technical lemma which will be useful in the following. The proof uses an idea due to Enss [6].

Lemma 2.2. *Let g be as in equation* (2.1) *and*

$$f(\lambda) := \begin{cases} 2\lambda^{\frac{1}{2}} g(\lambda^{\frac{1}{2}}) & \text{for } \lambda \geqq 0 \\ -2|\lambda|^{\frac{1}{2}} g(|\lambda|^{\frac{1}{2}}) & \text{for } \lambda < 0. \end{cases} \tag{2.5}$$

Then $f(H_L) - f(H_0)$ is a bounded operator.

Proof. Let $h(\lambda) := \lambda^{-1} f(\lambda)$. Then $h \in C^\infty(\mathbb{R})$, and vanishes at ∞ like $O(\lambda^{-\frac{1}{2}})$. We remark that some of the following calculations are formal only. They can be rigorously justified however by assuming that $h \in C_0^\infty$ and then by an approximation argument. We have

$$\begin{aligned} f(H_L) - f(H_0) &= H_L h(H_L) - H_0 h(H_0) \\ &= H_0\{h(H_L) - h(H_0)\} + V_L h(H_L). \end{aligned}$$

The last term in the second equality is bounded.

To show that the first term is also bounded we expand it by Duhamel's formula:

$$H_0\{h(H_L) - h(H_0)\} = H_0 \int ds\, e^{iH_0 s} s \cdot \hat{h}(s) \cdot \frac{1}{s} \{e^{-iH_0 s} e^{iH_L s} - 1\}$$

$$= H_0(-i) \int ds\, e^{iH_0 s} \hat{h}'(s) V_L \tag{2.6}$$

$$H_0(-i) \int ds\, e^{iH_0 s} \hat{h}'(s) \frac{1}{s} \int_0^s ds_1 (e^{-iH_0 s_1} V_L e^{iH_L s_1} - V_L). \tag{2.7}$$

The term (2.6) is equal to

$$H_0 h'(H_0) V_L$$

and therefore bounded. Thus we have to show that (2.7) is bounded. Using the identity

$$H_0 e^{iH_0 s} = -i \frac{d}{ds} e^{iH_0 s} \tag{2.8}$$

we can write (2.7) after a partial integration as

$$\int ds\, e^{iH_0 s} \left(i \frac{d}{ds}\right) \left\{\hat{h}'(s) \frac{1}{s} \int_0^s ds_1 \int_0^{s_1} ds_2\, e^{-iH_0 s_2} (V_L H_L - H_0 V_L) e^{iH_L s_2}\right\}. \tag{2.9}$$

Since the term in ( )-brackets is not bounded, we insert $(H_0+1)(H_0+1)^{-1}$ into (2.9), then use (2.8) and partial integration again.
Then we get for (2.9)

$$\int ds\, e^{iH_0 s}\left(1+\frac{d}{ds}\right)\left(i\frac{d}{ds}\right)\left\{\hat{h}'(s)\frac{1}{s}\int_0^s ds_1\int_0^{s_1} ds_2\, e^{-iH_0 s_2}B_1 e^{iH_L s_2}\right\} \tag{2.10}$$

where

$$B_1:=(H_0+1)^{-1}(V_L H_L - H_0 V_L)$$

is a bounded operator.

If we denote $h_1:=\hat{h}'$ and $f_1(s):=\frac{1}{s}\int_0^s ds_1\int_0^{s_1} ds_2\, e^{-iH_0 s_2}B_1 e^{iH_L s_2}$ we have for the integrand of (2.10)

$$\left(1+i\frac{d}{ds}\right)\left(i\frac{d}{ds}\right)(h_1(s)\,f_1(s))=i(h_1' f_1' + h_1 f_1') - (h_1'' f_1 + 2h_1 f_1 + h_1 f_1'').$$

Thus (2.10) consist of 5 integrals.

Since sh_1' and h_1 are in $L^1(\mathbb{R})$ and $\frac{1}{s}f_1(s)$ and $f_1'(s)$ are bounded functions, it is clear that

$$\left\|\int ds\, e^{iH_0 s} i(h_1' f_1 + h_1 f_1')\right\| \leqq c. \tag{2.11}$$

Now consider the third integral in (2.10)

$$\int ds\, e^{iH_0 s} h_1'' f_1$$

$$=\int ds\, e^{iH_0 s} s\left(\frac{d^2}{ds^2}\hat{h}'(s)\right)\frac{1}{s^2}\int_0^s ds_1\int_0^{s_1} ds_2\, e^{-iH_0 s_2}B_1 e^{iH_L s_2} \tag{2.12}$$

$$=\int ds\, e^{iH_0 s} s^2\left(\frac{d^2}{ds^2}\hat{h}'(s)\right)\frac{1}{s^3}\int_0^s ds_1\int_0^{s_1} ds_2\int_0^{s_2} ds_3\, e^{-iH_0 s_3}B_2 e^{iH_L s_3} \tag{2.13}$$

$$+\int ds\, e^{iH_0 s} s\left(\frac{d^2}{ds^2}\hat{h}'(s)\right)B_1 \tag{2.14}$$

where

$$B_2:=B_1 H_L - H_0 B_1$$

is also a bounded operator.

Thus $\frac{1}{s^3}\int_0^s ds_1\int_0^{s_1} ds_2\int_0^{s_2} ds_3\, e^{-iH_0 s_3}B_2 e^{iH_L s_3}$ is bounded and since $s\to s^2\left(\frac{d^2}{ds^2}\hat{h}'(s)\right)$ is in $L^1(\mathbb{R})$, (2.13) is bounded. Note that

$$s\frac{d^2}{ds^2}\hat{h}'(s)=\left(\frac{d}{dt}t^2\frac{d}{dt}h(\cdot)\right)^{\wedge}(s)=(2th'+h^2h'')^{\wedge}(s)$$

and therefore we have

$$\int ds\, e^{iH_0 s} s\left(\frac{d^2}{ds^2}\hat{h}'(s)\right)=2H_0 h'(H_0)+H_0^2 h''(H_0)$$

which is also bounded.

Therefore (2.14) and thus (2.12) is bounded.

By similar arguments one sees that the last two integrals in (2.10)

$$\int ds\, e^{iH_0 s}(2h_1' f_1' + h_1 f_1'')$$

are also bounded. Thus we can conclude that (2.7) is a bounded operator and this proves the lemma. □

Now we can prove some estimates for $G(\theta, z)$.

The first is an a priori estimate.

Lemma 2.3. *Let $z := \lambda + i\eta$; $\eta, \lambda > 0$; $\theta \in (0,1)$; $\delta \in [0, \frac{1}{2})$ and $G(\theta, z)$ as in* (2.3). *Then there exist $\lambda_1 > 0$ and $c > 0$ such that*

$$\|(H_0+1)^\delta G(\theta, z)\| \leqq c\theta^{-1}\lambda^{-(\frac{1}{2}-\delta)} \quad \text{for } \lambda \geqq \lambda_1. \tag{2.15}$$

Proof. Let $\tilde{\lambda}_0 > 0$, g and f as in (2.1) and Lemma 2.2 and denote

$$\tilde{H}(\theta) := H_L - i\theta f(H_L).$$

Let $\tilde{G}(\theta, z) := (\tilde{H}(\theta) - z)^{-1}$,

$$X(\theta) := H(\theta) - \tilde{H}(\theta)$$

and suppose that we can prove that for a suitable $\lambda_1 > 0$

$$\|(H_0+1)^\delta \tilde{G}(\theta, z)\| \leqq c\theta^{-1}\lambda^{-(\frac{1}{2}-\delta)} \tag{2.16}$$

$$\|X(\theta)\tilde{G}(\theta, z)\| \leqq \tfrac{1}{2} \tag{2.17}$$

for $\lambda \geqq \lambda_1$, $\theta \in (0,1)$, $\delta \in (0, \frac{1}{2})$.

Then by the identity

$$G(\theta, z) = \tilde{G}(\theta, z)(1 - X(\theta)\tilde{G}(\theta, z))^{-1}$$

the required estimate holds. Thus we need only check (2.16) and (2.17). To check (2.16) we observe that, choosing $c > 0$ so that $H_L + c > 0$ we need only bound $\|(H_L+c)^\delta \tilde{G}(\theta, z)\|$ since $(H_0+1)^\delta (H_L+c)^{-\delta}$ is a bounded operator for $\delta \in [0,1]$ (with bound uniform in δ). But writing $z = \lambda + i\eta$ we have by the functional calculus

$$\begin{aligned} &\|(H_L+c)^\delta(\tilde{H}(\theta)-z)^{-1} \\ &\quad = \sup_{x\in\sigma(H_L)} (x+c)^\delta |(x-\lambda) + i\theta f(x) + i\eta|^{-1} \\ &\quad \leqq \sup_{x\in\mathbb{R}} (x+c)^\delta [(x-\lambda)^2 + \theta^2 f^2(x)]^{-\frac{1}{2}}. \end{aligned} \tag{2.18}$$

Now choose $\lambda_1' \geqq 2\tilde{\lambda}_0$. For to estimate

$$F(x, \lambda) := (x+c)^\delta [(x-\lambda)^2 + \theta^2 f^2(x)]^{-\frac{1}{2}}, \quad \text{for } \lambda \geqq \lambda_1,\ x \in \mathbb{R}$$

we consider 3 cases.

Case 1. Let $x\in\left[\frac{\lambda}{2},\frac{3}{2}\lambda\right]$ and $\lambda\geqq\lambda_1'$.

Then, since $f(x)=x^{\frac{1}{2}}$ for $x\geqq\tilde{\lambda}_0$, we have

$$F(x,\lambda)\leqq\left(\frac{3}{2}\lambda+c\right)^{\delta}\left[\theta^2\frac{\lambda}{2}\right]^{-\frac{1}{2}}\leqq c_1\theta^{-1}\lambda^{\delta-\frac{1}{2}} \tag{2.19}$$

for $c_1>0$ suitably.

Case 2. Let $x\geqq\frac{3}{2}\lambda$. Then we have

$$F(x,\lambda)\leqq(x+c)^{\delta}(x-\lambda)^{-1}\leqq c_2\lambda^{\delta-1} \tag{2.20}$$

for $c_2>0$ suitably.

Case 3. Let $x\leqq\frac{1}{2}\lambda$. Then we have

$$F(x,\lambda)\leqq\left(\frac{\lambda}{2}+c\right)^{\delta}\left[\frac{\lambda}{2}\right]^{-1}\leqq c_3\lambda^{\delta-1} \tag{2.21}$$

for $c_3>0$ suitably.

From (2.18)-(2.21) we conclude that

$$\|(H_L+c)\tilde{G}(\theta,z)\|\leqq\max\{c_1\theta^{-1}\lambda^{\delta-\frac{1}{2}},(c_2+c_3)\lambda^{\delta-1}\}$$

which gives (2.16) for suitably large λ_1.

To check (2.17), we note that

$$X(\theta)=i\theta[f(H_L)-f(H_0)]-\theta Y(\theta) \tag{2.22}$$

where

$$Y(\theta):=\sum_{k=1}^{N}i\frac{(-i\theta)^{k-1}}{k}[V_L,iA]^{(k)}$$
$$+\sum_{k=2}^{N}i\frac{(-i\theta)^{k-1}}{k!}[H_0,iA]^{(k)} \tag{2.23}$$

is bounded uniformly in $\theta\in\left(0,\frac{\pi}{2}\right)$ by Remark 2.1. Since also $f(H_L)-f(H_0)$ is bounded by Lemma 2.2, we have, using (2.16), that

$$\|X(\theta)(\tilde{H}(\theta)-z)^{-1}\|\leqq c\lambda^{-\frac{1}{2}}\leqq\tfrac{1}{2} \tag{2.24}$$

for $\lambda\geqq\lambda_2'$ and some suitable $\lambda_2'>0$.

Choosing $\lambda_1:=\max\{\lambda_1',\lambda_2'\}$, we get (2.17), and hence the desired bound. □

The next lemma is closely related to the so-called Mourre-type estimate [14, 17].

Lemma 2.4. *Let $\theta\in(0,1)$, $z:=\lambda+i\eta$ and $H(\theta)$ as in* (2.3). *Then there exist $\rho_0>0$ and $c>0$ such that*

$$\frac{1}{\theta}E_\rho\frac{i}{2}\{(H(\theta)-z)-(H(\theta)-z)^*\}E_\rho\geqq cE_\rho \qquad (\text{for } \rho\geqq\rho_0) \tag{2.25}$$

holds, in the sense of quadratic forms, where $E_\rho := E^{H_L}_{(\rho,\infty)}$ *is the spectral projector of* H_L *on* (ρ, ∞).

Proof. Note that for g, f as in Lemma 2.2, we have (2.4). Then

$$\frac{i}{2}\{(H(\theta)-z)-(H(\theta)-z)^*\} = \theta f(H_0) + \theta[V_L, iA]^{(1)} + \sum_{\substack{3\leqq k\leqq N\\(k:\text{odd})}} \frac{(-i\theta)^k}{k!}[H_L, iA]^{(k)} + \eta \geqq \theta\{f(H_L)+B(\theta)\}$$

where

$$B(\theta) := (f(H_0)-f(H_L)) + [V_L, iA]^{(1)} + \sum_{\substack{3\leqq k\leqq N\\ k:\text{odd})}} i\frac{(-i\theta)^k}{k!}[H_L, iA]^{(k)}.$$

Note, that from Lemma 2.2 and Remark 2.1 we know, that $B(\theta)$ is a bounded operator (uniformly in $\theta\in[0,1]$). Thus we can estimate (as quadratic forms), if we choose $\rho_0 > 0$ suitably large

$$\begin{aligned} E_\rho \frac{i}{2}\{(H(\theta)-z)-(H(\theta)-z)^*\}E_\rho &\geqq \theta E_\rho\{f(H_L)-c_1\}E_\rho \\ &\geqq \theta E_\rho H_L^{\frac{1}{2}}\{g(H_L)-\rho^{-\frac{1}{2}}c_1\}E_\rho \\ &\geqq \theta E_\rho H_L^{\frac{1}{2}} c_2 E_\rho, \quad \text{for } \rho\geqq\rho_0 \text{ and suitable } c_1, c_2 > 0. \end{aligned}$$

Thus, since $E_\rho H_L^{\frac{1}{2}} \geqq \rho^{\frac{1}{2}} E_\rho$ we have

$$E_\rho \frac{i}{2}\{(H(\theta)-z)-\{H(\theta)-z)^*\}E_\rho \geqq \theta c E_\rho, \quad \text{for } c := c_2\rho_0^{\frac{1}{2}}$$

and this is the desired estimate (2.25). □

Using (2.25) we can show a resolvent estimate which is closely related to Mourre's "quadratic estimate" [14].

Lemma 2.5. *Let* $z := \lambda + i\eta$ *and* $H(\theta)$, $G(\theta, z)$ *as in* (2.3). *Then there exist* $\lambda_2 > 0$, $c > 0$ *and* $\theta_0 > 0$ *such that*

$$\|G(\theta,z)\langle x\rangle^{-1}\| \leqq c\theta^{-\frac{1}{2}}\{\|\langle x\rangle^{-1}G(\theta,z)\langle x\rangle^{-1}\|^{\frac{1}{2}} - (\lambda-\lambda_2)^{-\frac{1}{2}}\} \tag{2.26}$$

for $\lambda > \lambda_2$ *and* $\theta\in(0,\theta_0)$.

Proof. Let $\phi\in\mathscr{H}$, $\rho > 0$, $E_\rho := E^{H_L}_{(\rho,\infty)}$ as in Lemma 2.4. Then with $G := G(\theta, z)$

$$\begin{aligned} \|G\langle x\rangle^{-1}\phi\|^2 &= (\langle x\rangle^{-1}\phi, G^*G\langle x\rangle^{-1}\phi) \\ &= (\langle x\rangle^{-1}\phi, G^*E_\rho G\langle x\rangle^{-1}\phi) + (\langle x\rangle^{-1}\phi, G^*(1-E_\rho)G\langle x\rangle^{-1}\phi). \end{aligned} \tag{2.27}$$

Note, that for $\lambda > \lambda_2' := \rho$ we have for a suitable $c > 0$ and f as in Lemma 2.2

$$\|G^*(1-E_\rho)G\| \leqq \|(1-E_\rho)G\|^2 \leqq c(\lambda-\lambda_2')^{-2}, \tag{2.28}$$

since, by (2.17),

$$\begin{aligned}\|(1-E_\rho)G\| &= \|(1-E_\rho)(\tilde{H}(\theta)-z)^{-1}[1+X(\theta)(\tilde{H}(\theta)-z)^{-1}]^{-1}\| \\ &\leqq c\,\|(1-E_\rho)(H_L-i\theta f(H_L)-z)^{-1}\| \\ &\leqq c \sup_{x\leqq\rho} |x-i\theta f(x)-\lambda-i\eta|^{-1} \\ &\leqq c|\rho-\lambda|^{-1} \leqq c(\lambda-\lambda_2')^{-1}. \end{aligned} \tag{2.29}$$

Furthermore, we have by Lemma 2.4,

$$\begin{aligned}(\langle x\rangle^{-1}\phi, G^* E_\rho G\langle x\rangle^{-1}\phi) &\leqq \frac{c}{\theta}(\langle x\rangle^{-1}\phi, G^*\{(H(\theta)-z)-(H(\theta)-z)^*\}G\langle x\rangle^{-1}\phi) \\ &\quad +\frac{c}{\theta}(\langle x\rangle^{-1}\phi, (B_1+B_2+B_3)\langle x\rangle^{-1}\phi) \end{aligned} \tag{2.30}$$

where we used $E_\rho = 1-(1-E_\rho)$ and

$$\begin{aligned} B_1 &:= G^*(1-E_\rho)\{(H(\theta)-z)-(H(\theta)-z)^*\}(1-E_\rho)G \\ B_2 &:= G^*(1-E_\rho)\{(H(\theta)-z)-(H(\theta)-z)^*\}G \\ B_3 &:= G^*\{(H(\theta)-z)-(H(\theta)-z)^*\}(1-E_\rho)G. \end{aligned}$$

By arguments similar to those in (2.29) we can see that

$$\|B_1+B_2+B_3\| \leqq c(\lambda-\lambda_2'')^{-1}, \quad \text{for } \lambda>\lambda_2'' \text{ and } \lambda_2''>0 \text{ suitably.} \tag{2.31}$$

Inserting (2.28), (2.30) and (2.31) into (2.27) one obtains for $\lambda_2 := \max\{\lambda_2', \lambda_2''\}$

$$\|G\langle x\rangle^{-1}\|^2 = \frac{c}{\theta}\{\|\langle x\rangle^{-1}G\langle x\rangle^{-1}\| + (\lambda-\lambda_2)^{-1}\}, \tag{2.32}$$

for $\lambda>\lambda_2$ and this gives (2.26). □

We can now prove the main resolvent estimates for the long-range case.

Theorem 2.6. *Let $k, N\in\mathbb{N}$, $\eta>0$ and $H_L := H_0+V_L$ as above. Then there exist $\lambda_3>0$, $c>0$ and $\varepsilon_1>0$ such that*

$$\|\langle x\rangle^{-N}(H_L-\lambda-i\eta)^{-k}\langle x\rangle^{-N}\| \leqq c(\lambda-\lambda_3)^{-\frac{k-1}{2}-\varepsilon_1} \quad \textit{for } 1\leqq k\leqq N, \tag{2.33}$$

and

$$\|\langle x\rangle^{-N}(H_0+1)^\delta(H_L-\lambda-i\eta)^{-k}\langle x\rangle^{-N}\| \leqq c(\lambda)^{-\frac{k-1}{2}-\varepsilon_1} \tag{2.34}$$

for $N\geqq 2$, $1\leqq k\leqq N-1$, $\delta\in[0,\frac{1}{2})$ and $\lambda<\lambda_3$.

Proof. Let A, $G := G(\theta, z)$ be as in (2.2), (2.3) and $k, N\in\mathbb{N}$, $\delta\in[0,\frac{1}{2})$. Then a direct calculation shows

$$\frac{d}{d\theta}G = [iA, G] - \frac{(-i\theta)^N}{N!}G[H_L, iA]^{(N+1)}G. \tag{2.35}$$

Now, denote

$$F := \langle x\rangle^{-N}(H_0+1)^\delta G^k\langle x\rangle^{-N}.$$

Then we have

$$F' := \frac{d}{d\theta} F = \langle x\rangle^{-N}(H_0+1)^{\delta}[iA, G^k]\langle x\rangle^{-N}$$
$$-\frac{(-i\theta)^N}{N!}\langle x\rangle^{-N}(H_0+1)^{\delta}\sum_{j=1}^{k}(G^j[H_L, iA]^{(N+1)}G^{k+1-j})\langle x\rangle^{-N}$$

and we can estimate for a suitable $c>0$ (note that $[H_L, iA]^{(k)}$ is bounded for $k\geqq 2$)

$$\begin{aligned}\|F'\| \leqq c\{&\|\langle x\rangle^{-N}(H_0+1)^{\delta}A(H_0+1)^{-\delta}\langle x\rangle^{N-1}\|\,\|\langle x\rangle^{-(N-1)}(H_0+1)^{\delta}G^k\langle x\rangle^{-N}\|\\ &+\|\langle x\rangle^{-N}(H_0+1)^{\delta}G^k\langle x\rangle^{-(N-1)}\|\,\|\langle x\rangle^{(N-1)}A\langle x\rangle^{-N}\|\\ &+\Theta^N\|\langle x\rangle^{-1}(H_0+1)^{\delta}G\|\,\|G\|^{(k-1)}\|G\langle x\rangle^{-1}\|\}.\end{aligned} \tag{2.36}$$

Since $\langle x\rangle^{-N}(H_0+1)^{\delta}A(H_0+1)^{\delta}\langle x\rangle^{N-1}$ and $\langle x\rangle^{N-1}A\langle x\rangle^{-N}$ are pseudodifferential operators in the class $OPS(0,\{-i\}_{i=1}^{\infty})$, they are bounded (see Theorem A3, Appendix). Note that this is the technical reason for our choice of A in (2.2).

Furthermore, by an interpolation argument we can split

$$\|\langle x\rangle^{-(N-1)}(H_0+1)^{\delta}G^k\langle x\rangle^{-N}\| \leqq \|(H_0+1)^{\delta}G^k\langle x\rangle^{-N}\|^{\frac{1}{N}}\|F\|^{1-\frac{1}{N}} \tag{2.37}$$

and

$$\|\langle x\rangle^{-N}(H_0+1)^{\delta}G^k\langle x\rangle^{-(N-1)}\| \leqq \|\langle x\rangle^{-N}(H_0+1)^{\delta}G^k\|^{\frac{1}{N}}\|F\|^{1-\frac{1}{N}}. \tag{2.38}$$

Inserting (2.37) and (2.38) into (2.36), one gets for a suitable c

$$\begin{aligned}\|F'\| \leqq c\{&\|(H_0+1)^{\delta}G^k\langle x\rangle^{-1}\|^{\frac{1}{N}}+\|\langle x\rangle^{-1}(H_0+1)^{\delta}G^k\|^{\frac{1}{N}}\}\,\|F\|^{1-\frac{1}{N}}\\ &+c\theta^N\|\langle x\rangle^{-1}(H_0+1)^{\delta}G\|\,\|G\|^{k-1}\|G\langle x\rangle^{-1}\|.\end{aligned} \tag{2.39}$$

Now we consider three cases:

Case 1. Let $\delta=0$, $k=N=1$, then with

$$F_1 := \langle x\rangle^{-1}G\langle x\rangle^{-1}$$

we get from (2.39)

$$\|F_1'\| \leqq c(\|G\langle x\rangle^{-1}\|+\theta\|\langle x\rangle^{-1}G\|^2).$$

Now by Lemma 2.5 (2.26) and (2.32) we get the differential inequality

$$\|F_1'\| \leqq c\{\theta^{-\frac{1}{2}}(\|F_1\|^{\frac{1}{2}}+(\lambda-\lambda_2)^{-\frac{1}{4}})+\|F_1\|+(\lambda-\lambda_2)^{-\frac{1}{2}}\}.$$

Inserting the a priori estimate (2.15) from Lemma 2.3 (for $\delta=0$), integrating and inserting again we obtain

$$\|F_1\| \leqq c(\lambda-\lambda_3)^{-\frac{1}{8}} \quad \text{for } \lambda>\lambda_3 := \max\{\lambda_1, \lambda_2\}. \tag{2.40}$$

This implies (2.33) for $k=N=1$ and $\varepsilon_1=\frac{1}{8}$. Note that (2.40) and (2.26) in Lemma 2.5 gives

$$\|G\langle x\rangle^{-1}\| \leqq c\theta^{-\frac{1}{2}}(\lambda-\lambda_3)^{-\frac{1}{16}}, \quad \text{for suitable } c>0. \tag{2.41}$$

Case 2. Let $\delta=0$, $N\leqq 2$, $1\leqq k\leqq N$. Denote

$$F_2:=\langle x\rangle^{-N}G^k\langle x\rangle^{-N}.$$

Then we get from (2.39)

$$\|F_2'\|\leqq c\{\|G\|^{\frac{k-1}{N}}\|G\langle x\rangle^{-1}\|^{\frac{1}{N}}\|F_2\|^{1-\frac{1}{N}}+\theta^N\|G\|^{k-1}\|G\langle x\rangle^{-1}\|^2\}.$$

Thus by (2.15) and (2.41) we obtain the differential inequality

$$\|F_2'\|\leqq c\{\theta^{-1+\frac{1}{2N}}(\lambda-\lambda_3)^{-\frac{\varepsilon_2}{N}}\|F_2\|^{(1-\frac{1}{N})}+(\lambda-\lambda_3)^{-\varepsilon_2}\} \tag{2.42}$$

for $\varepsilon_2:=\dfrac{k-1}{2}+\dfrac{1}{16}$ and $\lambda>\lambda_3:=\max\{\lambda_1,\lambda_2\}$. Starting with the a priori estimate

$$\|F_2\|\leqq c\theta^{-k}\lambda^{-\frac{k}{2}} \quad \text{for } \lambda>\lambda_3$$

which follows from (2.15) in Lemma 2.3, we can iterate (2.42) and obtain after a finite numver of steps

$$\|F_2\|\leqq c(\lambda-\lambda_3)^{-\frac{k-1}{2}-\varepsilon_1}$$

for a suitable $\varepsilon_1>0$. This implies (2.33) for $1\leqq k\leqq N$, $N\geqq 2$.

Case 3. Let $\delta\in[0,\frac{1}{2})$, $N\geqq 2$, $1\leqq k\leqq N-1$. Denote

$$F_3:=\langle x\rangle^{-N}(H_0+1)^{\delta}G^k\langle x\rangle^{-N}.$$

Then we get from (2.39) and Lemma 2.3 (2.15) the differential inequality

$$\|F_3'\|\leqq c\{\theta^{-1+\frac{1}{N}}\lambda^{-\frac{\varepsilon_3}{N}}\|F_3\|^{1-\frac{1}{N}}+\lambda^{-\varepsilon_3}\}, \quad \text{for } \lambda>\lambda_1$$

and $\varepsilon_3:=\dfrac{k}{2}-\delta$. Now starting with the a priori estimate

$$\|F_3\|\leqq c\theta^{-k}\lambda^{-\varepsilon_3}$$

which follows also from (2.15), one gets by a similar iteration as above

$$\|F_3\|\leqq c\lambda^{-\frac{k}{2}+\delta+\varepsilon} \text{ for } \varepsilon>0,$$

which implies (2.34) for suitable ε_1. □

3. The Short-Range Perturbation

We will prove now an analog of the main resolvent estimate (2.33) for the whole Hamiltonian $H=H_L+V_S$. In order to do this we need a technical lemma.

Lemma 3.1. *Let $N\geqq 2$, $1\leqq k\leqq N-1$, $\eta>0$ and V_S as in Theorem 1. Let $\varepsilon_1>0$ and $\lambda_3>0$ as in Theorem 2.6. Then there exists a $c>0$ such that*

$$\|\langle x\rangle^N V_S(H_L-\lambda-i\eta)^{-k}\langle x\rangle^{-N}\|\leqq c(\lambda-\lambda_3)^{-\frac{k-1}{2}-\varepsilon_1} \quad \text{for } \lambda>\lambda_3. \tag{3.1}$$

Proof. Let $R>1$ and $\chi_R \in C_0^\infty(1R)$ such that

$$\chi_R(x) := \begin{cases} 1 & \text{if } |x| \leqq R \\ 0 & \text{if } |x| \geqq 2R. \end{cases}$$

Denote $R_L := (H_L - \lambda - i\eta)^{-1}$. Then

$$\begin{aligned} \|\langle x\rangle^N V_S R_L^k \langle x\rangle^{-N}\| &\leqq \|\langle x\rangle^N \chi_R V_S (H_0+1)^{-\delta}\langle x\rangle^N\| \, \|\langle x\rangle^{-N}(H_0+1)^\delta R_L^k \langle x\rangle^{-N}\| \\ &\quad + \|\langle x\rangle^{2N} V_S(1-\chi_R)\| \, \|\langle x\rangle^{-N} R_L^k \langle x\rangle^{-N}\|. \end{aligned}$$

Since by assumption (S) and Theorem A_3

$$\begin{aligned} \|\langle x\rangle^N \chi_R V_S(H_0+1)^{-\delta}\langle x\rangle^N\| &\leqq \|\langle x\rangle^{2N}\chi_R\| \, \|V_S(H_0+1)^{-\delta}\| \\ &\quad \|(H_0+1)^\delta \langle x\rangle^{-N}(H_0+1)^{-\delta}\langle x\rangle^N\| \leqq c, \end{aligned}$$

we obtain (using Theorem 2.6 and assumption (S)) for a suitable c and R suitably large

$$\|\langle x\rangle^N V_S R_L^k \langle x\rangle^{-N}\| \leqq c\{(\lambda-\lambda_3)^{-\left(\frac{k-1}{2}\right)-\varepsilon_1} + \lambda^{-\left(\frac{k-2}{2}\right)-\varepsilon_1}\}$$

and this gives (3.1). □

The next Theorem asserts the central estimate in this paper. It is proved by an induction argument similar to one used by Jensen and Kato [8].

Theorem 3.2. *Let $N \in \mathbb{N}$, $N \geqq 2$, $\eta > 0$ and $H = H_0 + V_S + V_L$ as in Theorem 1 and $\varepsilon_1 > 0$ as in Theorem 2.6. Then there exist $\lambda_4 > 0$ and $c > 0$ such that*

$$\|\langle x\rangle^{-N}(H-\lambda-i\eta)^{-N}\langle x\rangle^{-N}\| \leqq c(\lambda-\lambda_4)^{-\left(\frac{N-1}{2}\right)-\varepsilon_1} \quad \text{for } \lambda > \lambda_4. \tag{3.2}$$

Proof. Let $z := \lambda + i\eta$, $H_L := H_0 + V_L$. Denote $R_L := (H_L - z)^{-1}$ and $R := (H - z)^{-1}$. Then, if we differentiate the resolvent identity

$$R_L = R(1 + V_S R_L)$$

$(N-1)$-times with respect to λ, we get

$$R_L^N = R^N(1 + V_S R) + \binom{N-1}{1} R^{N-1} V_S R_L^2 + \dots .$$

Using the identity

$$1 - R V_S = (1 + R_L V_S)^{-1},$$

we get (compare [8])

$$R^N = \{(1 + R_L V_S)^{-1} R_L^N - \binom{N-1}{1} R^2 V_S R_L^{N-1} - \dots\}(1 - V_S R_L)^{-1}. \tag{3.3}$$

For $N=2$ we get by (3.3)

$$\begin{aligned} \langle x\rangle^{-2} R^2 \langle x\rangle^{-2} &= \{(1 + \langle x\rangle^{-2} R_L V_S \langle x\rangle^2)^{-1} \langle x\rangle^{-2} R_L^2 \langle x\rangle^{-2}\} \\ &\quad \cdot (1 - \langle x\rangle^2 V_S R_L \langle x\rangle^{-2})^{-1}. \end{aligned}$$

By Lemma 3.1 we know that $\|\langle x\rangle^2 V_S R_L \langle x\rangle^{-2}\|<1$ if λ is large enough.

Therefore using Theorem 2.6 (2.33) we obtain

$$\|\langle x\rangle^{-2} R^2 \langle x\rangle^{-2}\| \leqq c(\lambda-\lambda_4)^{-\frac{1}{2}-\varepsilon_1} \quad \text{for } \lambda>\lambda_4$$

and λ_4 suitably large.

Now the assertion (3.2) follows by induction on N. Indeed, suppose it holds for all k with $2 \leqq k \leqq N-1$ and consider a typical term in (3.3) (multiplied by the weights $\langle x\rangle^{-N}$). Then we can estimate

$$\begin{aligned}
&\|\langle x\rangle^{-N} R^k V_S R_L^{(N+1-k)}(1-V_S R_L)^{-1}\langle x\rangle^{-N}\| \\
&\quad \leqq \|\langle x\rangle^{-N} R^k \langle x\rangle^{-N}\| \, \|\langle x\rangle^N V_S R^{(N+1-k)}\langle x\rangle^{-N}\| \, \|(1-\langle x\rangle^N V_S R_L \langle x\rangle^{-N})^{-1}\| \\
&\quad \leqq c(\lambda-\lambda_4)^{-\left(\frac{k-1}{2}\right)-\varepsilon_1} \cdot (\lambda-\lambda_4)^{-\left(\frac{N-k}{2}\right)-\varepsilon_1} \quad \text{for } \lambda>\lambda_4
\end{aligned}$$

and suitably large λ_4.

The last inequality holds by the induction hypothesis and Lemma 3.1 and since

$$\|\langle x\rangle^N V_S R_L \langle x\rangle^{-N}\|<1 \quad \text{for } \lambda \text{ large enough.}$$

Thus we have

$$\|\langle x\rangle^{-N} R^k V_S R_L^{N+1-k}(1-V_S R_L)^{-1}\langle x\rangle^{-N}\| \leqq c(\lambda-\lambda_4)^{-\left(\frac{N-1}{2}\right)-\varepsilon_1}. \tag{3.4}$$

The first term in (3.3) is exceptional but can be estimated in a way similar to the others, if one uses Theorem 2.6 (2.33), by

$$\|\langle x\rangle^{-N}(1+R_L V_S)^{-1} R_L^N \langle x\rangle^{-N}\| \leqq c(\lambda-\lambda_4)^{-\left(\frac{N-1}{2}\right)-\varepsilon_1} \tag{3.5}$$

for $\lambda>\lambda_4$ and λ_4 suitably large.

Collecting all terms (3.4) and (3.5) one gets (3.2). □

4. Proof of Theorem 1

We first prove a lemma which will be needed in the following.

Lemma 4.1. *Let* $N \in \mathbb{N}$, $\lambda_0>0$, $H=H_0+V$ *and* $\chi \in C^\infty(\mathbb{R})$ *with* $\chi(\lambda)=0$ *if* $\lambda \leqq \lambda_0$ *and* $\chi(\lambda)=1$ *if* $\lambda \geqq 2\lambda_0$ *as in Theorem* 1. *Then*

$$\langle x\rangle^N \chi(H) \langle x\rangle^{-N}$$

is a bounded operator in $\mathscr{H}$.

Proof. Denote

$$B(\alpha) := e^{i\alpha\langle x\rangle}(H+\lambda_0)^{-1} e^{-i\alpha\langle x\rangle}. \quad \text{for } \alpha \in \mathbb{R}$$

and

$$\tilde{\chi}(\lambda) := \begin{cases} \chi\left(\frac{1}{\lambda}-\lambda_0\right) & \text{if } \lambda \in \left[0, \frac{1}{\lambda_0}\right] \\ \chi\left(\lambda_0-\frac{1}{\lambda}\right) & \text{if } \lambda \in (-\infty, 0) \cup \left(\frac{1}{\lambda_0}, \infty\right). \end{cases}$$

Note that $\tilde{\chi}\in C_0^\infty(\mathbb{R})$ and $\alpha\mapsto B(\alpha)$ is norm $-C^N$ (since V commutes with $\langle x\rangle$ and $\alpha\mapsto e^{i\alpha\langle x\rangle}H_0e^{-i\alpha\langle x\rangle}$ is an analytic family of type (A) [10]). Then $\tilde{\chi}(B(\alpha))$ preserves $D(\langle x\rangle^N)$ for $\alpha\in\mathbb{R}$ (see [16, Prop. A4]). Since $\tilde{\chi}(B(0))=\chi(H)$, Lemma 4.1 follows by the closed graph theorem. □

Now we reexpress e^{-itH} in terms of the resolvent of H and use the estimate (3.2) in Theorem 3.2 to prove Theorem 1.

Proof (*of Theorem* 1). Let $N\geqq 3$, $\eta>0$ and χ be as above. By Cauchy's integral formula and the functional calculus we have, for $\pm t\in(0,\infty)$,

$$e^{-itH\mp\eta t}\chi(H)=\frac{(N-1)!}{2\pi i}(-it)^{-(N-1)}\int_{\mathbb{R}} d\lambda\, e^{-it\lambda}(H-\lambda\mp i\eta)^{-N}\chi(H). \tag{4.1}$$

We consider only the case $t>0$ in the following (the case $t<0$ can be treated similarly by "dilating" the essential spectrum of $H(\theta)$ into the positive half-plane).

From (4.1) follows for suitable $c>0$

$$\begin{aligned}&\|\langle x\rangle^{-N}e^{-itH-\eta t}\chi(H)\langle x\rangle^{-N}\|\\&\quad\leqq ct^{-(N-1)}\int d\lambda\|\langle x\rangle^{-N}(H-\lambda-i\eta)^{-N}\chi(H)\langle x\rangle^{-N}\|\end{aligned} \tag{4.2}$$

Let $\lambda_0>0$. Then we split the R.H.S. of (4.2) into two parts

$$ct^{-(N-1)}\{I_1+I_2\}=ct^{-(N-1)}\int_{-\infty}^{\frac{\lambda_0}{2}}\dots+\int_{\frac{\lambda_0}{2}}^{\infty}\dots. \tag{4.3}$$

The first integral I_1 is obviously bounded uniformly in η and I_2 can be estimated by

$$I_2\leqq\int_{\frac{\lambda_0}{2}}^{\infty}d\lambda\|\langle x\rangle^{-N}(H-\lambda-i\eta)^{-N}\langle x\rangle^{-N}\|\,\|\langle x\rangle^{N}\chi(H)\langle x\rangle^{-N}\|. \tag{4.4}$$

The second factor of the integrand in (4.4) is bounded by Lemma 4.1 and if we choose $\lambda_0:=4\lambda_4$ (for $\lambda_4>0$ as in Theorem 3.2) we obtain for suitable $c>0$

$$I_2\leqq c\int_{\lambda_0}^{\infty}\left(\lambda-\frac{\lambda_0}{4}\right)^{-\left(\frac{N-1}{2}\right)-\varepsilon_1}\leqq c\left(\frac{\lambda_0}{4}\right)^{-\left(\frac{N-1}{2}\right)-\varepsilon_1}\qquad\text{for } \varepsilon_1>0 \text{ as in Theorem 3.2.}$$

Thus I_2 and therefore the R.H.S. of (4.2) is also bounded uniformly in η and by taking limits in (4.2) we get

$$\|\langle x\rangle^{-N}e^{-itH}\chi(H)\langle x\rangle^{-N}\|\leqq c(1+t)^{-(N-1)},\qquad t>0 \tag{4.5}$$

for suitable $c>0$.

Now let $s\in[0,N]$. Then an interpolation argument shows that

$$\|\langle x\rangle^{-s}e^{-itH}\chi(H)\langle x\rangle^{-s}\|\leqq c(1+t)^{-s+\frac{s}{N}}\qquad\text{for } t>0$$

and together with a similar estimate for $t<0$ one gets (1.2). □

5. Appendix. Some Facts About Pseudodifferential Operators

For the convenience of the reader, we collect some basic results on the classes of pseudodifferential operators used here.

We are using the notation of Agmon [1]. Some results are proved in Taylor's book [22] but we are mainly quoting from a thesis of Klein [12].

Definition. Let $k, m \in \mathbb{N}$ and $\{\mu(i)\}_{i=1}^{k}$ be a non-increasing sequence of real numbers. Then we define the symbol class $S^m_{\{\mu(i)\}}$ to consist of $p(x,\xi) \in C^k(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leqq c_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|} \langle x \rangle^{\mu(|\alpha|)}$$

holds for $x, \xi \in \mathbb{R}^n$, multiindices α, β with $|\alpha|, |\beta| \leqq k$ and suitable constants $c_{\alpha\beta} > 0$.

Definition 2. Let $m \in \mathbb{N}$ and $\{\mu(i)\}$ as above. Then we define the class of pseudodifferential operators

$$\mathrm{OPS}(m, \{\mu(i)\})$$

to consist of operators $p(x,D)$ defined by

$$p(x,D)\,\phi(x) := \int p(x,\xi)\,\hat{\phi}(\xi)\,e^{ix\xi}\,d\xi, \qquad \text{for } \phi \in L^2(\mathbb{R}^n)$$

and

$$p \in S^m_{\{\mu(i)\}}.$$

Then we have the symbolic calculus:

Theorem A$_1$. *Let $m_1, m_2 \in \mathbb{N}$ and $\{\mu_1(\cdot)\}$, $\{\mu_2(\cdot)\}$ be non-increasing sequences. Let*

$$a \in S^{m_1}_{\{\mu_1(i)\}}, \qquad b \in S^{m_2}_{\{\mu_2(i)\}}$$

and $A \in \mathrm{OPS}(m_1, \{\mu_1(i)\})$ and $B \in \mathrm{OPS}(m_2, \{\mu_2(i)\})$ be the operators arising from the symbols a and b respectively.

Then the operator AB is a pseudodifferential operator with

$$AB \in \mathrm{OPS}(m_1 + m_2, \{\mu_1(0) + \mu_2(0) - i\})$$

and the symbol $a \odot b$ of AB has an expansion for any $N \in \mathbb{N}$

$$a \odot b = \sum_{|\alpha| < N} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\beta a\, D_x^\alpha b + r_N(a,b)$$

where the remainder $r_N(a,b)$ is a symbol

$$r_N(a,b) \in S^{\tilde{m}}_{\{\tilde{\mu}(i)\}},$$

with $\tilde{m} := m_1 + m_2 - N - |\beta|$ and

$$\tilde{\mu}(i) := \max_{j+k=i} \{\mu_1(j) + \mu_2(N+k)\}, \qquad i \in \mathbb{N}.$$

For a proof, see [12].

Corollary A$_2$. *Let $a \in S^{m_1}_{\{\mu_1(i)\}}$, $b \in S^{m_2}_{\{\mu_2(i)\}}$ and A and B the corresponding operators respectively. Then we have for the commutator*

$$[A, B] \in \mathrm{OPS}(m_1 + m_2 - 1, \{\tilde{\mu}(i)\})$$

$$\tilde{\mu}(i) := \max_{j+k=i} \{\mu_1(j) + \mu_2(1+k),\ \mu_1(1+j) + \mu_2(k)\}, \qquad i \in \mathbb{N}.$$

One of the central problems in pseudodifferential operator theory is the L^2-continuity. The first result of this type was shown by Calderon and Vaillancourt, but the optimized version we are quoting here i due to Cordes [3]. Compare also [22, Ch. XIII].

Theorem A$_3$. *Let $p \in S^0_{\{0\}}$ such that*

$$|D^\alpha_x D^\beta_\xi p(x, \xi)| \leqq c_{\alpha\beta} \tag{A$_1$}$$

for $(x, \xi) \in \mathbb{R}^{2n}$ and $|\alpha|, |\beta| \leqq \left[\frac{n}{2}\right] + 1$ and suitable $c_{\alpha\beta} > 0$.

Then the associated operator is bounded, i.e.

$$\|p(x, D)\| \leqq c \sup \left\{ c_{\alpha\beta} \,\middle|\, |\alpha|, |\beta| \leqq \left[\frac{n}{2}\right] + 1 \right\}$$

where c depends only on n.

We give also a useful

Corollary A$_4$ [22]. *If $p \in S^0_{\{0\}}$ satisfies (A_1) and*

$$|p(x, \xi)| \to 0 \qquad \text{as } |x| + |\xi| \to \infty,$$

then $P(x, D)$ is compact on $L^2(\mathbb{R}^n)$.

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